**Math 55b, Assignment #8, April 18, 2006**
**(due April 24, 2006)**

*Notations and Convention.* $\mathbb{C}$ is the set of all complex numbers. $\mathbb{R}$ is the set of all real numbers. $\mathbb{N}$ is the set of all positive integers. For a domain $D$ in $\mathbb{R}^n$ with coordinates $x_1, \cdots, x_n$, the orientation of $D$ is defined by the differential $n$-form $dx_1 \wedge \cdots \wedge dz_n$ unless explicitly specified otherwise. For a function $f$ on $D$ the integral $\int_D f$ means the integration of the differential $n$-form $f dx_1 \wedge \cdots \wedge dx_n$ over the oriented domain $D$.

*Problem 1 (Differential Forms and Volumes).* Let $1 \leq k < n$ be integers and $G$ be an open subset of $\mathbb{R}^k$ and $f : G \to \mathbb{R}^n$ be a continuously differentiable map which is injective and whose derivative $f'$ as an $n \times r$ matrix has rank $r$ at every point of $G$. Let $u_1, \cdots, u_k$ be the coordinates of $\mathbb{R}^k$ and $x_1, \cdots, x_n$ be the coordinates of $\mathbb{R}^n$ and let

$$f(u_1, \cdots, u_k) = (f_1(u_1, \cdots, u_k), \cdots, f_n(u_1, \cdots, u_k)).$$

Define the pullback $f^*\left(\sum_{j=1}^n dx_j \otimes dx_j\right)$ by the map $f$ of the Euclidean metric $\sum_{j=1}^n dx_j \otimes dx_j$ as

$$f^*\left(\sum_{j=1}^n dx_j \otimes dx_j\right) = \sum_{j=1}^n df_j \otimes df_j$$

which is equal to

$$\sum_{\mu,\nu=1}^k \left(\sum_{j=1}^n \frac{\partial f_j}{\partial u_\mu} \frac{\partial f_j}{\partial u_\nu}\right)(du_\mu \otimes du_\nu).$$

Define the Jacobian (determinant)

$$\frac{\partial(f_{j_1}, \cdots, f_{j_k})}{\partial(u_1, \cdots, u_k)}$$

to be the determinant of the $k \times k$ matrix whose $(\mu, \nu)$-th entry is

$$\frac{\partial f_{j_\mu}}{\partial u_\nu}$$

for $1 \leq \mu, \nu \leq k$.

(a) Show that the $k \times k$ determinant

$$\det\left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial u_\mu}\frac{\partial f_j}{\partial u_\nu}\right)_{1\le\mu,\nu\le k} = \begin{vmatrix} \sum_{j=1}^{n} \frac{\partial f_j}{\partial u_1}\frac{\partial f_j}{\partial u_1} & \cdots & \sum_{j=1}^{n} \frac{\partial f_j}{\partial u_1}\frac{\partial f_j}{\partial u_k} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \sum_{j=1}^{n} \frac{\partial f_f}{\partial u_k}\frac{\partial f_j}{\partial u_1} & \cdots & \sum_{j=1}^{n} \frac{\partial f_j}{\partial u_k}\frac{\partial f_j}{\partial u_k} \end{vmatrix}$$

of the coefficients of $f^*\left(\sum_{j=1}^{n} dx_j \otimes dx_j\right)$ in terms of $du_\mu \otimes du_\nu$ is equal to

$$\sum_{1\le j_1<\cdots<j_k\le n} \left|\frac{\partial\left(f_{j_1},\cdots,f_{j_k}\right)}{\partial\left(u_1,\cdots,u_k\right)}\right|^2.$$

Define the *volume* of the image $f(G)$ of $f: G \to \mathbb{R}^n$ as

$$\int_G \left(\det\left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial u_\mu}\frac{\partial f_j}{\partial u_\nu}\right)_{1\le\mu,\nu\le k}\right)^{\frac{1}{2}} du_1\cdots du_k.$$

(b) Consider the following special case.

(i) $k = 1$;

(ii) $n = 2$;

(iii) $G = (a, b)$;

(iv) $f : G \to \mathbb{R}^2$ is given by $x = x(t)$ and $y = y(t)$.

For this special case, verify that the above definition of the arc-length of the image of $f : (a, b) \to \mathbb{R}^2$ agrees with

$$\int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(c) Consider the following special case.

