Math 55a, Fall 2004

Fourth Assignment, Solutions
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Problem 1.

First assume that (i) holds and we'll show how it implies (ii), (iii), and (iv).

For any closed set $S \subset Y$, $Y \setminus S$ is open so that $F^{-1}(Y \setminus S)$ is open. But F maps each element of X to either an element in S or an element in $Y \setminus S$, so $F^{-1}(S) = X \setminus F^{-1}(Y \setminus S)$ is open, so (ii) holds.

Also, if $V \in \mathcal{S}_Y \subset \mathcal{T}_Y$, then from (i) we know $F^{-1}(V) \in \mathcal{T}_X$, so (iii) also holds.

Given any $x \in X$ and any neighborhood $N_{F(x)}$ of $F(x)$ in Y, then $U = \operatorname{int}(N_{F(x)})$ is an open set containing $F(x)$. Thus by (i), $N_x = F^{-1}(U)$ is an open set, and thus a neighborhood (since an open set equals its interior), containing x. Clearly $F(N_x) = U \subset N_{F(x)}$, so (iv) holds as well.

Now suppose (ii) is true. Then for any open set $U \subset Y$, $Y \setminus U$ is closed so that $F^{-1}(Y \setminus U)$ is closed. But F maps each element of X to either an element in U or an element in $Y \setminus U$, so $F^{-1}(U) = X \setminus F^{-1}(Y \setminus U)$ is open, i.e. (i) holds.

Next suppose (iii) is true. Given finitely many open sets $\{U_k\}$ such that $\{F^{-1}(U_k)\}$ are open, then $F^{-1}(\bigcap_k U_k) = \bigcap_k F^{-1}(U_k)$ is open as well (since any finite intersection of open sets in X is open). So form a base $\mathcal{B}_Y$ of Y by taking all finite intersections of sets in $\mathcal{S}_Y$; then any set $U \in \mathcal{B}_Y$ has an open preimage. Now given any open sets $\{U_\alpha\}$ such that $\{F^{-1}(U_\alpha)\}$ are open, then $F^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha F^{-1}(U_\alpha)$ is open as well (since any union of open sets in X is open). And since every open set in Y is the union of sets in $\mathcal{B}_Y$, every open set in Y has open preimage, so (i) holds.

Finally, suppose (iv) is true. Then given an open set U, $U = \operatorname{int}(U)$. Thus for every point $x \in F^{-1}(U)$, U is a neighborhood of $F(x)$. Then from (iv), there exists a neighborhood $N_x \subset X$ such that $x \in \operatorname{int}(N_x)$ and $F(N_x) \subset U$. Thus the open set

$$V = \bigcup_{x \in F^{-1}(U)} \operatorname{int}(N_x)$$

contains $F^{-1}(U)$ since for each $x \in F^{-1}(U)$ we have $x \in \operatorname{int}(N_x)$. But on the other hand V is *contained* in $F^{-1}(U)$ since for each x we have $F(\operatorname{int}(N_x)) \subset F(N_x) \subset U$. Thus V *equals* $F^{-1}(U)$, proving (i).

Thus, (i) is equivalent to each of the other three statements, and therefore they are all equivalent.

Problem 2.
(a) If $x \neq y$ are different points, consider any open neighborhoods U_x and U_y of x and y respectively. Since U_x and U_y are nonempty, their complements must be finite, so clearly $U_x \cap U_y \neq \emptyset$, so $(\mathbb{R}, \mathcal{T}_{\text{cof}})$ is not Hausdorff.

(b) We claim that a sequence converges if and only if at most one real appears in the sequence infinitely many times.

Indeed, for the "if " part suppose we have a neighborhood N whose interior U contains x. Then since U is open it must contain all but finitely many reals $\{r_1, r_2, \ldots, r_n\}$. But each of $r_1, r_2, \ldots, r_n$ appears only finitely many times in the sequence; so U (and thus N) contains all but finitely many of the sequence terms, as desired.

Let us now treat the "only if" part. Suppose that y_0, distinct from x, appears infinitely many times in the sequence. Let U be the open set containing all reals except y_0. Then U is a neighborhood of x, but because it does not contain y, it *does not* contain all but finitely many terms of the sequence, which is a contradiction.

