Take-Home Final Examination of Math 55b
(Noon May 5 to Noon May 12, 2006)

Access to Books and Homework Papers. *You are allowed to use your own homework papers and the material from textbooks and references which you have already studied and learned before this examination. You are not allowed to look up any new material in print or electronic form.*

Discussion of Problems With Others Not Allowed. *You are not allowed to consult any person or discuss with any person the problems in this examination.*

Quoting From Homework Problems. *For problems which are similar to those on the homework assignments, complete self-contained solutions are required and homework problems cannot be quoted simply as known facts in the solutions.*

Where to Hand in Your Paper. *Put your finished paper in the mailbox of Professor Yum-Tong Siu on the third floor of the Science Center outside the Main Math Office. If you prefer, you can also send in your paper electronically by e-mail to siu@math.harvard.edu.*

Notations. $\mathbb{N} =$ all positive integers.
$\mathbb{Z} =$ all integers.
$\mathbb{R} =$ all real numbers.
$\mathbb{C} =$ all complex integers.

Problem 1 *(Wirtinger's and Isoperimetric Inequalities).*

(a) *(Wirtinger's Integral Inequality).* Let f be a real-valued function on $\mathbb{R}$ of period 2π. Suppose the derivative $f'(x)$ exists for every $x \in \mathbb{R}$ and is Lebesgue measurable on $\mathbb{R}$ such that

$$\int_{x=0}^{2\pi} f(x)dx = 0 \quad \text{and} \quad \int_{x=0}^{2\pi} f'(x)^2 dx < \infty.$$

Use Parseval's Identity to show that

$$\int_{x=0}^{2\pi} f(x)^2 dx \leq \int_{x=0}^{2\pi} f'(x)^2 dx$$

and that equality holds if and only if $f(x) = a\sin(x+b)$ for some a, $b \in \mathbb{R}$.

(b) *(Hurwitz's Proof of the Isoperimetric Inequality).* Let D be a bounded domain in $\mathbb{R}^2$ (with coordinates x, y) whose boundary has length 2π and is parametrized in the counter-clockwise sense by its arc-length θ by two continuously differentiable functions

$$x = f(\theta) \quad \text{and} \quad y = g(\theta) \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

with

$$f(2\pi) = f(0), \quad f'(2\pi) = f'(0), \quad g(2\pi) = g(0), \quad g'(2\pi) = g'(0)$$

so that $f'^2 + g'^2 \equiv 1$.

(i) By considering the differentiable 1-form xdy on $\mathbb{R}^2$ and applying Stokes's Theorem, show that the area of D is equal to

$$\int_{\theta=0}^{2\pi} f(\theta) g'(\theta) d\theta.$$

(ii) Let a be $\frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\theta)\, d\theta$. By applying Wirtinger's inequality from Part(a) to $f(\theta) - a$ and using

$$\int_{\theta=0}^{2\pi} g'(\theta) d\theta = 0,$$

show that the area of D is no more than π. Moreover, show that the area of D is equal to π if and only if D is a disk of radius 1 in $\mathbb{R}^2$.

Problem 2 *(Steiner Symmetrization of a Solid With Respect to a Plane).* Let Ω be a bounded solid in $\mathbb{R}^3$ with coordinates x, y, z. Let $D \subset \mathbb{R}^2$ be the image of Ω by orthogonal projection from $\mathbb{R}^3$ to the xy-plane. Assume that D is a bounded domain in $\mathbb{R}^2$ with continuously differentiable boundary. Assume that the intersection of Ω with every line $\{x = a,\ y = b\}$ with $(a, b) \in D$ is an interval $[-f(a, b), g(a, b)]$, where both $f(x, y)$ and $g(x, y)$ are nonnegative continuously differentiable functions on D which vanish on the boundary of D.

