Math 55a, Fall 2004

Second Assignment, Solutions
Adapted from Andrew Cotton and George Lee

Problem 1. (i) First suppose that $\mathbb{R}$ is complete. We prove that the least upper bound axiom is true.

Suppose we have a set S of real numbers bounded from above by the real number M — that is, $s \in S \Rightarrow s \leq M$. If there is a maximum element m in S, then it is an upper bound; and it is a least upper bound, because any smaller number is smaller than the element m. So, assume there is no maximum element.

Then for each element $s \in S$, there exists a smallest k such that there exists $t \in S$ bigger than $s + \frac{1}{k}$. (If no such k exists, then no element in S can be bigger than s — so s is maximal, a contradiction.)

So let s_1 be any element in S; find the corresponding k and t; and let $k_1 = k$ and $t_1 = t$. Then for $i \geq 2$, write $s_i = t_{i-1}$ and find corresponding k_i and t_i. Then observe that:

• $s_1, s_2, \ldots$ is an increasing sequence since $s_{i+1} > s_i + \frac{1}{k_i}$.

• $\{k_i\}$ is a nondecreasing sequence because if $k_i > k_{i+1} = k$, then we could have picked $k_i = k$ instead — contradicting its minimal definition.

• $\{k_i\}$ is unbounded. Otherwise, if $k_i \leq K$ for all i, then for each i we would have $s_{i+1} \geq s_1 + \frac{i}{K}$. But this is bigger than M for large enough i, contradicting the assumption that S is bounded.

Now we prove that $\{s_i\}$ is a Cauchy sequence. Given any $\epsilon > 0$, we can find k such that $\frac{1}{k-1} < \epsilon$ and we can find N such that $k_N \geq k$. Then for any $i, j \geq N$ with $i \leq j$, we have $|a_j - a_i| = a_j - a_i \leq a_j - a_N$ since $\{a_i\}$ is an increasing sequence. But we must have $a_j - a_N \leq \frac{1}{k-1}$, or else we could have picked $k_N = k - 1$, contradicting its minimal definition. Thus $i, j \geq N \Rightarrow |a_i - a_j| \leq \frac{1}{k-1} < \epsilon$, and $\{s_i\}$ is a Cauchy sequence.

Since $\mathbb{R}$ is complete, this Cauchy sequence converges to some real number U. Suppose by way of contradiction there existed $s \in S$ bigger than U, where $s - U > \frac{1}{n}$ for some integer n. There exists some $k_i > n$ (since $\{k_i\}$ is not bounded); so we could have picked $k_i = n$ and $t_i = s$, contradicting the minimal choice of k_i.

Thus U is an upper bound. And for any smaller $U' = U - \delta$, since $\{s_i\}$ converges to U there exists $s_i > U - \delta = U'$. Therefore U is a *least* upper bound, proving that every bounded set of reals indeed has a least upper bound.

(ii) Now suppose the least upper bound axiom is true. Then given any set of reals $\{a_\alpha\}$ bounded from *below* by M, the set $\{-a_\alpha\}$ is bounded from above by $-M$ and thus has a least upper bound B. But then $\{a_\alpha\}$ has a greatest lower bound $-B$. Therefore, any set of reals bounded from below has a greatest lower bound.

Lemma 1. *Any nondecreasing sequence of reals $\{x_k\}$ with a least upper bound U converges to U. Also, any nonincreasing sequence of reals with a greatest lower bound L converges to L.*

Proof: Given any $\delta > 0$, $U - \delta$ is not a least upper bound so there must be some $x_N > U - \delta$. But since $\{x_k\}$ is nondecreasing, this implies that $U \geq x_N, x_{N+1}, \cdots > U - \delta$ so that $\forall i \geq N, |x_i - U| < \delta$. Therefore $\{x_k\}$ indeed converges to U, and a similar proof shows the other half of the lemma.

Lemma 2. *Any infinite sequence of reals has an infinite monotone subsequence.*

Proof: Consider a sequence $\{x_i\}$. We call x_i a "giant" if $\forall j > i, x_i \geq x_j$. If there are infinitely many "giants" then the subsequence $\{x_i \mid x_i \text{ is a giant}\}$ is monotonically decreasing. Also, if there are finitely many giants, there exists an index N such that for all $n > N$, x_n is not a "giant." Thus there exists an m such that $m > n$ and $x_m > x_n$. Thus we can form a monotonically increasing sequence.

