Math 55b, Assignment #9, April 27, 2006
(due May 4, 2006)

Notations and Terminology. $\mathbb{R}$ = the set of all real numbers.

The symbol $*$ denotes the *star operator* which, for an $\mathbb{R}$-vector space V of dimension n, with an inner product $\langle \cdot, \cdot \rangle$ and an orientation given by its ordered orthonormal basis $e_1, \cdots, e_n$, is an $\mathbb{R}$-linear map from $\bigwedge^p V$ to $\bigwedge^{n-p} V$ characterized by $\langle \xi, \eta \rangle (*1) = \xi \wedge *\eta$ for any $\xi, \eta \in \bigwedge^p V$, where $*1$ means $e_1 \wedge \cdots \wedge e_n$ and the inner product $\langle \cdot, \cdot \rangle$ for $\bigwedge^p V$ is defined by using $e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $1 \leq i_1 < \cdots < i_p \leq n$ as an orthonormal basis.

The coordinates of $\mathbb{R}^n$ are $x_1, \cdots, x_n$. The orientation of $\mathbb{R}^n$ is defined by the differential n-form $dx_1 \wedge \cdots \wedge dx_n$. The $*$ operator for differential p-forms on $\mathbb{R}^n$ are defined with respect to the inner product for which $dx_1, \cdots, dx_n$ form an orthonormal basis.

For a vector field $\mathbf{F} = \sum_{j=1}^3 F_j \frac{\partial}{\partial x_j}$ on an open subset of $\mathbb{R}^3$, the divergence $\sum_{j=1}^3 \frac{\partial F_j}{\partial x_j}$ of $\mathbf{F}$ is denoted by $\nabla \cdot \mathbf{F}$ and the curl of $\mathbf{F}$ is denoted by $\nabla \times \mathbf{F}$ whose i-th component is $\frac{\partial F_k}{\partial x_j} - \frac{\partial F_j}{\partial x_k}$ when (i, j, k) is an orientation-preserving cyclic permutation of $(1, 2, 3)$.

For a function f on an open subset of $\mathbb{R}^3$ the gradient $\sum_{j=1}^3 \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}$ of f is denoted by ∇f and the Laplacian $\sum_{j=1}^3 \frac{\partial^2 f}{\partial {x_j}^2}$ of f is denoted by $\nabla^2 f$ or Δf.

A differential k-form ω on a domain U of $\mathbb{R}^n$ is said to be *closed* on U if its exterior differentiation $d\omega$ vanishes identically on U. The differential k-form ω is said to be *exact* on U if $\omega = d\sigma$ for some differential $(k-1)$-form on U.

Problem 1 (Cartan's Formula Relating Exterior Derivatives of Differential Forms to Lie Derivatives of Vector Fields). Let $1 \leq k < n$ be integers. Let G be an open subset of $\mathbb{R}^n$ and ω be an infinitely differentiable k-form on G and $\xi_1, \cdots, \xi_{k+1}$ be infinitely differentiable vector fields on G. Verify the following formula of Cartan.

$$(k+1)\, d\omega\left(\xi_1, \cdots, \xi_{k+1}\right) = \sum_{j=1}^{k+1} (-1)^{j+1} \xi_j \left(\omega\left(\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_{k+1}\right)\right)$$

$$+ \sum_{1 \leq j < \ell \leq k+1} (-1)^{j+\ell} \omega\left(\left[\xi_j, \xi_\ell\right], \xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_{\ell-1}, \xi_{\ell+1}, \cdots, \xi_{k+1}\right).$$

Note that the Lie bracket $[\xi, \eta]$ of two vector fields is defined by

$$[\xi, \eta]\, f = \xi(\eta(f)) - \eta(\xi(f))$$

for any function f and that the factor $k+1$ comes from one kind of normalization used in the skew-symmetrization of the tensor product to yield the exterior product.

Hint: check the linearity of the formula when vector fields are multiplied by functions and check the formula for vector fields defined by partial differentiation with respect to the coordinates.