(i) $k = 2$;

(ii) $n = 3$;

(iii) $f : G \to \mathbb{R}^3$ is given by the vector-valued function $(x, y, z) = \vec{r}(u, v)$ of two variables $u, v$.

Let

$$E(u,v) = \left| \frac{\partial \vec{r}}{\partial u} \right|^2,$$
$$F(u,v) = \left( \frac{\partial \vec{r}}{\partial u} \right) \cdot \left( \frac{\partial \vec{r}}{\partial v} \right),$$
$$G(u,v) = \left| \frac{\partial \vec{r}}{\partial v} \right|^2.$$

For this special case, verify that the volume of $f(G)$ is given by

$$\int_G \sqrt{EG - F^2}\, du \wedge dv$$

and that it is also given by

$$\int_G \left| \left( \frac{\partial \vec{r}}{\partial u} \right) \times \left( \frac{\partial \vec{r}}{\partial v} \right) \right| du \wedge dv,$$

where $\vec{\xi} \times \vec{\eta}$ means the vector product of the two vectors $\vec{\xi}$ and $\vec{\eta}$.

(d) In the special case of $k = n - 1$ and $f_j = u_j$ for $1 \leq j \leq n - 1$ and $f_n = F(u_1, \cdots, u_{n-1})$ (in other words, the image of $f : G \to \mathbb{R}^n$ is defined as the graph of the function

$$(x_1, \cdots, x_{n-1}) \to F(x_1, \cdots, x_{n-1})$$

over $G \subset \mathbb{R}^{n-1}$), verify that the volume of the image $f(G)$ of $f : G \to \mathbb{R}^n$ is equal to

$$\int_G \sqrt{1 + \sum_{j=1}^{n-1} \left( \frac{\partial F}{\partial x_j} \right)^2}\, dx_1 \cdots dx_{n-1}.$$

(e) In the special case when each $f_j$ is an $\mathbb{R}$-linear function

$$f_j(u_1, \cdots, u_k) = \sum_{\nu=1}^{k} a_{j,\nu} u_\nu$$

for some $a_{j,\nu} \in \mathbb{R}$, verify that the volume of $f(G)$ is equal to

$$\sqrt{\sum_{1 \leq j_1 < \cdots < j_k \leq n} \left(V_{j_1, \cdots, j_k}\right)^2}\,,$$

where $\pi_{j_1, \cdots, j_k} : \mathbb{R}^n \to \mathbb{R}^k$ is the projection

$$(x_1, \cdots, x_n) \mapsto (x_{j_1}, \cdots, x_{j_k})$$

and $V_{j_1, \cdots, j_k}$ is the volume of the subset $\left(\pi_{j_1, \cdots, j_k} \circ f\right)(G)$ of $\mathbb{R}^k$.

(f) Consider the following special case:

(i) $k = 2\ell$;

(ii) $n = 2m$;

(iii) $u_{2\mu-1}$ equals the real part of $\zeta_\mu$ and $u_{2\mu}$ equals the imaginary part of $\zeta_\mu$ for $1 \leq \mu \leq \ell$, where $(\zeta_1, \cdots, \zeta_\ell) \in \mathbb{C}^\ell = \mathbb{R}^k$;

(iv) $x_{2\nu-1}$ equals the real part of $z_\nu$ and $x_{2\nu}$ equals the imaginary part of $z_\nu$ for $1 \leq \nu \leq m$, where $(z_1, \cdots, z_m) \in \mathbb{C}^m = \mathbb{R}^n$;

(v) $G$ is the ball of radius $R > 0$ in $\mathbb{C}^\ell = \mathbb{R}^k$ centered at the origin;

(vi) The function
$$f_{2\nu-1} + \sqrt{-1}\, f_{2\nu}$$
of $\zeta_1, \cdots, \zeta_\ell$ is a power series in $(\zeta_1, \cdots, \zeta_\ell)$ which converges at every point of $G$.