(c) The only such functions map all of $\mathbb{R}$ to a single number x. These functions are clearly continuous since the preimage of any open set U either equals $\mathbb{R}$ or $\emptyset$ (depending on whether $x \in U$), which are both open.

But suppose by way of contradiction that the $f(\mathbb{R})$ contained at least two different numbers x and y with $|x - y| = 2\epsilon$. Then $B_x = B(x, \epsilon)$ and $B_y = B(y, \epsilon)$ are open and disjoint, so their preimages P_x and P_y are open and disjoint as well. Because x and y are in $f(\mathbb{R})$, P_x and P_y are also nonempty. Therefore, P_x contains all but finitely many reals; so P_y can only contain finitely many reals and is not open, a contradiction.

Problem 3.

Remarks: The most common mistake on this problem was confusing different notions of "open" and "compact." A set may be open or compact as a subset of one space, but not as a subset of another. For example, at first glance, it's not obvious that an open subset of $\mathbb{A}_S$ is also an open subset of $\mathbb{A}$. Similarly, Tychonoff's Theorem states that the product of compact *spaces* is a compact *space*; it does not say anything about compact *subspaces.* Some notation: in this solution, if we say that $A \subset B$ is open or compact, we mean that A is open or compact when viewed as a subset of B.

How do we deal with these issues? First note that as proved in class, given a metric space, the "metric subspace topology" on a subset is the same as the "general subspace topology." This, for instance, would tell you that any subset of $\mathbb{Z}_p$ which is open in $\mathbb{Q}_p$, is also open in $\mathbb{Z}_p$; in this solution, we say "the topology of $\mathbb{Z}_p$" (as its own space) with the understanding that whether we use the metric notion or the general notion makes no difference.

Also, if X is a Hausdorff topological space, and $Y \subset X$ is equipped with the subspace topology, then Y is compact as its own space if and only if it is compact as a subset of X. (Do you remember why?) We use this in the proof with $X = \mathbb{A}$.

Also, one base for the product topology of $\prod_{\alpha \in A} X_\alpha$ is the collection of all sets $\prod_{\alpha \in A} U_\alpha$ where each U_α is open in X_α and where all but finitely U_α equal the corresponding X_α. I had thought this was proven in class, but some people still proved it on their homework sets, so I'm guessing I was mistaken. A quick proof: if p_α is the projection function from the product to X_α, then $\bigcap_{\alpha \in B} p^{-1}(U_\alpha) = \prod_{\alpha \in B} U_\alpha \times \prod_{\alpha \notin B} X_\alpha$. For all possible B's and open sets U_α, we get the base described above.

Lemma 1. *Consider the family $\mathcal{F}$ of all sets of the form*

$$U_0 \times \prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p,$$

where S is a finite set of primes, U_0 is an open set in $\mathbb{R}$, and each U_p is an open set in $\mathbb{Q}_p$. (S and the U's are not fixed; $\mathcal{F}$ contains all sets for any such S and U's.) Then $\mathcal{F}$ is a base for $\mathbb{A}$.

Proof: First we prove that any set $V = U_0 \times \prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p$ in $\mathcal{F}$ is open. For convenience, for $p \notin S$ write $U_p = \mathbb{Z}_p$. Then given any finite set of primes S',

$$\begin{aligned} V \cap \mathbb{A}_{S'} &= (U_0 \cap \mathbb{R}) \times \prod_{p \in S'} (U_p \cap \mathbb{Q}_p) \times \prod_{p \notin S'} (U_p \cap \mathbb{Z}_p) \\ &= U_0 \times \prod_{p \in S'} U_p \times \prod_{p \notin S'} (U_p \cap \mathbb{Z}_p). \end{aligned}$$

Now, U_0 is open in $\mathbb{R}$; and for each $p \in S'$, U_p is open in $\mathbb{Q}_p$ by definition. As for $p \notin S'$, since the finite intersections of open sets in $\mathbb{Q}_p$ is open in $\mathbb{Q}_p$, $U_p \cap \mathbb{Z}_p$ is open in $\mathbb{Q}_p$ (and hence $\mathbb{Z}_p$) as well. Thus, $V \cap \mathbb{A}_{S'} \subset \mathbb{A}_{S'}$ is the product of open sets at each of its coordinates. Furthermore, all but finitely many of the $U_p \cap \mathbb{Z}_p$ equal *all* of $\mathbb{Z}_p$, so $V \cap \mathbb{A}_{S'}$ is indeed open in $\mathbb{A}_{S'}$. And since this is true for any $\mathbb{A}_{S'}$, V itself is open in $\mathbb{A}$ (by definition), as claimed.