PROBLEM 2 *(Lie Brackets of Vector Fields From Commutators of Trajectory Functions).* Let G and G' be bounded open subsets of $\mathbb{R}^n$ with coordinates $(x^1, \cdots, x^n)$ such that the closure of G is contained in G'. Let

$$\xi = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j},$$
$$\eta = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

be infinitely differentiable $\mathbb{R}$-valued vector fields on G'. Let $[\xi, \eta]$ denote the Lie bracket of ξ and η which is defined as the vector field whose k-th component is

$$\sum_{j=1}^{n} \xi^j \frac{\partial \eta^k}{\partial x^j} - \sum_{j=1}^{n} \eta^j \frac{\partial \xi^k}{\partial x^j}$$

for $1 \leq k \leq n$. Let T be a positive number and

$$\Phi_t(x) = \left(\Phi_t^1(x), \cdots, \Phi_t^n(x)\right)$$

be the n-tuple of functions on G which describes the integral trajectories of ξ in the sense that

$$\frac{\partial}{\partial t} \Phi_t^k(x) = \xi^k \left(\Phi_t(x)\right)$$

and $\Phi_t(x)|_{t=0} = x$ for $x \in G$ and $|t| \leq T$. Here we assume that $\Phi_t(x) \in G'$ for $x \in G$ and $|t| \leq T$. Likewise, let

$$\Psi_t(x) = \left(\Psi_t^1(x), \cdots, \Psi_t^n(x)\right)$$

be the n-tuple of functions on G which describes the integral trajectories of η in a similar sense. Let $\varphi(t)$ and $\psi(t)$ be continuously differentiable $\mathbb{R}$-valued functions defined in a neighborhood of 0 in $\mathbb{R}$ vanishing at 0 such that $\varphi'(0)$ and $\psi'(0)$ are nonzero. Compute

$$\lim_{t\to 0}\frac{1}{t^2}\left(-x+\left(\Phi_{\varphi(t)}\circ\Psi_{\psi(t)}\circ\Phi_{-\varphi(t)}\circ\Psi_{-\psi(t)}\right)(x)\right)$$

in terms of the components of $[\xi,\eta]$ at x and $\varphi'(0)$ and $\psi'(0)$ for $x\in G$.
Hint: Use

$$\Phi_u^k(x)=x^k+\xi^k(x)\,u+\frac{1}{2}\left(\sum_{j=1}^n\xi^j(x)\,\frac{\partial\xi^k}{\partial x^j}(x)\right)u^2+O\left(u^3\right),$$
$$\Psi_v^k(x)=x^k+\eta^k(x)\,v+\frac{1}{2}\left(\sum_{j=1}^n\eta^j(x)\,\frac{\partial\eta^k}{\partial x^j}(x)\right)v^2+O\left(v^3\right)$$

to compute $\left(\Phi_u\circ\Psi_v-\Psi_v\circ\Phi_u\right)^k(x)$ in terms of the components of ξ and η and their first-order partial derivatives, up to an error term of order ≥ 3 in u and v. Then compose $\Phi_u\circ\Psi_v-\Psi_v\circ\Phi_u$ with $\Phi_{-u}\circ\Psi_{-v}$.

Problem 3 (Maxwell's Equations in Terms of Closed Forms and the Star Operator.) Let $\mathbf{E}=(E_x,E_y,E_z)$ be the electric field and $\mathbf{B}=(B_x,B_y,B_z)$ be the magnetic field. Let ρ be the charge density and $\mathbf{j}=(j_x,j_y,j_z)$ be the current density. They are all functions of the variables (x,y,z) of space and the variable t of time. The Maxwell equations (with all the constants set to be 1) are the following.

$$\nabla\cdot\mathbf{E}=\rho,$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{B}=\mathbf{j}+\frac{\partial\mathbf{E}}{\partial t}.$$

Introduce the differential 2-form

$$F=(E_xdx+E_ydy+E_zdz)\wedge dt-(B_xdy\wedge dz+B_ydz\wedge dx+B_zdx\wedge dy)$$

on $\mathbb{R}^4$ with variables (t,x,y,z).

