Math 55a, Fall 2004

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5 th Assignment, due October 26

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1. Prove that all compact Hausdorff spaces and all metric spaces are normal. Hint: in the case of a metric space (X, d), it may be helpful to consider the function $d_S : X \to \mathbb{R}$, $\quad d_S(x) = \inf\{\, d(x, y) \mid y \in S\,\}$, associated to the choice of a closed subset $S \subset X$.

2. It was remarked in class that sequences cannot be used to characterize closure, compactness, etc., in general topological spaces. This problem introduces the notion of a net, which is analogous to the notion of a sequence (and, roughly speaking, generalizes notion of a sequence). Many arguments using sequences to establish properties of metric spaces can be translated into the language of nets, so as to apply to general topological spaces. A *directed system* is a pair $(D, \geq)$ consisting of a set D and a partial order $\geq$ on D (so $\geq$ is reflexive, transitive and antisymmetric, in the sense that $m \geq n$ and $n \geq m$ together imply $m = n$), which satisfies the additional condition that any finite subset of D must have an upper bound. By definition, a *net* in a topological space $(X, \mathcal{T})$ consists of a triple $(S, D, \geq)$, where $(D, \geq)$ is a directed system and $S : D \to X$ a map from D to X. A net $(S, D, \geq)$ in $(X, \mathcal{T})$ is said to converge to a point x in X – equivalently, x is the limit of the net – if, for every neighborhood N_x of x, there exists $n_0 \in D$ such that $S(n) \in N_x$ whenever $n \geq n_0$. The point x is a *cluster point* of the net $(S, D, \geq)$ if, for every neighborhood N_x of x and every $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $S(n) \in N_x$. A net $(T, E, \geq_E)$ is a *subnet* of the net $(S, D, \geq_D)$ if there exists a function $F : E \to D$ such that $T = S \circ F$ and such that $F(k)$ gets large as k gets large, in the following sense: for each $n_0 \in D$, there exists $k_0 \in E$ with the property that $F(k) \geq_D n_0$ whenever $k \geq_E k_0$. Note that sequences are particular types of nets, with $(\mathbb{N}, \geq)$ playing the role of the directed system $(D, \geq)$. Show:

a) If X is Hausdorff, the limit of any convergent net is uniquely determined.

b) A subset B of a topological space X is closed if and only if no net in B converges to a point in the complement of B.

c) A point $x \in X$ lies in the closure of a set $B \subset X$ if and only if x is the limit of some net in B.

d) A point $x \in X$ is a cluster point of a net $(S, D, \geq)$ in X if and only if there exists a subnet which converges to x.

Remark: The family of all neighborhoods of a point is directed by the relation $U_1 \geq U_2 \iff U_1 \subset U_2$; this directed system is particularly useful in arguments involving nets. Another helpful idea is to consider the family of subsets $B_n = \{S(m) \mid m \geq n\}$ of X defined by a net $(S, D, \geq)$ in X.