Next we prove that any open set U can be written as the union of sets in $\mathcal{F}$. But since every element of U is in one of the $\mathbb{A}_S$'s, we have

$$U = \bigcup_S U \cap \mathbb{A}_S,$$

where the union runs over all finite sets of primes S. Since U is open in $\mathbb{A}$, each $U \cap \mathbb{A}_S$ is open in $\mathbb{A}_S$ and can be written as the union of base sets in $\mathbb{A}_S$. And for a set T to be in the usual base of the product topology for $\mathbb{A}_S$, it must be the product of open sets $U_0 \subset \mathbb{R}$, $U_p \subset \mathbb{Q}_p$ (for $p \in S$), and $U_p \subset \mathbb{Z}_p$ (for $p \notin S$) — but furthermore, all but finitely many of these open sets must be *all* of the corresponding spaces $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Z}_p$. So letting S' be the finite set of primes where U_p doesn't equal exactly all of $\mathbb{Z}_p$, each T can be written in the form

$$T = U_0 \times \prod_{p \in S'} U_p \times \prod_{p \notin S'} \mathbb{Z}_p \in \mathcal{F}$$

Therefore each $U \cap \mathbb{A}_S$ is the union of sets of this form; and $U = \bigcup_S U \cap \mathbb{A}_S$ is as well, as claimed. □

Observation 1. *The topology "decreed" for $\mathbb{A}$ is really a topology.*

Proof: We found a base, and the union of the sets in the base is all of $\mathbb{A}$.□

Corollary 1. *The topology of $\mathbb{A}_S$, viewed as its own space equipped with the product topology, is the same as the subspace topology of $\mathbb{A}_S$, viewed as a subset of $\mathbb{A}$.*

Proof: The topology of a set Y viewed as its own space has the subspace topology of $Y \subset X$ iff the following two conditions hold: (i) given any open subset $U \subset X$, the set $U \cap Y \subset Y$ is open; and (ii) given any open subset $U \subset Y$, there exists an open set $V \subset X$ such that $U = V \cap Y$. We wish to show these conditions hold for $X = \mathbb{A}$ and $Y = \mathbb{A}_S$.

Condition (i) holds by the definition of "open" in $\mathbb{A}$. As for condition (ii), any open subset U of $\mathbb{A}_S$ can be written as a union $\bigcup_{\beta \in B} U_\beta$ of sets in the standard base of $\mathbb{A}_S$; but from the above lemma, each U_β is also open in $\mathbb{A}_S$. Therefore, $U \subset \mathbb{A}$ is open, implying that we can set $V = U$ to show that condition (ii) holds. □

(a)

- $\mathbb{A}$ *is locally compact.*

  Intuitively, given any point a in $\mathbb{A}$ only finitely many coordinates of a are not p-adic integers in a corresponding space $\mathbb{Q}_p$. For each of these coordinates, we find a compact neighborhood in the corresponding space; then multiply these compact spaces by $\mathbb{Z}_p$ in the remaining spaces. This, we will show, is a compact neighborhood of a in $\mathbb{A}$.

  Lemma 2. $\mathbb{R}$ *is locally compact, and* $\mathbb{Q}_p$ *and* $\mathbb{Z}_p$ *are locally compact for every prime* p.

  Proof: Given any $r \in \mathbb{R}$, the set $[r-1, r+1]$ is a compact neighborhood of r; and for any $z \in \mathbb{Z}_p$, we proved in the last homework assignment that $\mathbb{Z}_p$ is both open (so it equals its interior, and thus $z \in \text{int}(\mathbb{Z}_p)$) and compact in $\mathbb{Q}_p$. Because $\mathbb{Q}_p$ is Hausdorff, $\mathbb{Z}_p$ is compact when viewed as its own space; also, $\mathbb{Z}_p$ (like any space) is open when viewed as its own space. Thus $\mathbb{Z}_p$ is a compact neighborhood of z, and both $\mathbb{R}$ and $\mathbb{Z}_p$ are locally compact spaces.