For $(x, y) \in D$ let $h(x, y) = \frac{1}{2}(f(x, y) + g(x, y))$. Let $\tilde{\Omega}$ be another bounded solid in $\mathbb{R}^3$ such that be the image of $\tilde{\Omega}$ by projection from $\mathbb{R}^3$ to the xy-plane is also equal to D but the intersection of Ω with every line $\{x = a,\ y = b\}$ with $(a, b) \in D$ is the interval $[-h(a, b), h(a, b)]$ so that $\tilde{\Omega}$ is symmetric with respect to the xy-plane. The new solid $\tilde{\Omega}$ is known as the *Steiner symmetrization* of the original solid Ω with respect to the xy-plane. Show that

(a) the volume of $\tilde{\Omega}$ is equal to the volume of Ω, and

(b) the surface area of $\tilde{\Omega}$ is no greater than the surface area of Ω.

Hint: The surface area of Ω is

$$\int_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy + \int_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dxdy$$

Problem 3.

(a) *(Fundamental Theorem of Calculus for Certain Functions).* Let $-\infty < a < b < \infty$ and $f(x)$ be a real-valued continuous function on $[a, b]$ whose first-order derivative $f'(x)$ exists at every point of (a, b). Suppose for some $M > 0$ and some $0 < \gamma < 1$ the following inequality holds.

$$|f'(x)| \leq \frac{M}{(x-a)^\gamma (b-x)^\gamma} \quad \text{for} \quad a < x < b.$$

Show, by using the Mean-Value Theorem and Lebesgue's Theorem of Dominated Convergence, that

$$\int_{x=a}^{b} f'(x)\, dx = f(b) - f(a),$$

where the integral on the left-hand side is a Lebesgue integral.

(b) *(Riemann-Stieltjes Integrals).* Let $-\infty < a < b < \infty$ and $f(x)$ and $\alpha(x)$ be $\mathbb{R}$-valued functions on $[a, b]$ such that

(i) $f(x)$ is bounded on $[a, b]$ and is continuous almost everywhere on $[a, b]$,

(ii) $\alpha(x)$ is a nondecreasing function on $[a, b]$ whose derivative $\alpha'(x)$ exists everywhere on $[a, b]$ and is bounded.

Prove that the Riemann-Stieltjes integral $\int_a^b f(x) d\alpha$ exists and is equal to the Lebesgue integral $\int_a^b f(x)\alpha'(x)\, dx$.

Problem 4 *(Riemann's Continuous Nowhere Differentiable Function and the Growth of Derivatives of Delayed Means of Differentiable Functions).* For a $\mathbb{C}$-valued function $f(x)$ on $\mathbb{R}$ with period 2π whose Fourier series is

$\sum_{n\in\mathbb{Z}} a_n e^{inx}$ (that is, $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$), the n-th *partial sum* of the Fourier series of f means

$$s_n(f) = \sum_{|k|\leq n} a_k e^{ikx}$$

and its *Cesàro means* is

$$\sigma_n(f) = \frac{s_0(f) + s_1(f) + \cdots + s_{n-1}(f)}{n}$$

whose value at x_0 is given by the following convolution

$$\int_{-\pi}^{\pi} F_n(t) f\left(x_0 - t\right) dt$$

with the Féjer kernel

$$F_n(x) = \frac{1}{2\pi}\,\frac{\sin^2\left(\frac{nx}{2}\right)}{n\,\sin^2\left(\frac{x}{2}\right)}.$$

Define the *delayed means* $\Delta_n(f)$ of the Fourier series of f by

$$\Delta_n(f) = 2\sigma_{2n}(f) - \sigma_n(f).$$

(a) *(Inequalities of the Derivative of the Féjer Kernel).* By using the inequality

$$c|x| \leq \left|\sin\left(\frac{x}{2}\right)\right| \leq C|x| \quad \text{for} \quad |x| \leq \pi$$

(where c and C are positive numbers), show that there exists a positive number A such that the following inequalities for the derivative of the Féjer kernel hold.

$$|F_n{}'(x)| \leq An^2 \quad \text{and} \quad |F_n{}'(x)| \leq \frac{A}{|x|^2} \quad \text{for} \quad 0 < |x| \leq \pi.$$

(b) *(Growth of the Derivative of Delayed Means of Differentiable Functions).* By using

$$\sigma_n(f)'\left(x_0\right) = \int_{-\pi}^{\pi} F_n{}'(t)\left(f\left(x_0 - t\right) - f\left(x_0\right)\right) dt$$

and breaking up the interval of integration $\{|t| \leq \pi\}$ into the union of $\left\{|t| \geq \frac{1}{n}\right\}$ and $\left\{|t| \leq \frac{1}{n}\right\}$ and applying the inequalities for the derivative of the Féjer kernel from Part (a), prove the following.