(a) Verify that the equation $dF = 0$ is equivalent to the second and third equations of Maxwell.

(b) Let the space-time with variables (t, x, y, z) be oriented by the order dt, dx, dy, dz. The star operator, with respect to the first fundamental form $dt^2 - dx^2 - dy^2 - dx^2$, is defined as the star operator with respect to the first fundamental form $dt^2 + dx^2 + dy^2 + dx^2$ followed by changing dx, dy, dz to $-dx, -dy, -dz$. (Here the first fundamental form means the inner product for the tangent space and it defines also at the same time the inner product for the dual of the tangent space. It is a length element in the sense that the integration of its square root along a curve gives the length of the curve with respect to it.) Verify that the star operator, with respect to the first fundamental form $dt^2 - dx^2 - dy^2 - dx^2$, which transforms F to $*F$ has the same effect as interchanging the pair $(\mathbf{E}, \mathbf{B})$ with the pair $(-\mathbf{B}, \mathbf{E})$

(c) Verify that the equation $d{*F} = *\,(\rho dt - j_x dx - j_y - j_z dz)$ is equivalent to the first and last equations of Maxwell. Here the star operator is with respect to $dt^2 - dx^2 - dy^2 - dx^2$.

Problem 4 (Möbius Band – Problem 32 on Page 298 of Rudin's Book). Fix $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in \mathbb{R}^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in $\mathbb{R}^3$, with parameter domain D, given by

$$\begin{cases} x = (1 - t\sin\theta)\cos 2\theta, \\ y = (1 - t\sin\theta)\sin 2\theta, \\ z = t\cos\theta, \end{cases}$$

where $(x, y, z) = \Phi(\theta, t)$. Note that $\Phi(\pi, t) = \Phi(0, -t)$, and that Φ is one-to-one on the rest of D.

The range of $M = \Phi(D)$ of Φ is known as a *Möbius band*. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$, $\mathbf{p}_2 = (\pi, -\delta)$, $\mathbf{p}_3 = (\pi, \delta)$, $\mathbf{p}_4 = (0, \delta)$, $\mathbf{p}_5 = \mathbf{p}_1$. Put $\gamma_i = [\mathbf{p}_i, \mathbf{p}_i]$, $i = 1, 2, 3, 4$, and put $\Gamma_i = \Phi \circ \gamma_i$. Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put $\mathbf{a} = (1, 0, -\delta)$, $\mathbf{b} = (1, 0, \delta)$. Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ can be described as follows.

Γ_1 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y)-plane has winding number $+1$ around the origin.

$\Gamma_2 = [\mathbf{b}, \mathbf{a}]$.

Γ_3 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y)-plane has winding number -1 around the origin.

$\Gamma_4 = [\mathbf{b}, \mathbf{a}]$.

Thus $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$.

If we go from $\mathbf{a}$ to $\mathbf{b}$ along Γ_1 and continue along the "edge" of M until we return to $\mathbf{a}$, the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval $[0, 2\pi]$ by the equations

$$\begin{cases} x = (1 + \delta \sin\theta) \cos 2\theta, \\ \\ y = (1 + \delta \sin\theta) \sin 2\theta, \\ \\ z = -\delta \cos\theta. \end{cases}$$

It should be emphasized that $\Gamma \neq \partial\Phi$. Let

$$\eta = \frac{x dy - y dx}{x^2 + y^2}.$$

Since $d\eta = 0$, Stokes's theorem shows that

$$\int_{\partial\Phi} \eta = 0.$$

But although Γ is the "geometric" boundary of M, we have

$$\int_{\Gamma} \eta = 4\pi.$$

Remark. In order to avoid this possible source of confusion, Stokes's formula is frequently stated only for orientable sufaces Φ.

Problem 5 (Green's Identities). Let E be an open subset of $\mathbb{R}^3$.