Let

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\nu=1}^{m} dz_\nu \wedge d\overline{z_\nu}$$

and $\Theta = f^*\omega$, where

$$dz_\nu = dx_{2\nu-1} + \sqrt{-1}\, dx_{2\nu},$$
$$d\overline{z_\nu} = dx_{2\nu-1} - \sqrt{-1}\, dx_{2\nu}.$$

so that

$$\frac{\sqrt{-1}}{2}\, dz_\nu \wedge d\overline{z_\nu} = dx_{2\nu-1} \wedge\, dx_{2\nu}$$

and

$$f^*\left(\frac{\sqrt{-1}}{2}\,dz_\nu \wedge d\overline{z_\nu}\right) = df_{2\nu-1} \wedge df_{2\nu}.$$

Show that for this special case the volume of $f(G)$ is equal to the integration of the differential $k$-form

$$\frac{1}{\ell!}(\underbrace{\Theta \wedge \cdots \wedge \Theta}_{\ell \text{ copies}})$$

over $G$. (The significance of this special case is that while in general the volume is not computed by the integration of a differential form on the ambient space $\mathbb{R}^n$ which is independent of $f(G)$, for this special case the volume does come from the integration of such a differential form.)

(*Hint:* use the $\mathbb{C}$-basis

$$\frac{\partial}{\partial u_{2\mu-1}} + \sqrt{-1}\,\frac{\partial}{\partial u_{2\mu}}, \quad \frac{\partial}{\partial u_{2\mu-1}} - \sqrt{-1}\,\frac{\partial}{\partial u_{2\mu}} \quad (1 \le \mu \le \ell)$$

instead of

$$\frac{\partial}{\partial u_j} \quad (1 \le j \le k)$$

and use the $\mathbb{C}$-basis

$$\frac{\partial}{\partial x_{2\nu-1}} + \sqrt{-1}\,\frac{\partial}{\partial x_{2\mu}}, \quad \frac{\partial}{\partial x_{2\nu-1}} - \sqrt{-1}\,\frac{\partial}{\partial x_{2\nu}} \quad (1 \le \nu \le \ m)$$

instead of

$$\frac{\partial}{\partial x_j} \quad (1 \le j \le n)$$

for the calculation and observe from direct computation that

$$\left(\frac{\partial}{\partial u_{2\mu-1}} + \sqrt{-1}\,\frac{\partial}{\partial u_{2\mu}}\right)\left(f_{2\nu-1} + \sqrt{-1}\,f_{2\nu}\right) \equiv 0,$$

because

$$f_{2\nu-1} + \sqrt{-1}\,f_{2\nu}$$

is a power series in $\zeta_\mu = u_{2\mu-1} + \sqrt{-1}\,u_{2\mu}$ when the other variables

$$u_1, \cdots, u_{2\mu-2}, u_{2\mu+1}, \cdots, u_n$$

are fixed.)

*Problem 2 (Volume of n-Sphere Computed by Integrating a Radial Function of Quadratic Exponent Decay).*

(a) Suppose $G$ is a domain in $\mathbb{R}^{n+1}$. Let $F(x_1, \cdots, x_n, x_{n+1})$ be a continuously differentiable real-valued function on $G$ so that the length $|dF|$ of the differential of $F$ is identically 1 at every point of the zero-set $Z$ of $F$. Here $|dF|$ equals $\left(\sum_{j=1}^{n+1} \left|\frac{\partial F}{\partial x_j}\right|^2\right)^{\frac{1}{2}}$. Assume that $\frac{\partial F}{\partial x_{n+1}} > 0$ at every point of $Z$. Give $Z$ the orientation defined by the differential $n$-form $dx_1 \wedge \cdots \wedge dx_n$ on $Z$. Show that the volume of $Z$ is equal to

$$\int_Z \frac{1}{\frac{\partial F}{\partial x_{n+1}}} dx_1 \wedge \cdots \wedge dx_n,$$

when locally $Z$ is represented as a graph over a domain in $\mathbb{R}^n$ by the implicit function theorem and the volume of that graph is as defined in Problem 1 above.

*Hint:* When $F(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) \equiv 0$ in $x_1, \cdots, x_n$, use implicit differentiation and the chain rule to express $\frac{\partial f}{\partial x_j}$ in terms of $\frac{\partial F}{\partial x_{n+1}}$ and $\frac{\partial F}{\partial x_j}$ for $1 \leq j \leq n$.

*Remark:* Note that, unlike the form $\omega$ on $\mathbb{R}^n$ in Problem 1(f) used to compute the volume of $f(G)$ there, the form $\frac{1}{\frac{\partial F}{\partial x_{n+1}}} dx_1 \wedge \cdots \wedge dx_n$ used here to compute the volume of $Z$ depends on $Z$.