If $f(x)$ is a continuous $\mathbb{C}$-valued function on $\mathbb{R}$ with period 2π and if its derivative at x_0 exists for some $|x_0| \leq \pi$, then there exist some positive numbers K and L such that

$$\left|\sigma_n(f)'(x_0)\right| \leq L + K \log n \quad \text{for all} \quad n \in \mathbb{N}.$$

(c) *(Riemann's Continuous Nowhere Differentiable Function).* Let $0 < \alpha < 1$ and

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n\alpha}} e^{i2^n x}.$$

Show that $F(x)$ is continuous on $\mathbb{R}$. Show that $F(x)$ is nowhere differentiable by verifying that

$$\Delta_{2N}(F) - \Delta_N(F) = \frac{1}{2^{n\alpha}} e^{i2^n x}$$

when $N = 2^{n-1}$ so that the differentiability of F at x_0 would imply

$$\left|\Delta_{2N}(F)'(x_0) - \Delta_N(F)'(x_0)\right| = 2^{n(1-\alpha)}$$

and contradict the inequality of Part (b).

Problem 5 *(Perturbation of the Defining Equation for an Implicitly Defined Function).* Let a and b be positive numbers and $F(x, y)$ be an $\mathbb{R}$-valued infinitely differentiable function on $\{|x| < a, |y| < b\}$ with $F(0,0) = 0$ and $F_y(0,0) = 0$, where F_y is the first-order partial derivative of $F(x, y)$ with respect to y. Let $y = f(x)$ be an $\mathbb{R}$-valued infinitely differentiable function on $|x| < a$ with $f(0) = 0$ and $\sup_{|x|<a} |f(x)| < b$ such that $F(x, f(x)) \equiv 0$ for $|x| < a$. Let $G(x, y)$ an $\mathbb{R}$-valued infinitely differentiable function on $\{|x| < a, |y| < b\}$ with $G(0,0) = 0$. Let $H(x,y) = F(x,y) + (F_y(x,y))^2 G(x,y)$. Prove that there exists an $\mathbb{R}$-valued infinitely differentiable function $h(x)$ on $|x| < \eta$ for some positive number η such that $h(0) = 0$ and $H(x, h(x)) \equiv 0$ on $|x| < \eta$.

Hint: Use the generalization of the Implicit Function Theorem for smooth functions adapted from Michael Artin's 1968 Inventiones Mathematicae paper and given in Problem 2 of Homework Assignment #2.

Problem 6 *(Spherical Coordinates in Higher Dimensions, the Laplacian, and the Volume of the Unit Ball).*

(a) *(Orthogonal Curvilinear Coordinates).* Let $y = (y_1, \cdots, y_n)$ be a curvilinear coordinate system for a domain in $\mathbb{R}^n$ whose Euclidean coordinates are $x_1, \cdots, x_n$. Assume that the two coordinate systems have the same orientation. Let $\vec{e}_1, \cdots, \vec{e}_n$ be the unit vectors along the Euclidean coordinate axes of $\mathbb{R}^n$. Let $\vec{R} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$. Assume that

$$\frac{\partial \vec{R}}{\partial y_1}, \cdots, \frac{\partial \vec{R}}{\partial y_n}$$

are mutually perpendicular so that

$$\left(d\vec{R}\right)^2 = \sum_{j=1}^{n} E_j(y)^2 {dy_j}^2,$$

where $E_j(y) = \sqrt{\left(\frac{\partial \vec{R}}{\partial y_j}\right)^2}$ for $1 \leq j \leq n$. Verify that

$$dx_1 \wedge \cdots \wedge dx_n = E_1 \cdots E_n \, dy_1 \wedge \cdots \wedge dy_n.$$