(a) Let $\mathbf{F}$ be a (differentiable) vector field on E. Let ω be the 1-form on E characterized by $\omega(\mathbf{v}) = \langle \mathbf{v}, \mathbf{F} \rangle$ for any tangent vector $\mathbf{v}$ of $\mathbb{R}^3$ at any point of E.

(i) Verify that the coefficient of $dx_1 \wedge dx_2 \wedge dx_3$ in the 3-form $d * \omega$ is equal to the divergence $\nabla \cdot \mathbf{F}$ of the vector field $\mathbf{F}$ on E.

(ii) Verify that the coefficients of dx_1, dx_2, dx_3 in the 1-form $*d\omega$ are the components of the curl $\nabla \times \mathbf{F}$ of the vector field $\mathbf{F}$ on E with respect to $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$.

(b) Let g, h be two twice continuously differentiable $\mathbb{R}$-valued functions on E and consider the vector field $\mathbf{F} = g\nabla h$.

(i) Prove that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h).$$

(ii) Let Ω be a bounded closed subset of E with positively oriented continuously differentiable boundary $\partial\Omega$. Prove that

$$(*) \qquad \int_\Omega \big(g\nabla^2 h + (\nabla g) \cdot (\nabla h)\big)\, dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n}\, dA,$$

where $\frac{\partial h}{\partial n}$ means the dot product of ∇h with the outward normal vector $\mathbf{n}$ of Ω. Interchange g and h and subtract the resulting formula from the first one to obtain

$$(**) \qquad \int_\Omega \big(g\nabla^2 h - h\nabla^2 g\big)\, dV = \int_{\partial\Omega} \left(g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n}\right) dA.$$

Both $(*)$ and $(**)$ are usually called Green's Identities.

(iii) Assume that h is harmonic in E; this means that $\nabla^2 h = 0$. Take $g = 1$ in $(*)$ to conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n}\, dA = 0.$$

In other words, the integration, over the boundary of the domain, of the normal derivative of a function harmonic on a domain is zero.

Take $g = h$ in $(**)$ to conclude that $h = 0$ in Ω if $h = 0$ in $\partial\Omega$. In other words, any harmonic function on Ω whose boundary value is zero must be identically zero.

(iv) Let f be a twice continuously differentiable $\mathbb{R}$-valued function on Ω. Assume that Ω is connected. If f is harmonic on Ω and if at every point of $\partial\Omega$ either f is zero or its normal derivative is zero, prove that f must be identically constant on Ω.

(c) Let Ω be a bounded closed subset of $\mathbb{R}^n$ with positively oriented continuously differentiable boundary $\partial\Omega$. Let g, h be twice continuously differentiable $\mathbb{R}$-valued functions on Ω. Prove the following analogues of the two Green's Identities in general dimension.

(i)

$$\int_\Omega (g\, d * dh + dg \wedge *dh) = \int_{\partial\Omega} g \, * dh.$$

(ii)

$$\int_\Omega (g\, d * dh - h\, d * dg) = \int_{\partial\Omega} (g \, * dh - h \, * dg)\,.$$

Verify that they are reduced to $(*)$ and $(**)$ in Part (b)(ii) when $n = 3$.

Problem 6 (Inexact Closed Forms). Fix an integer $n \geq 2$. Let

$$r_k = \left(\sum_{j=1}^{k} x_j^2\right)^{\frac{1}{2}}$$

for $1 \leq k \leq n$. Let E_k be the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $r_k > 0$. Let ω_k be the $(k-1)$-form defined in E_k by

$$\omega = \frac{1}{(r_k)^k} \sum_{j=1}^{k} (-1)^{j-1}\, x_j\, dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_k.$$

(a) For $3 \leq k \leq n$ determine the real number c_k such that

$$\omega_k = c_k \, * d\, \frac{1}{(r_k)^{k-2}}.$$

(b) For $k = 2$ determine the real number c_k such that $\omega_k = c_k * d \log r_k$.