It is easy to see that any Cauchy sequence must be bounded and thus, by the least upper bound axiom have both a least upper bound and a greatest lower bound. Thus by the above lemmas, any Cauchy sequence has a monotone subsequence, and this subsequence converges to the upper or lower bound. If a subsequence converges to a certain limit, then the original Cauchy sequence must converge to that same limit, and therefore, the least upper bound axiom implies that the reals are complete . . . and combined with part (a), this implies that the two claims are equivalent.

Problem 2. We begin with the following lemma:

Lemma 3. *Given a metric space (X, d), an element $x \in X$, and a positive real ϵ, the open epsilon-ball $B(x_0, \epsilon) = \{x \mid d(x_0, x) < \epsilon)$ is open.*

Proof: Suppose that $x \in B(x_0, \epsilon)$. Then we claim that $B' = B(x, \epsilon - d(x, x_0)) \subseteq B(x_0, \epsilon)$. Indeed, if $y \in B'$ then $d(x, y) < \epsilon - d(x, x_0)$; so by the triangle inequality, $\epsilon > d(x_0, x) + d(x, y) \geq d(x_0, y)$ and $y \in B(x_0, \epsilon)$. Thus for every element x in the given set, there is an epsilon-ball centered at x in the set — which is exactly the definition of "open."

Now we prove that the statements are equivalent:

• (a) $\Rightarrow$ (b). Given a closed set $S \subset Y$, let $U = Y - S$; by the definition of "closed," U is open. Also, every element of X maps to an element in either S or U, so every element of X is in exactly one of $F^{-1}(S)$ or $F^{-1}(U)$ and thus $F^{-1}(S) = X - F^{-1}(U)$. From (a), $F^{-1}(U)$ is open, which implies that $F^{-1}(S)$ is closed.

• (b) $\Rightarrow$ (a). This proof is the same as (a) $\Rightarrow$ (b), with the words "open" and "closed" interchanged.

• (a) $\Rightarrow$ (c). Given $x_0 \in X$ and $\epsilon > 0$, the lemma proved above implies that $U = B(F(x_0), \epsilon)$ is open. Thus $F^{-1}(U)$ is open and contains x_0, which (by the definition of "open") implies that some epsilon-ball $B(x_0, \delta) \subseteq F^{-1}(U)$, as desired.

• (c) $\Rightarrow$ (a). Given an open set U, for each $x_0 \in F^{-1}(U)$ we must prove there exists $\delta > 0$ such that $B(x_0, \delta) \in F^{-1}(U)$. Since U is open, there exists $\epsilon > 0$ such that $B(F(x_0), \epsilon) \subseteq U$. Then by (c), there exists $\delta > 0$ such that $x \in B(x_0, \delta) \Rightarrow F(x) \in B(F(x_0), \epsilon) \subseteq U$. Therefore, $B(x_0, \delta) \subseteq F^{-1}(U)$, as desired.

• (c) $\Rightarrow$ (d). Suppose that $\{x_k\}$ converges to x_∞. Then (c) implies that $\forall \epsilon > 0, \exists \delta > 0$ such that $d_X(x, x_\infty) < \delta \Rightarrow d_Y(F(x), F(x_\infty)) < \epsilon$. And for this δ, because $\{x_k\}$ is convergent there exists N such that $k \geq N \Rightarrow d_X(x_k, x_\infty) < \delta \Rightarrow d_Y(F(x_k), F(x_\infty)) < \epsilon$. Therefore $\{F(x_k)\}$ indeed converges to $F(x_\infty) \in Y$, and we know (from a proof in class) that this limit is unique.

