Math 55a, Fall 2004

Seventh Assignment, Solutions
Adapted from Andrew Cotton and George Lee

Notation.

To prevent confusion between cosets as sets in G and as elements in G/H, elements of G/H will usually be written in the form $g\overline{H}$.

Given a topological group G, an element $g \in G$, and a set $U \subset G$, let gU denote the set $\{gu \mid u \in U\}$. Similarly define Ug.

Let $\Phi : G \to G/H$ be defined by $\Phi(g) = g\overline{H}$. Also, let $\mu : G \times G \to G$ denote multiplication $(a,b) \mapsto ab$; let $\nu : G - \{0\} \to G$ denote the multiplicative inverse $a \mapsto a^{-1}$; and let $\alpha : G \times G/H \to G/H$ denotes the natural action of G on G/H, with $\alpha(a, B) = (ab)\overline{H}$, where $B = b\overline{H}$. A quick check shows that α is well-defined and that $B_1 \neq B_2 \Rightarrow \alpha(a, B_1) \neq \alpha(a, B_2)$.

Preliminary Results.

Lemma 1 (Continuous Manifesto)**.** *History is a series of continuous struggles between topological spaces:*

(i) **On Oppressive Restrictions.** *Suppose that $f : X \to Y$ is a continuous function between spaces $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$.*

Give subsets $S_x \subset X$ and $S_y \subset Y$ (with $f(S_x) \subset S_y$) the restricted topologies. Then $f|_{S_x}$ is continuous.

(ii) **Functions of the World Unite!** *Given topological spaces X and Y_α ($\alpha \in A$), and continuous functions $f_\alpha : X \to Y_\alpha$, the function $f : X \to \prod_{\alpha \in A} Y_\alpha$ defined by $f(x) = (f_\alpha(x))_{\alpha \in A}$ is continuous.*

(iii) **Bloody Coordinate-Killing.** *Given a collection of topological spaces $\{X_\alpha \mid \alpha \in A\}$ and a nonempty subset $B \subset A$, the function $f : \prod_{\alpha \in A} X_\alpha \to \prod_{\alpha \in B} X_\alpha$ defined by $(x_\alpha)_{\alpha \in A} \mapsto (x_\alpha)_{\alpha \in B}$ is continuous.*

Proof:

(i) Given an open set $U \subset S_y$, there is an open set $U' \subset Y$ such that $U' \cap S_y = U$. Now for $v \in S_x$, we have $v \in f^{-1}(U) \iff v \in f^{-1}(U')$, so $f'^{-1}(U) = f^{-1}(U') \cap S_x$. Because $f^{-1}(U')$ is open in X, $f^{-1}(U') \cap S_x$ is open in S_x, as desired.

(ii) The sets $U_{\alpha_0} \times \prod_{\alpha \in A - \{\alpha_0\}} Y_\alpha$, where U_{α_0} is open, constitute a sub-base for $\prod_{\alpha \in A} Y_\alpha$. Thus, it suffices to prove that the preimage of such a set is open. Indeed,

$$f^{-1}\left(U_{\alpha_0} \times \prod_{\alpha \in A - \{\alpha_0\}} Y_\alpha\right) = f_{\alpha_0}^{-1}(U_{\alpha_0}),$$

is open because f_{α_0} is continuous.

(iii) Each projection function $p_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ is continuous, so (iii) follows from (ii) applied with $X = \prod_{\alpha \in A} X_\alpha \to X_\alpha$, $Y_\alpha = X_\alpha$, and $f_\alpha = p_\alpha$. ∎

Lemma 2 ("S.P.", the shifting property)**.** *Suppose G is a topological group. If U is an open (resp. closed) subset, then $g_0 U$ and $U g_0$ is open (resp. closed) for any $g_0 \in G$.*

Proof: Fix $g_0 \in G$. Then from the Continuous Manifesto, $g \mapsto (g_0^{-1}, g)$ is continuous as is $\mu_{\{g_0^{-1}\} \times G}$. Hence f, their composition $g \mapsto g_0^{-1} g$, is continuous. Thus, if U is open (resp. closed) then $f^{-1}(U) = g_0 U$ is open (resp. closed). Similarly, if U is open (resp. closed) then so is $U g_0$. ∎

