Math 55a, Fall 2004

Midterm Solutions

1a) Carefully state Urysohn's Lemma:

Let X be a normal topological space and S_0 , $S_1 \subset X$ disjoint closed subsets. Then there exists a continuous function $f : X \to \mathbb{R}$ such that $0 \le f(x) \le 1$ for all $x \in X$, $f \equiv 0$ on S_0, and $f \equiv 1$ on S_1 .

1b) Prove that every compact Hausdorff space is normal:

This is (one half of) problem 1) of the fifth assignment – see the fifth solution set.

1c) Let X be a compact Hausdorff space, $X \neq \emptyset$, and $C(X)$ the ring of all continuous real-valued functions on X. Let $I \subset C(X)$ denote a proper ideal (i.e., $I \subset C(X)$; $I \neq C(X)$; $f, g \in I \Longrightarrow f + g \in I$; $f \in I, g \in C(X) \Longrightarrow fg \in I$). Show that

$$V(I) =_{\text{def}} \{\, x \in X \mid f(x) = 0 \ \text{ for all } f \in I \,\} \qquad \text{(this is the } \textit{variety} \text{ of } I)$$

is non-empty:

We argue by contradiction. Because of the assumption that $V(I) = \emptyset$, for each $x \in X$, there exists $f_x \in I$ such that $f_x(x) \neq 0$. The sets $U_x = \{y \in X \mid f_x(y) \neq 0\}$ are open and cover X, since $x \in U_x$ by construction. Finitely many of the U_x cover X. Let f denote the sum of the *squares* of the finitely many corresponding functions f_x. Since each f_x^2 is strictly positive on U_x, the function f is strictly positive, and therefore $1/f \in C(X)$. As a sum of functions in I, f lies in the ideal I, as does the constant function $1 = f \cdot 1/f \in I \cdot C(X) = I$. But any ideal which contains the constant function 1 coincides with $C(X)$. Thus $I = C(X)$, completing the proof by contradiction.

1d) With X and $C(X)$ as in the previous part of the problem, show that for every $x \in X$,

$$I_x =_{\text{def}} \{\, f \in C(X) \mid f(x) = 0 \,\}$$

is a maximal ideal in $C(X)$, and that conversely every maximal ideal equals I_x, for some uniquely determined $x \in X$:

Evidently I_x is an ideal. If it were not maximal, there would exist a proper ideal J containing I_x and some function $f \in J - I_x$. But then $f(x) \cdot 1 = f - (f - f(x))$, $f \in J$, $f - f(x) \in I_x \subset J$, and $f(x) \neq 0$ since $f \notin I_x$ by hypothesis. This would imply that J contains a non-zero constant function, hence the constant function

1, contradicting the assumption that J is a proper ideal.
To prove the converse, I consider a maximal ideal $I \subset C(X)$. We just saw that there exists some point $x \in V(I)$ – in other words, a point at which all functions in I vanish. Thus $I \subset I_x$, hence $I = I_x$ by maximality. It remains to be shown that $I_x \neq I_y$ if $x \neq y$, but that follows from Urysohn's lemma.

Addendum to problem 1. According to the problem, the algebraic structure of $C(X)$ determines X as a set. But more can be said: the algebraic structure determines X even as a topological space. To see this, note first of all that we can recover the values of functions at points $x \in X$ from the algebraic data. Indeed, the value of $f \in C(X)$ at the point $x \in X$ is the unique constant $c \in \mathbb{R}$ such that $f - c \cdot 1 \in I_x$. This implies that the sets $U_{f,\epsilon} = f^{-1}(-\epsilon, \epsilon)$, with $f \in C(X)$ and $\epsilon > 0$, are completely determined by the algebraic structure of $C(X)$. These sets, on the other hand, constitute a base for the topology of X, as can be inferred from Urysohn's lemma.

2a) Recall the definition of Sym($\{1, 2, \ldots, n\}$) as the group – under composition of maps – of all bijective maps from the set $\{1, 2, 3, \ldots, n\}$ to itself. One commonly denotes this group by S_n and calls it the "symmetry group on n letters". Note that $\{\sigma \in S_n \mid \sigma(n) = n\}$, the isotropy group at n, can be naturally identified with S_{n-1}. Thus, for every subgroup $H \subset S_n$, $\{\sigma \in H \mid \sigma(n) = n\}$ can be viewed as a subgroup of S_{n-1}. Let $H \subset S_n$ be a subgroup which acts (via the tautological action of S_n) transitively on $\{1, 2, 3, \ldots, n\}$. Show that H coincides with S_n if and only if $\{\sigma \in H \mid \sigma(n) = n\} = S_{n-1}$:

If $H = S_n$, the isotropy subgroup of H at n must coincide with the isotropy subgroup of S_n at n, i.e., it must coincide with S_{n-1}. Conversely, suppose that a subgroup $H \subset S_n$ acts transitively on $\{1, 2, 3, \ldots, n\}$, and that the isotropy subgroup of H at n is all of S_{n-1}. Let $\sigma \in S_n$ be given. Choose $\tau \in H$ so that $\tau(n) = \sigma(n)$; here we use the transitivity assumption. Then $(\tau^{-1}\sigma)(n) = n$, i.e., $\tau^{-1}\sigma$ lies in the isotropy subgroup of S_n at n. That isotropy subgroup coincides with the isotropy subgroup of H at n by hypothesis, hence $\tau^{-1}\sigma \in H$. But then $\sigma = \tau\tau^{-1}\sigma \in \tau H = H$, as needed to be shown.

2b) A transposition is an element of S_n which interchanges two distinct indices and leaves all the others unchanged. In other words, for $1 \leq k < \ell \leq n$, the transformation $\sigma_{k,\ell} \in S_n$, defined by the rule

$$\sigma_{k,\ell}(k) = \ell\,, \quad \sigma_{k,\ell}(\ell) = k\,, \quad \sigma_{k,\ell}(j) = j \ \text{ if } \ j \neq k, \ell\,,$$

is a transposition, and every transposition is of this type. Let T_n denote the subgroup of S_n generated by all transpositions. Show that T_n acts transitively on the set $\{1, 2, 3, \ldots, n\}$:

Let $\mathcal{O}_1$ denote the orbit of T_n containing 1. For each integer k, $2 \leq k \leq n$, the transposition $\sigma_{1,k}$ maps 1 to k. But then $k \in \mathcal{O}_1$. Thus $\mathcal{O}_1$ contains all integers between k and n; in other words, T_n acts transitively.

2c) Prove that the subgroup $T_n \subset S_n$ defined in the previous part of the problem equals S_n.

Let us argue by induction on n. For $n = 1$, there is nothing to prove since $S_1 = \{e\}$ is the trivial group. Because of a) and b), it suffices to show that the isotropy subgroup of T_n at n coincides with S_{n-1}. But transpositions generate S_{n-1}, according to the induction hypothesis. Thus T_n contains all of S_{n-1}, viewed as subgroup of S_n, as before. This completes the inductive step.