Suppose that we have a p-adic q in $\mathbb{Q}_p$, and consider the function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by $x \mapsto q + x$. It's easy to verify that this is bijective; using the ϵ-δ definition, we see that this map and its inverses are continuous (in fact, for each $\epsilon > 0$, the value $\delta = \epsilon$ suffices). Thus f maps compact sets to compact sets, and open sets to open sets, so $\mathbb{Z}_p$ gets mapped to a compact neighborhood of q. Thus, $\mathbb{Q}_p$ is locally compact. □

Now, given any $a \in \mathbb{A}$, there is a finite set S of primes such that $a \in \mathbb{A}_S$. Thus for all primes $p \notin S$, the coordinate of a corresponding to $\mathbb{Q}_p$ is a p-adic integer.

For the coordinates $a_0 \in \mathbb{R}$ and $a_p \in \mathbb{Q}_p$ (for $p \in S$), since $\mathbb{R}$ and each $\mathbb{Q}_p$ are locally compact there are compact neighborhoods $C_0 \subset \mathbb{R}$ of a_0 and $C_p \subset \mathbb{Q}_p$ of a_p (for $p \in S$). Then we claim that

$$N = C_0 \times \prod_{p \in S} C_p \times \prod_{p \notin S} \mathbb{Z}_p$$

is a compact neighborhood of a. This is compact when viewed as its own space by Tychonoff's Theorem. If we can show that its topology as its own space is the same as the subspace topology of N, viewed as a subset of the Hausorff space $\mathbb{A}$, this would imply that it is compact as a subset of $\mathbb{A}$. We already know that the topology of the space $\mathbb{A}_S$ is the subspace topology of $\mathbb{A}_S \subset \mathbb{A}$. We now show that the topology of the space N is the subspace topology of $N \subset \mathbb{A}_S$, by proving a more general result; it then easily follow that properties (i) and (ii), described in the corollary proven before, apply with $X = \mathbb{A}$ and $Y = N$.

Lemma 3. *Suppose we have a collection of topological spaces $\{X_\alpha \mid \alpha \in A\}$ and subsets $Y_\alpha \in X_\alpha$ for all $\alpha \in A$, where each Y_α is equipped with the induced subspace topology. Equip $Y = \prod_{\alpha \in A} Y_\alpha$ and $X = \prod_{\alpha \in A} X_\alpha$, each viewed as its own space, with the respective product topologies. Then the topology of Y is the same as the subspace topology of $Y \subset X$*

Proof: We must prove the properties (i) and (ii) set out in the corollary from before (conveniently, without changing any variable names!). First we prove property (i). Each open set U in X is the union $\bigcup_{\beta \in B} U_\beta$ of sets in the standard base of X. Each set U_β equals $\prod_{\alpha \in A} \mathcal{O}_\alpha$ for open sets $\mathcal{O}_\alpha \subset X_\alpha$, where $\mathcal{O}_\alpha = X_\alpha$ for all but finitely many α. Then $V_\beta = U_\beta \cap Y = \prod_{\alpha \in A} (\mathcal{O}_\alpha \cap Y_\alpha)$. By the definition of the topology of each Y_α, this is a product of open sets $\mathcal{O}'_\alpha \subset Y_\alpha$; and as all but finitely many $\mathcal{O}_\alpha$ equal X_α, we have $\mathcal{O}'_\alpha = Y_\alpha$ for the same all but finitely many α. Hence, each V_β is open as a subset of Y, implying that $U \cap Y = \bigcup_{\beta \in B} (U_\beta \cap Y) = \bigcup_\beta V_\beta$ is open as a subset of Y, as desired.