Lemma 3. *Φ is an open map (a map that sends open sets to open sets).*

Proof: Let $U \subset G$ be open. We show that $\Phi^{-1}(\Phi(U))$ is open, which implies $\Phi(U)$ is open since Φ is a quotient map.

$$\begin{aligned}\Phi^{-1}(\Phi(U)) &= \Phi^{-1}\{u\overline{H} \mid u \in U\} \\ &= \{uh \mid u \in U, h \in H\} = \bigcup_{h \in H} Uh.\end{aligned}$$

And this set is open since each Uh is open, by the shifting property. ∎

Lemma 4 (Amazing Lemma)**.** *Suppose we have functions $f_1 : P \to Q$, $f_2 : Q \to R$, and $f_3 = f_2 \circ f_1$. If f_3 is continuous and f_1 is open (mapping open sets to open sets), then f_2 is continuous.*

Proof: Given an open set $U \subset R$, $V = f_3^{-1}(U)$ is open so $f_1(V) = f_2^{-1}(U)$ is open. ∎

Problem 1.

(a) Suppose that $x, y \in G$ with $x \neq y$. Without loss of generality, there exists an open set U containing x but not y. By S.P., $x^{-1}U$ is open, and hence there exists a set $A \times B$ (A, B open subsets of G) in the standard base of $G \times G$ containing (e, e) such that $\mu(A \times B) \subset x^{-1}U$. Let $\mathcal{O} = A \cap \nu^{-1}(B)$. Because $\mathcal{O}$ contains e, $x\mathcal{O}$ and $y\mathcal{O}$ are open neighborhoods of x and y, respectively. Suppose, for sake of contradiction, that $xg_1 = yg_2$ for $g_1, g_2 \in \mathcal{O}$. Then $g_1 \in A$ and $g_2^{-1} \in B$, so $x^{-1}y = g_1g_2^{-1} \in x^{-1}U$. This is impossible, because $y \notin U$. Therefore, we have separated x and y with open sets, as desired.

(b) If $\{U_\alpha\}$ are sets in $\mathcal{T}_Z$, then their preimages $F^{-1}(U_\alpha)$ are open in Y so their union $\bigcup F^{-1}(U_\alpha) = F^{-1}(\bigcup U_\alpha)$ is open in Y. Thus $\bigcup U_\alpha$ is open as well.

And because $\bigcap F^{-1}(U_i) = F^{-1}(\bigcap U_i)$, a similar argument shows that the intersection of a finite, nonempty collection of sets in $\mathcal{T}_Z$ is still in $\mathcal{T}_Z$.

Also, $F^{-1}(\emptyset) = \emptyset$ and
$F^{-1}(Z) = Y$ are open in Y, so $\emptyset$ and Z are in
$\mathcal{T}_Z$. Thus, $\mathcal{T}_Z$ is a topology.

Next, if a set in Z is open then its preimage under F is open by the definition of $\mathcal{T}_Z$. Thus, F is continuous.

Finally, the following conditions are equivalent: Z is T_1; $Z - \{z\} \in \mathcal{T}_Z$ for each $z \in Z$; $Y - F^{-1}(\{z\}) - F^{-1}(Z - \{z\}) \in \mathcal{T}_Y$ for each $z \in Z$; $F^{-1}(\{z\})$ is closed. We proved that the first two conditions are equivalent in class, and the second and third conditions are equivalent by the definition of $\mathcal{T}_Z$. ■

(c) The (left) cosets of the subgroup H partition the topological group G,
so $G \setminus H$ is the union of the other cosets. But by
S.P., all these cosets are open; so $G \setminus H$ is the
union of open sets, and thus open itself. Therefore H is
closed.

(d) Because $H \subset G$ is closed, any coset of H is closed by
S.P. Thus,

$$x = g\overline{H} \in G/H \Longrightarrow \Phi^{-1}(\{x\}) = gH \text{ is closed for all } x \in G/H.$$

Hence, from part (a), G/H is T_1.