(b) Let $\omega_n$ be the volume of the unit $n$-sphere in $\mathbb{R}^{n+1}$ with coordinates $x_1, \cdots, x_{n+1}$. Show that

$$\int_{\mathbb{R}^{n+1}} e^{-\sum_{j=1}^{n+1}(x_j)^2} = \omega_n \int_{r=0}^{\infty} r^n e^{-r^2} dr.$$

*Hint:* Let $r = \left(\sum_{j=1}^{n+1}(x_j)^2\right)^{\frac{1}{2}}$. Use local coordinates $x_1, \cdots, x_n, r$ at a point where $\frac{\partial r}{\partial x_n}$ is nonzero and apply Part(a) by observing that $|dr| \equiv 1$. The factor $r^n$ comes from the fact that the volume of an $n$-sphere of radius $a$ is equal to $\omega_n a^n$ as one can see by considering the map $(x_1, \cdots, x_{n+1}) \mapsto (ax_1, \cdots, ax_{n+1})$ of $\mathbb{R}^{n+1}$.

(c) Use Part(b) to show that

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)},$$

where the denominator is the value of the Gamma function at $\frac{n}{2}$.

*Hint:* Write the integral on the left-hand side of the equation in Part(b) as the $(n+1)$-st power of $\int_{-\infty}^{\infty} e^{-t^2} dt$.

*Problem 3 (Expressing a 1-Form as the Differential of a Function).*

(a) Let $P(x,y)dx + Q(x,y)dy$ be a continuously differentiable differential 1-form on $(-1,1) \times (-1,1) \subset \mathbb{R}^2$. Define

$$g(x) = \int_0^x P(s,0)ds \quad \text{for } x \in (-1,1)$$

and

$$f(x,y) = g(x) + \int_0^y Q(x,t)dt \quad \text{for } x, y \in (-1,1).$$

Assume that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Show that $df = P\,dx + Q\,dy$ on $(-1,1) \times (-1,1)$.

*Hint:* Use

$$\frac{\partial}{\partial x} \int_0^y Q(x,t)dt = \int_0^y \frac{\partial}{\partial x} Q(x,t)dt = \int_0^y \frac{\partial}{\partial t} P(x,t)dt.$$

(b) For $1 \leq j \leq n$ let $-\infty < a_j < b_j < \infty$. Let $\Theta = \sum_{j=1}^n \varphi_j(x_1, \cdots, x_n)\, dx_j$ be a continuously differentiable differential 1-form on $(a_1,b_1) \times \cdots \times (a_n,b_n) \subset \mathbb{R}^n$ such that

$$\frac{\partial \varphi_j}{\partial x_k} = \frac{\partial \varphi_k}{\partial x_j}$$

on $(a_1,b_1) \times \cdots \times (a_n,b_n)$ for $1 \leq j,k \leq n$. Define $f_0 = 0$ and for $1 \leq j \leq n$ inductively define

$$f_j(x_1, \cdots, x_j) = f_{j-1}(x_1, \cdots, x_{j-1}) + \int_{t_j=0}^{x_j} \varphi_j(x_1, \cdots, x_{j-1}, t_j, 0, \cdots, 0)\, dt_j.$$

Let $f = f_n$. Verify that $df = \sum_{j=1}^n \varphi_j\, dx_j$ on $(a_1,b_1) \times \cdots \times (a_n,b_n)$.

(c) Use the notations of Part(b). Let $-\infty < \alpha < \beta < \infty$ and let $\Phi : [\alpha,\beta] \to (a_1,b_1) \times \cdots \times (a_n,b_n)$ be a continuously differentiable map with $\Phi(\alpha) = \Phi(\beta)$. Show that $\int_{[\alpha,\beta]} \Phi^*\Theta = 0$, where $\Phi^*\Theta$ is the pullback of the differential 1-form $\Theta$ by the map $\Phi$.

*Hint:* Use the fundamental theorem of calculus and $\Phi(\alpha) = \Phi(\beta)$.

*Problem 4 (Riemann Double and Iterated Integrals for a Function Continuous Outside a Closed Subset of Zero Measure).* Let $-\infty < a < b < \infty$ and $-\infty < \alpha < \beta < \infty$. Let $A$ be a *closed* subset of $[a,b] \times [\alpha,\beta]$ of measure zero in the sense that given any $\varepsilon > 0$ there exists a sequence of open disks $B(P_j, r_j)$ (for $j \in \mathbb{N}$) of radius $r_j > 0$ centered at $P_j \in \mathbb{R}^2$ such that $A \subset \bigcup_{j\in\mathbb{N}} B(P_j, r_j)$ and $\sum_{j\in\mathbb{N}} {r_j}^2 < \varepsilon$.