(c) For $2 \leq k \leq n$ verify that $d\omega_k = 0$ in E_k.

(d) Verify that, if $\xi = \sum_{j=1}^{k} \xi_k \, dx_k$ satisfies $\sum_{j=1}^{k} x_k \, \xi_k = 0$ and

$$\xi_k = (-1)^k \frac{\left(r_{k-1}\right)^{k-1}}{\left(r_k\right)^k},$$

then $\omega_k = \xi \wedge \omega_{k-1}$.

(e) Let

$$g_k(t) = \int_{s=-1}^{t} \left(1 - s^2\right)^{\frac{k-3}{2}} ds$$

for $-1 < t < 1$. Let $f_k(\mathbf{x}) = (-1)^k g_k\left(\frac{x_k}{r_k}\right)$. Use Part(d) to show that $\omega_k = d\left(f_k \omega_{k-1}\right)$ and hence ω_k is exact on E_{k-1}.

(f) Compute the integral of ω_k over the $(k-1)$-surface

$$\{r_k = 1\} \cap \{x_{k+1} = \cdots = x_n = 0\}$$

to show that ω_k is not exact in E_k.

PROBLEM 7 *(Computation of Laplacian from the Divergence Theorem)*. The following is the procedure of computation of the Laplacian required for this problem. The Divergence Theorem on $\mathbb{R}^2$ states that if Ω is a bounded domain in $\mathbb{R}^2$ with piecewise continuously differentiable boundary $\partial\Omega$ and if $\vec{v}$ is a continuously differentiable $\mathbb{R}$-valued vector field on some open neighborhood G of the closure $\bar{\Omega}$ of Ω in $\mathbb{R}^2$, then

$$\int_{\partial\Omega} \vec{v} \cdot \vec{n} = \int_{\Omega} \operatorname{div} \vec{v},$$

where $\vec{n}$ is the outward-pointing unit normal vector to $\partial\Omega$ and $\operatorname{div} \vec{v}$ is the divergence of $\vec{v}$. In particular, if φ is a twice continuously differentiable $\mathbb{R}$-valued function on G, then

$$\int_{\partial\Omega} \operatorname{grad} \varphi \cdot \vec{n} = \int_{\Omega} \Delta\varphi,$$

where $\operatorname{grad}\varphi$ is the gradient of φ and $\Delta\varphi$ is the Laplacian of φ, when $\vec{v}$ is chosen to be the gradient of φ. As a consequence, if Ω is replaced by a sequence of Ω_ν which shrinks to a point P_0 as $\nu \to \infty$, then

$$(\dagger) \qquad (\Delta\varphi)(P_0) = \lim_{\nu\to\infty} \frac{1}{\operatorname{Area}(\Omega_\nu)} \int_{\partial\Omega_\nu} \operatorname{grad}\varphi \cdot \vec{n}_\nu,$$

where $\operatorname{Area}(\Omega_\nu)$ is the area of Ω_ν and $\vec{n}_\nu$ is the outward-pointing unit normal vector to $\partial\Omega_\nu$.

Let (r, θ) be the polar coordinates of $\mathbb{R}^2$ so that the Euclidean coordinates x, y of $\mathbb{R}^2$ are related to (r, θ) by $x = r\cos\theta$ and $y = r\sin\theta$. Let φ be a twice continuously differentiable $\mathbb{R}$-valued function on an open neighborhood G of a point P_0 of $\mathbb{R}^2$ different from the origin. Let η_ν $(\nu \in \mathbb{N})$ be a sequence of positive numbers decreasing to 0. Let Ω_ν be the domain defined by $|r - r(P_0)| < \eta_\nu$ and $|\theta - \theta(P_0)| < \eta_\nu$ in polar coordinates. Use $(\dagger)$ to compute the Laplacian $\Delta\varphi$ of φ in terms of r, θ, and the first-order and second-order partial derivatives of φ with respect to the polar coordinates (r, θ).