• (d) $\Rightarrow$ (c). Suppose by way of contradiction that there exist $x_0 \in X$ and $\epsilon > 0$ such that $\forall \delta > 0$, $\exists y \in B(x_0, \delta)$ such that $F(y) \notin B(F(x_0), \epsilon)$. Then for each $\delta_i = \frac{1}{i}$, let y_i be such an element. For any value $r > 0$, there exists i such that $\frac{1}{i} < r$ so that $y_i, y_{i+1}, \ldots$ are all in $B(x_0, r)$; thus, $\{y_k\}$ converges to x_0. Therefore, by (d), $F(y_1), F(y_2), \ldots$ must converge to $F(x_0)$. But all of the $F(y_i)$ are outside the epsilon-ball $B(F(x_0), \epsilon)$, so they *can't* converge to $F(x_0)$ — a contradiction. Thus our assumption was false, and (c) is indeed true.

Problem 3. Notice that $r \in [0, 1]$ is in U if and only if it cannot be written without a 1 in base 3: Each of the intervals in U of length $\frac{1}{3^k}$

contains exactly the numbers that must have a 1 in base-3, with the leftmost 1 in the $\frac{1}{3^k}$ place.

(a) Each interval in U of length $\frac{1}{3^k}$ corresponds to the numbers whose first $k+1$ digits in base 3 are $.a_1a_2\ldots a_k1$ for some fixed $a_i \in \{0,2\}$ (and whose remaining digits are not all 0, since in that case $.a_1a_2\ldots a_k1\overline{0} = .a_1a_2\ldots a_k0\overline{2}$ is not in U). There are 2^k such possible beginning strings of $k+1$ digits (the first k digits can each be either 0 or 2), so there are exactly 2^k intervals of length $\frac{1}{3^k}$. Thus the total length of the I_k is

$$\sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = 1.$$

(b) For any "partition" of $[0,1]$ into $n$open intervals I_i of length $\Delta_i > 0$ (the "partition" actually doesn't include the $n+1$ endpoints of the intervals), define

$$S_L = \sum_{i=1}^{n} \Delta_i \cdot \inf_{x\in I_i}\left(\chi_U(x)\right)$$

and

$$S_U = \sum_{i=1}^{n} \Delta_i \cdot \sup_{x\in I_i}\left(\chi_U(x)\right).$$

We claim that over all partitions, the supremum of S_L and the infimum of S_U both equal 1, implying that χ_U is indeed Riemann integrable, with the integral equal to 1.

First we prove that $\inf(S_U) = 1$. Given a partition, each interval I_i contains some element of U — otherwise, the total length of the intervals in U would be at most $1 - \Delta_i < 1$, contradicting (a). Thus $\sup_{x\in I_i}\left(\chi_U(x)\right) = 1$ for each i, and $S_U = \sum_{i=1}^{n}\Delta_i = 1$. It follows immediately that $\inf(S_U) = 1$.

Next we prove that $\sup(S_L) = 1$. To do this, choose a positive integer k and split $[0,1]$ into the intervals $I_1, \ldots, I_{3^k}$ as follows:

$$\left(0, \frac{1}{3^k}\right), \left(\frac{1}{3^k}, \frac{2}{3^k}\right), \ldots, \left(\frac{3^k-1}{3^k}, 1\right).$$

If a number x is *not* in U (that is, if $\chi_U(x) = 0$), then its first k base-3 digits to the right of the decimal point (in the $\frac{1}{3}$, $\frac{1}{9}$, …, $\frac{1}{3^k}$ places) all must be 0 or 2 — so, x has one of 2^k possible strings of first k digits.

On the other hand, each of the intervals corresponds to *different* beginning strings of first k digits. So at most 2^k intervals contain a number not in U, and the other $3^k - 2^k$ intervals are contained in U. For these intervals I we have $\inf_{x \in I} (\chi_U(x)) = 1$. Therefore,

$$S_L = \sum_{i=1}^{3^k} \frac{1}{3^k} \cdot \inf_{x \in I_i} (\chi_U(x)) \geq \frac{1}{3^k} \cdot \left(3^k - 2^k\right) = 1 - \left(\frac{2}{3}\right)^k.$$

As $k \to \infty$, $\left(\frac{2}{3}\right)^k \to 0$ and S_L gets arbitrarily close to 1. Also, S_L clearly never *surpasses* 1 since $S_L \leq \sum_{i=1}^{n} \Delta_i = 1$. Therefore, $\sup(S_L) = 1$, completing the proof.