Show in the following two ways (i) and (ii) that the Laplacian $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial {x_j}^2}$ of a function f is given by the formula (†) below when expressed in terms of the coordinate system $y = (y_1, \cdots, y_n)$.

$$(\dagger) \qquad \Delta f = \frac{1}{E_1 \cdots E_n} \sum_{j=1}^{n} \frac{\partial}{\partial y_j} \left(\frac{E_1 \cdots E_{j-1} E_{j+1} \cdots E_n}{E_j} \frac{\partial f}{\partial y_j} \right).$$

(i) The first way uses the star operator $*$ as described in Homework Assignment #9 and

$$(\Delta f) \, dx_1 \wedge \cdots \wedge dx_n = d * df$$

and the definition of $E_1, \cdots, E_n$ to expresses df, $*df$, $d * df$ in terms of $y_1, \cdots, y_n, dy_1, \cdots, dy_n$.

(ii) The second way applies the Divergence Theorem (as given in Problem 7 in Homework Assignment #9) to the integration of Δf over the domain $b_j \leq y_j \leq b_j + h_j$ $(1 \leq j \leq n)$ and dividing the integral by $h_1 \cdots h_n$ and letting $h_j \to 0$ for $1 \leq j \leq n$ while keeping $b_1, \cdots, b_n$ unchanged.

(b) *(Spherical Coordinates in Higher Dimensions).* The spherical coordinates $r, \theta_1, \cdots, \theta_{n-1}$ for $\mathbb{R}^n$ are introduced by defining

$$\begin{aligned}
&\vec{\rho}_1 = \vec{e}_1, \\
&\vec{\rho}_2 = (\sin\theta_1)\,\vec{\rho}_1 + (\cos\theta_1)\,\vec{e}_2, \\
&\vec{\rho}_3 = (\sin\theta_2)\,\vec{\rho}_2 + (\cos\theta_2)\,\vec{e}_3, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
&\vec{\rho}_j = (\sin\theta_{j-1})\,\vec{\rho}_{j-1} + (\cos\theta_{j-1})\,\vec{e}_j, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
&\vec{\rho}_{n-1} = (\sin\theta_{n-2})\,\vec{\rho}_{n-2} + (\cos\theta_{n-2})\,\vec{e}_{n-1}, \\
&\vec{\rho}_n = (\sin\theta_{n-1})\,\vec{\rho}_{n-1} + (\cos\theta_{n-1})\,\vec{e}_n
\end{aligned}$$

and

$$\vec{R} = r\,\vec{\rho}_n\,(\theta_1, \cdots, \theta_{n-1})\,.$$

(i) Verify that the n vectors

$$\frac{\partial \vec{R}}{\partial r}, \frac{\partial \vec{R}}{\partial \theta_1}, \frac{\partial \vec{R}}{\partial \theta_2}, \cdots, \frac{\partial \vec{R}}{\partial \theta_{n-1}},$$

are mutually perpendicular.

(ii) Verify that

$$\frac{\partial\,(x_1, x_2, \cdots, x_n)}{\partial\,(r, \theta_1, \cdots, \theta_{n-1})} = r^{n-1} \sin^{n-2}\theta_{n-1}\, \sin^{n-3}\theta_{n-2}\, \cdots \sin^2\theta_3\, \sin\theta_2.$$

(iii) Use (†) to express the Laplacian Δf of a function f in terms of $r, \theta_1, \cdots, \theta_{n-1}$ and the first and second partial derivatives of f with respect to $r, \theta_1, \cdots, \theta_{n-1}$.

(c) *(Volume of the Unit Ball).* Show that the volume of the unit ball in $\mathbb{R}^{2n}$ is $\frac{\pi^n}{n!}$ and the volume of the unit ball in $\mathbb{R}^{2n+1}$ is

$$\frac{2^{n+1}\pi^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

by using the spherical coordinates and Part (b)(ii) and by computing

$$\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta.$$