*Remark.* For the statements of this problem the condition of $A$ being closed can be removed (when $E$ is not required to be closed). It is used here so that the problem can be directly handled from the definition of Riemann integrability without the use of a more complicated theory such as Lebesgue integration.

(a) Show that there exists a closed subset $E$ of measure zero in $[a,b]$ such that $(\{x\} \times [\alpha,\beta]) \cap A$ is of measure zero in the interval $\{x\} \times [\alpha,\beta]$ for $x \in [a,b] - E$.

(b) Suppose $f$ is a bounded real-valued function on $[a,b] \times [\alpha,\beta]$ which is continuous on $[a,b] \times [\alpha,\beta] - A$. Show that $f$ is Riemann integrable on $[a,b] \times [\alpha,\beta]$ in the sense that given any $\varepsilon > 0$ there exist

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_{p-1} < x_p = b, \\ \alpha = y_0 < y_1 < \cdots < y_{q-1} < y_q = \beta, \end{aligned}$$

such that

$$\sum_{k=1}^{p} \sum_{\ell=1}^{q} (M_{k,\ell} - m_{k.\ell})\,(x_k - x_{k-1})\,(y_\ell - y_{\ell-1}) < \varepsilon,$$

where $M_{k,\ell}$ is the supremum of $f(P)$ for $P \in [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell]$ and $m_{k,\ell}$ is the infimum of $f(P)$ for $P \in [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell]$. We denote by

$$\int_{[a,b]\times[\alpha,\beta]} f(x,y)dxdy$$

the supremum of

$$\sum_{k=1}^{p} \sum_{\ell=1}^{q} m_{k.\ell}\,(x_k - x_{k-1})\,(y_\ell - y_{\ell-1})\,,$$

or equivalently the infimum of

$$\sum_{k=1}^{p}\sum_{\ell=1}^{q} M_{k.\ell}\left(x_k - x_{k-1}\right)\left(y_\ell - y_{\ell-1}\right),$$

over all possible choices of

$$\begin{aligned} a &= x_0 < x_1 < \cdots < x_{p-1} < x_p = b, \\ \alpha &= y_0 < y_1 < \cdots < y_{q-1} < y_q = \beta. \end{aligned}$$

(c) Verify that the function

$$\int_{y=\alpha}^{\beta} f(x,y)dy$$

on $[a,b]-E$ of the variable $x$ defined by Riemann integration can be extended to a function on $[a,b]$ which is Riemann integrable on $[a,b]$.

(d) Verify that

$$\int_{x=a}^{b}\left(\int_{y=\alpha}^{\beta} f(x,y)dy\right)dx = \int_{[a,b]\times[\alpha,\beta]} f(x,y)dxdy,$$

where the left-hand side is the Riemann integral of the function in Part(c).

*Remark.* For a closed subset $S$ in $[a,b]\times[\alpha,\beta]$ whose boundary $A$ is of measure zero in $[a,b]\times[\alpha,\beta]$ and a continuous function $g$ on $S$, the integral $\int_S g(x,y)dxdy$ can be defined as the Riemann integral of the function $f$ over $[a,b]\times[\alpha,\beta]$ which is defined by $f(P)=g(P)$ for $P\in S$ and $f(P)=0$ for $P\notin S$.

*Problem 5 (Representing an Interated Indefinite Integral as a Single Integral).*

(a) Let $c>0$ and let $f(x)$ be a continuous function on $[0,c]$. Use two ways to prove by induction on $n$ that for $n\geq 1$ and $0\leq x\leq c$ the iterated integral

$$\int_0^x\left(\int_0^{t_n}\left(\int_0^{t_{n-1}}\left(\cdots\left(\int_0^{t_2} f\left(t_1\right)dt_1\right)\cdots\right)dt_{n-2}\right)dt_{n-1}\right)dt_n$$

is equal to

$$\int_0^x \frac{(x-t)^n}{n!} f(t)\,dt,$$

the first way differentiating both sides with respect to $x$ and the second way by switching the order of the iterated integration on the left-hand side.