We now prove that $\alpha : G\times G/H \to G/H$ is continuous. Consider the functions $f_1 : G \times G \to G \times G/H$ defined by $f_1(a, b) = (a, a\overline{h})$; $f_2 = \alpha$; and $f_3 = \Phi \circ \mu$. Given $(a, b) \in G \times G$, both $f_3(a, b)$ and $(f_2 \circ f_1)(a, b)$ equal $(ab)\overline{H}$; so, $f_3 = f_2 \circ f_1$.

Furthermore, suppose we have an open set $U_1 \times U_2$ in the base of $G \times G$. Then $\Phi(U_2)$ is open since Φ is an open map, so $f_1(U_1 \times U_2) = U_1 \times \Phi(U_2)$ is open as well. Thus, f_1 is an open map.

Also, $f_3 = \Phi \circ \mu$ is a composition of two continuous maps, so it is continuous itself. Then applying the Amazing Lemma, f_2 must be continuous — as desired.

(e) First we show we can separate $\overline{H}$ and $g_0\overline{H}$ with open sets whenever they are distinct.

We first find an open set $V \subset G$ containing e where ${v_1}^{-1}v_2 \notin g_0H$ for all $v_1, v_2 \in V$. Since H is closed, by S.P. g_0H is closed and thus $U = G \setminus g_0H$ is open. Because multiplication is continuous, $\mu^{-1}(U)$ is open and contains (e, e). Then there must be open sets $A, B \subset G$ containing e such that $A \times B \subset \mu^{-1}(U)$. Then $V = \nu^{-1}(A) \cap B$ is an open set containing e; and by construction, ${v_1}^{-1}v_2 \in \mu(A \times B) \subset U$ for all $v_1, v_2 \in V$. (Alternatively, we could let $W = A \cap B$ and then take $V = W \cap \nu^{-1}(W)$.)

Consider the sets $\Phi(V)$ and $\phi(Vg_0)$ in G/H. Remembering that $e \in V$ we know that $\overline{H} \in \Phi(V)$ and $g_0\overline{H} \in \phi(Vg_0)$. Furthermore, these sets are open because Φ is open and V and Vg_0 are open (by S.P.); and they are disjoint because if $\Phi(v_1g_0) = \Phi(v_2)$ for $v_1, v_2 \in V$, then ${v_1}^{-1}v_2 \in g_0H$, a contradiction. Thus we have found two disjoint open sets in G/H containing $\overline{H}$ and

$g_0\overline{H}$, as desired.

Now if we have two different elements $g_1\overline{H}$ and $g_2\overline{H}$

in G/H, then we can separate $\overline{H}$ and $(g_1^{-1}g_2)\overline{H}$

with two open sets V and W. From the Continuous Manifesto, the map $g\overline{H} \mapsto (g_1^{-1}, g\overline{H})$ is continuous as is the restriction $\alpha|_{\{g_1^{-1}\}\times G/H}$ of the natural action. Hence f, these two functions' composition $g\overline{H} \mapsto (g_1^{-1}g)\overline{H}$, is continuous. Thus, $f^{-1}(V)$ and $f^{-1}(W)$ are open disjoint sets containing $g_1\overline{H}$ and $g_2\overline{H}$, respectively. Therefore we can

separate any two different elements of G/H with disjoint open sets; this

completes the proof.

(f) Because H is normal, G/H is a group. Now we prove that multiplication is continuous.

Define $\mu' : G/H \times G/H \to G/H$ by $\mu'(a\overline{H}, b\overline{H}) = (ab)\overline{H}$. Now look at the functions $f_1 : G \times G \to G/H \times G/H$ defined by $f_1(a, b) = (\Phi(a), \Phi(b))$; $f_2 = \mu'$; and $f_3 = \Phi \circ \mu$. Given $(a, b) \in G \times G$, both $f_3(a, b)$ and $(f_2 \circ f_1(a, b)$ equal $(ab)\overline{H}$; so, $f_3 = f_2 \circ f_1$.

Given an open set $U_1 \times U_2$ in the base of $G \times G$, $\Phi(U_1)$ and $\Phi(U_2)$ are open since Φ is an open map. Thus $f_1(U_1 \times U_2) = \Phi(U_1) \times \Phi(U_2)$ is open, so f_1 is an open map. And as in part (c), f_3 is continuous since it is the composition of two continuous maps.