Now for property (ii): suppose $U \in Y$ is open. Then it is the union $\bigcup_{\beta \in B} U_\beta$ of sets in the standard base of Y. Each U_β equals $\prod_{\alpha \in A} \mathcal{O}_\alpha$ for open $\mathcal{O}_\alpha \subset Y_\alpha$, where $\mathcal{O}_\alpha = Y_\alpha$ for all but finitely many α. For these all but finitely many α define $\mathcal{O}'_\alpha = X_\alpha$; for the other α, by definition of "open" in Y_α we have $\mathcal{O}_\alpha = \mathcal{O}'_\alpha \cap Y_\alpha$ for some open $\mathcal{O}'_\alpha \subset X_\alpha$. Then $V_\beta = \prod_{\alpha \in A} \mathcal{O}'_\alpha$ is open as a subset of X, as is $V = \bigcup_{\beta \in B} V_\beta$. As $U = V \cap Y$, property (ii) is true. $\square$

Thus, $N \subset \mathbb{A}$ is indeed compact. Also, N contains

$$N' = \operatorname{int}(C_0) \times \prod_{p \in S} \operatorname{int}(C_p) \times \prod_{p \notin S} \mathbb{Z}_p,$$

which contains a since $a_0 \in \operatorname{int}(C_0)$, $a_p \in \operatorname{int}(C_p)$ $\forall p \in S$, and $a_p \in \mathbb{Z}_p$ $\forall p \notin S$. The first lemma implies that N' is open, which means that $a \in N' \subset \operatorname{int}(N)$ so that N is indeed a compact neighborhood of a. Therefore, $\mathbb{A}$ is locally compact, as desired.

- $\mathbb{A}$ *is a Hausdorff space.*

  Intuitively, given any two distinct points in $\mathbb{A}$ they differ at some coordinate in either $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{Z}_p$. Then in the space $\mathbb{S}$ corresponding to this coordinate, we find disjoint open sets containing the coordinates; and then we multiply these sets by all the rest of $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{Z}_p$ to find disjoint open sets in $\mathbb{A}$ containing our two original points. Let's make this rigorous and precise.

  Given $a, b \in \mathbb{A}$, we have $a \in \mathbb{A}_{S_a}$ and $b \in \mathbb{A}_{S_b}$ for some finite sets of primes S_a and S_b. Then for all the primes p not in $S = S_a \cup S_b$, the corresponding coordinates of both a and b are from $\mathbb{Z}_p$. For convenience, write $\mathbb{S}_1 = \mathbb{R}$, $\{\mathbb{S}_2, \dots, \mathbb{S}_k\} = \{\mathbb{Q}_p \mid p \in S\}$, and $\{\mathbb{S}_{k+1}, \mathbb{S}_{k+2}, \dots\} = \{\mathbb{Z}_p \mid p \notin S\}$.

  Now, if $a, b \in \mathbb{A}$ with $a \neq b$, then they differ in at least one coordinate $a_n = b_n$ from some space $\mathbb{S}_n$ (which is either $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{Z}_p$). $\mathbb{S}_n$ is Hausdorff (since it is a metric space) so there are disjoint open sets $U_a, U_b \subset \mathbb{S}_n$ containing a, b. Then

  $$U_a \times \prod_{i \neq n} \mathbb{S}_i \quad \text{and} \quad U_b \times \prod_{i \neq n} \mathbb{S}_i$$

  are disjoint sets in $\mathbb{A}$ containing a and b, respectively. Furthermore, they are open since they are of the form described by the first lemma (all the spaces at each coordinate are open, and all but finitely many of them are all of $\mathbb{Z}_p$). Thus, $\mathbb{A}$ is indeed a Hausdorff space.

(b)

Lemma 4. *Addition and multiplication are continuous in each of* $\mathbb{R}$ *and* $\mathbb{Q}_p$.

Proof: This proof applies to any metric space X where (writing $|x| = d(0, x)$) we have $|ab| = |a||b|$ and $d(a, b) = |a - b|$ for all $a, b \in X$. (Recall that the first fact is true for rationals with the p-adic absolute value; to prove that it is true for all p-adics, use this fact in addition with the definition of p-adic absolute value that you created in the last problem set.)

Say we have an open set $U \subset X$ with additive preimage $U^{-1} \subset X \times X$. Then for any point $(a,b) \in U^{-1}$, we have $a+b \in U$ so there is an open ball $B(a+b,\epsilon)$ contained in U. Then $\beta = B(a,\frac{\epsilon}{2}) \times B(b,\frac{\epsilon}{2})$ is an open set in $X \times X$ containing (a,b), and it is in the preimage because $(c,d) \in \beta$ implies $|(c+d)-(a+b)| \leq |c-a|+|d-b| < \epsilon$ so that $(c+d) \in U$. Therefore we can find an open set β in U^{-1} around any point in U^{-1}, which implies U^{-1} is open.