(b) Use Part(a) to obtain the following Taylor expansion with integral remainder

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \quad \text{for } 0 \le x \le c$$

for any $(n+1)$-times continuously differentiable function $f$ on $[0,c]$

(c) Use Part(b) and the Mean Value Theorem to obtain the following Taylor expansion with derivative remainder

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1} \quad \text{for } 0 \le x \le c \text{ and some } 0 < \xi < x$$

for any $(n+1)$-times continuously differentiable function $f$ on $[0,c\,]$.

*Problem 6 (Sobolev Imbedding).* Let $f$ be a real-valued continuously differentiable function on $\mathbb{R}^n$ which is identically zero outside some ball of radius $R > 0$ centered at the origin. Let

$$g(P) = \left( \sum_{j=1}^{n} \left| \frac{\partial f}{\partial x_j}(P) \right|^2 \right)^{\frac{1}{2}}.$$

(a) Let $\omega_{n-1}$ be the volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. By applying the Fundamental Theorem of Calculus radially from $x$ to $y$ and by averaging the unit vector from $x$ to $y$ over the unit $(n-1)$-sphere, show that

$$(\dagger) \qquad |f(x)| \le \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}}\, g(y)\,dy,$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ and $|x-y| = \left(\sum_{j=1}^n |x_j - y_j|^2\right)^{\frac{1}{2}}$ and $dy = dy_1 \cdots dy_n$.

(b) Let $1 \le p < n$ and $\mu > n$. Let $h(x) = \frac{1}{|x|^{n-1}}$ for $x \in \mathbb{R}^n - \{0\}$. By Hölder's inequality show that

$$|f(x)| \le \frac{1}{\omega_{n-1}} \left( \int_{\mathbb{R}^n} h(x-y)^{\frac{\mu}{\mu-1}} g(y)^{\frac{\mu-p}{\mu-1}} dy \right)^{\frac{\mu-1}{\mu}} \left( \int_{\mathbb{R}^n} g(y)^p \right)^{\frac{1}{\mu}}.$$

*Hint:* Write $h(x-y)g(y) = \left(h(x-y)g(y)^{\frac{\mu-p}{\mu}}\right)\left(g(y)^{\frac{p}{\mu}}\right)$ and apply Hölder's inequality to raise the exponent of the first factor $h(x-y)g(y)^{\frac{\mu-p}{\mu}}$ to $\frac{\mu}{\mu-1}$.

(c) Let $q = \frac{p\mu}{\mu - p}$. Use Part(b) to show that

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, dx\right)^{\frac{1}{q}} \leq \frac{C_{R,\mu,n}}{\omega_{n-1}} \left(\int_{\mathbb{R}^n} g(y)^p dy\right)^{\frac{1}{p}},$$

where

$$C_{R,\mu,n} = \left(\int_{|x| \leq R} |x|^{-\frac{\mu(n-1)}{\mu-1}} dx\right)^{\frac{\mu}{\mu-1}} < \infty.$$

*Hint:* For $s \geq 1$ and for functions $\varphi(x)$, $\psi(x)$ on $\mathbb{R}^n$ with compact support, their convolution

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy$$

satisfies

$$\|\varphi * \psi\|_{L^s} \leq \|\varphi\|_{L^1} \|\psi\|_{L^s},$$

which is the limiting case of

$$\left\|\sum_{j \in J} c_j \left(T_{P_j}\psi\right)\right\|_{L^s} \leq \left(\sum_{j \in J} |c_j|\right) \|\psi\|_{L^s}$$

as $\sum_{j \in J} |c_j|$ approaches $\|\varphi\|_{L^1}$, where $c_j \in \mathbb{R}$ and $P_j \in \mathbb{R}^n$ and $\left(T_{P_j}\psi\right)(x) = \psi(x + P_j)$ is the translate of $\psi$ by $P_j$. The norms $\|\cdot\|_{L^s}$ and $\|\cdot\|_{L^1}$ denote respectively the $L^s$ norm and the $L^1$ norm over $\mathbb{R}^n$.

*Remark.* This inequality gives the following statement concerning Sobolev norms. Any function on $\mathbb{R}^n$ with compact support whose first-order derivatives are all $L^p$ for some $1 \leq p < n$ is $L^q$ for any $1 \leq q < \frac{np}{n-p}$. Actually the Sobolev imbedding theorem can conclude that the function is $L^q$ for $q = \frac{np}{n-p}$, but the procedure given here gives only the weaker result of the function being $L^q$ for any $1 \leq q < \frac{np}{n-p}$.