And as in part (c), $f_3 = \Phi \circ \mu$ is a composition of two continuous maps, so it is continuous itself. Then applying the Amazing Lemma, f_2 must be continuous — as desired.

We prove similarly that the multiplicative inverse $\nu' : G/H \to G/H$ defined by $\nu'(a\overline{H}) = a^{-1}\overline{H}$ is continuous. This time, we let $f_1 = \Phi$, $f_2 = \nu'$, and $f_3 = \Phi \circ \nu$. Given $a \in G$, both $f_3(a)$ and $(f_2 \circ f_1)(a)$ equal $a^{-1}\overline{H}$; so, $f_3 = f_2 \circ f_1$. Then since f_1 is open and f_3 (the composition of two continuous maps) is continuous, f_2 is continuous as well.

(g) Write $H = \text{clos}(\{e\})$; then any closed set containing e also contains H.

Since the multiplicative inverse ν is continuous, $\nu^{-1}(H)$ is a closed set containing e. So $a \in H \Rightarrow a \in \nu^{-1}(H) \Rightarrow a^{-1} \in H$.

Next, suppose that $a, b \in H$. Then by S.P., $a^{-1}H$ is a closed set containing e. Thus $b \in a^{-1}H \Rightarrow ab \in H$.

Thus, H is closed under multiplication and inverses, so it is a subgroup of G. Furthermore, given $a \in H$ and $g \in G$, by S.P. gHg^{-1} is a closed set containing e. Thus $a \in gHg^{-1} \Rightarrow g^{-1}ag \in H$, so H is normal.

Then because $H = \text{clos}(\{e\})$ is a closed subgroup, from part (c) G/H is T_1 and hence T_0. And because it is closed and normal, from part (e) it is a topological group. But since G/H is a T_0 topological group, from (a) it follows that G/H it a T_2 topological group, as desired.

Note: if a topological group is not Hausdorff, this result allows us to identify inseparable elements to form a Hausdorff topological group.

(h) We assume that matrix multiplication is associative in the set $\mathbb{R}^4$ of all 2×2 matrices; we also assume that if a matrix is invertible, its determinant is nonzero (this follows from $\det(A)\det(B) = \det(AB)$, which is easy to show for 2×2 matrices). Furthermore, any matrix $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with nonzero determinant has inverse

$$\triangleleft(x) = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix},$$

which can be verified with simple arithmetic. Hence $\mathbb{G} = GL(2, \mathbb{R})$ consists of precisely those matrices with nonzero determinant.

$\mathbb{G}$ has identity element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, multiplication of matrices in $\mathbb{G}$ can be interpreted as the composition of corresponding linear transformations; because composition of functions is associative, so is matrix multiplication. If $A, B \in \mathbb{G}$ then let A^{-1}, B^{-1} be their inverses; then $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$, so $\mathbb{G}$ is closed under matrix multiplication. Finally, every $A \in \mathbb{G}$ is invertible by the definition of $\mathbb{G}$ — that is, there exists $A^{-1} \in \mathbb{G}$ such that $AA^{-1} = A^{-1}A = I$ — and $\mathbb{G}$ is closed under the multiplicative inverse. It follows that $\mathbb{G}$ is a group.

We now show that multiplication $\cdot$ and the matrix inverse $\triangleleft$ are continuous in $\mathbb{G}$. To show this, we recruit a hefty collection of continuous functions using the Continuous Manifesto, then show that $\cdot$ and $\triangleleft$ are just compositions of these functions.

First consider $f : \mathbb{R}^8 \to \mathbb{R}^4$ defined by $f(x) = (a_1a_2 + b_1c_2, a_1b_2 + b_1d_2, c_1a_2 + d_1c_2, c_1b_2 + d_1d_2)$, where $x = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2)$. The function $f_1 : \mathbb{R}^8 \to \mathbb{R}^2$ defined by $f_1(x) = (a_1, a_2)$ is continuous; and since multiplication $\times$ in the reals is continuous, $f_2 : \mathbb{R}^8 \to \mathbb{R}$ defined by $f_2(x) = (\times \circ f_1)(x) = a_1a_2$ is continuous.