Now suppose we have an open set $U \subset X$ with *multiplicative* preimage $U^{-1} \subset X \times X$. Then for any point $(a,b) \in U^{-1}$, we have $ab \in U$ so there is an open ball $B(ab,3\epsilon)$ contained in U. There exists $\epsilon_1 > 0$ such that $\epsilon_1|b| < \epsilon$ and $\epsilon_1 < \sqrt{\epsilon}$; let $B_a = B(a,\epsilon_1)$. Similarly, there exists $\epsilon_2 > 0$ such that $\epsilon_2|a| < \epsilon$ and $\epsilon_2 < \sqrt{\epsilon}$; let $B_b = B(b,\epsilon_2)$. Then $\beta = B_a \times B_b$ is an open set in $X \times X$ containing (a,b). Also, if $(c,d) \in \beta$ then either $|b| = 0 \Rightarrow |c-a||b| = 0 < \epsilon$ or else $|c-a||b| < \frac{\epsilon}{|b|} \cdot |b| = \epsilon$; similarly, $|a(d-b)| < \epsilon$. Thus

$$\begin{aligned}
|cd-ab| &\leq |(c-a)b| + |(c-a)(d-b)| + |a(d-b)| \\
&\leq |c-a||b| + |c-a||d-b| + |a||d-b| \\
&< \epsilon + \sqrt{\epsilon}\cdot\sqrt{\epsilon} + \epsilon \\
&= 3\epsilon
\end{aligned}$$

so that $(c,d) \in U^{-1}$. Since this is true for *all* $(c,d) \in \beta$ we have $(a,b) \in \beta \subset U^{-1}$. Therefore as before we can find an open set β in U^{-1} around any point in U^{-1}, which implies U^{-1} is open. This completes the proof. □

Look at the base $\mathcal{F}$ of the adeles described before in the first lemma. It is also a sub-base so from part (iii) of problem one, to prove that addition and multiplication are continuous it suffices to prove that the inverse image of each subset of $\mathcal{F}$ is open. We state the proof in terms of addition; the proof is exactly the same for multiplication. Also, by “preimage” we mean the preimage of a set under addition in $\mathbb{R}$ or $\mathbb{Q}_p$ or $\mathbb{A}$.

Suppose that we have some base subset

$$V = U_0 \times \prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p$$

in $\mathcal{F}$. (For convenience, we will sometimes write $U_p = \mathbb{Z}_p$ for all $p \notin S$.) Also suppose we have some point $x = (a,b) \in \mathbb{A} \times \mathbb{A}$ in the preimage of V. Because addition is continuous in each $\mathbb{Q}_p$, we know that the preimage U_p^{-1} of U_p is open and contains the $\mathbb{Q}_p \times \mathbb{Q}_p$ coordinate (a_p,b_p) of x. U_p^{-1}, though, is a union of sets in the base $\{A \times B \mid A, B \subset \mathbb{Q}_p \text{ are open}\}$, so there is some open set $A_p \times B_p \subset U_p^{-1}$ containing (a_p,b_p). Similarly, we can find an open set $A_0 \times B_0$ containing the $\mathbb{R} \times \mathbb{R}$ coordinate (a_0, b_0) of x and contained in the preimage of U_0.

Now let S' be the finite set of primes p such that either $U_p \neq \mathbb{Z}_p$, or the coordinates a_p and b_p are not both p-adic integers. Then consider the set

$$\left(A_0 \times \prod_{p \in S'} A_p \times \prod_{p \notin S'} \mathbb{Z}_p\right) \times \left(B_0 \times \prod_{p \in S'} B_p \times \prod_{p \notin S'} \mathbb{Z}_p\right).$$

Both factors are open by the lemma, so this set is indeed open in $\mathbb{A} \times \mathbb{A}$. Also, this set is in the preimage P of V; and finally, this set contains x. Thus every point x in P is contained in some set $U_x \subset P$ which is open in $\mathbb{A} \times \mathbb{A}$, so $P = \bigcup_{x \in P} U_x$ is open. Therefore addition (and also multiplication) are indeed continuous in the adeles, as desired.