Math 55a, Fall 2004

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6 th Assignment, due November 2

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1. Prove that a topological space is compact if and only if each net has a convergent subnet. Note: the remark at the end of the 5 th assignment is relevant for this problem also.

2. Let X be a locally compact Hausdorff space and $C(X)$ the space of all continuous real-valued (or alternatively, complex-valued) functions on X. For any compact set $K \subset X$ and $\epsilon > 0$, define

$$U_{K,\epsilon} = \{\, f \in C(X) \ \mid \ \sup_{x \in K} |f(x)| < \epsilon \,\}\,.$$

a) Show that the sets $V_{f,K,\epsilon} = \{f+g \mid g \in U_{K,\epsilon}\}$, indexed by $f \in C(X)$, compact subsets $K \subset X$, and choices of $\epsilon > 0$, constitute a base of a topology on $C(X)$, the so-called *topology of convergence on compacta.* Verify that the sets $U_{K,\epsilon}$ constitute a neighborhood base for 0 in this topology.
b) There is an obvious notion of Cauchy net in $C(X)$ – a net $(S, D, \geq)$ such that for every neighborhood U of 0, there exists $n_0 \in D$ such that $S(n) - S(m) \in U$ whenever $n, m \geq n_0$. Show that that $C(X)$ is complete, i.e., that every Cauchy net conveges.
c) Verify that addition and multiplication of functions, considered as maps from $C(X) \times C(X)$ to $C(X)$, are continuous. Also verify the continuity of the map $f \mapsto 1/f$ from

$$C^*(X) = \{f \in C(X) \mid f(x) \neq 0 \text{ for all } x \in X\}$$

to itself when $C^*(X) \subset C(X)$ is equipped with the restricted topology.
d) Now consider two locally compact Hausdorff spaces X, Y, and a continuous map $F : X \to Y$ between them. Show that the map

$$F^* \,:\, C(Y) \longrightarrow C(X)\,, \qquad F^* f \,=\, f \circ F\,,$$

is continuous with respect to the topologies of convergence on compacta.