Similarly, $f_3 : \mathbb{R}^8 \to \mathbb{R}$ defined by $f_3(x) = b_1c_2$ is continuous, so $f_4(x) = (f_2(x), f_3(x)) = (a_1a_2, b_1c_2)$ is continuous. And since addition $+$ in the reals is continuous, $f_a(x) = (+ \circ f_4)(x) = a_1a_2 + b_1c_2$ is continuous.

Likewise, $f_b(x) = a_1b_2 + b_1d_2$, $f_c(x) = c_1a_2 + d_1c_2$, and $f_d(x) = c_1b_2 + d_1d_2$ from $\mathbb{R}^8 \to R$ are continuous. But then $f(x) = (f_a(x), f_b(x), f_c(x), f_d(x))$, so matrix multiplication from $\mathbb{R}^8$ to $\mathbb{R}^4$ is continuous. Restricting multiplication to the domain $S_x = \mathbb{G} \times \mathbb{G}$ and the range $S_y = \mathbb{G}$ preserves continuity, so multiplication in $\mathbb{G}$ is indeed continuous.

Very similar arguments show that $\triangleleft$ is continuous; although I won't give too many details, here's a sketch. which can be verified with simple arithmetic. It's easy to show that the function $g : \mathbb{G} \to \mathbb{R}^4$ is continuous, where $g(x) = (\frac{d}{ad-bc}, \frac{-b}{ad-bc}, \frac{-c}{ad-bc}, \frac{a}{ad-bc})$ when $x = (a, b, c, d)$.

Specifically, the proof would use that multiplication, addition, and subtraction (or alternatively, negation) in $\mathbb{R}$ are continuous, and that reciprocation in $\mathbb{R}-\{0\}$ is continuous. Also, at times the continuous functions in the proof need to be restricted to domains and ranges where $ad-bc\neq 0$.

Finally, we prove that $\mathbb{S}=SL(\mathbb{R},2)$ is a topological group. Because each matrix in $\mathbb{S}$ has nonzero determinant, $\mathbb{S}\subset\mathbb{G}$. Also, $\mathbb{S}$ is closed under matrix multiplication because if $\det(A)=\det(B)=1$, then $\det(AB)=\det(A)\det(B)=1$. Furthermore, it is closed under the multiplicative inverse of $\mathbb{G}$ because if $\det(A)=1$ then from $\det(I)=\det(A\triangleleft(A))=\det(A)\det(\triangleleft(A))$ we have that $\det(\triangleleft(A))=1$ as well. Hence, $\mathbb{S}$ is a subgroup of $\mathbb{G}$ and is thus a group in its own right. From the Continuous Manifesto, because the multiplicative inverse on $\mathbb{G}$ is continuous its restriction on $\mathbb{S}$ is continuous; because the product topology on $\mathbb{S}\times\mathbb{S}$ is the same as the subspace topology on $\mathbb{S}\times\mathbb{S}\subset\mathbb{G}\times\mathbb{G}$, it follows that matrix multiplication on $\mathbb{S}\times\mathbb{S}$ is continuous as well.

Problem 2.

First note that $SL(2,\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $GL(2,\mathbb{Z}/2\mathbb{Z})$ since the determinant of a matrix over $\mathbb{Z}/2\mathbb{Z}$ must be 0 or 1.

Consider the three non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^2$, $(1,0)$, $(0,1)$, and $(1,1)$. Any invertible linear transformation must permute these (since it must be bijective and map $(0,0)$ to itself).

Thus we have an action of $SL(2,\mathbb{Z}/2\mathbb{Z})$ on these three non-zero elements. It is faithful because the linear transformation is determined by where it sends these three elements.

Also, note that the sum of any two of these three non-zero elements equals the third. Thus any mapping fixing $(0,0)$ and permuting the three non-zero elements is an invertible linear transformation (linearity under scalar multiplication is trivial).

Thus the faithful action of $SL(2,\mathbb{Z}/2\mathbb{Z})$ on the three non-zero elements gives an injective homomorphism into S_3, and we see from the discussion above it is also surjective, so it is a group isomorphism.