Math 55b, Assignment #7, April 8, 2006
(due April 14, 2006)

Notations and Terminology. $\mathbb{R}$ is the set of all real numbers. $\mathbb{N}$ is the set of all positive integers. For two functions $f(x)$ and $g(x)$ on $\mathbb{R}$ with periodicity 2π the *convolution* $f * g$ of f and g is a function on $\mathbb{R}$ with periodicity 2π and is defined by

$$(f * g)(x) = \int_{t=-\pi}^{\pi} f(x-t) g(t)\, dt.$$

The Fourier transform $\hat{f}(\xi)$ of a function $f(x)$ on $(-\infty, \infty)$ is defined by

$$\hat{f}(\xi) = \int_{x=-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx.$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ is defined as consisting of all complex-valued functions $f(x)$ on $\mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^\ell f(x)}{dx^\ell} \right| < \infty \quad \text{for all nonnegative integers} \quad k \quad \text{and} \quad \ell.$$

Problem 1 (Pointwise Convergence of the Cesàro Sum of the Fourier Series of a Continuous Periodic Function). Let $f(x)$ be a continuous function on $\mathbb{R}$ with period 2π. Let s_n be the n-th partial sum of the Fourier series of $f(x)$ so that

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx \right),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx,$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx.$$

Let σ_n denote the n-th Cesàro sum of the Fourier series of $f(x)$ defined by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n}.$$

Let $D_n(x)$ denote the n-th Dirichlet kernel which is defined by

$$D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}.$$

Define the Féjer kernel $F_n(x)$ by

$$F_n(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{n-1}(x)}{n}.$$

(a) Verify that $\sigma_n = f * F_n$.

(b) Verify that

$$F_n(x) = \frac{1}{2\pi n} \frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}.$$

(c) Verify that the family of functions $\{F_n(x)\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ with period 2π is an approximate identity on $[-\pi, \pi]$ in the sense that the following three conditions are satisfied.

(i) *(Nonnegativity)* $F_n(x) \geq 0$ for $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$.

(ii) *(Unit Integral)* $\int_{-\pi}^{\pi} F_n(x)\, dx = 1$ for all $n \in \mathbb{N}$.

(iii) *(Integral Outside Any Neighborhood of the Origin Approaching 0)* For any $\eta > 0$ the integral

$$\int_{\substack{-\pi\leq x\leq\pi \\ |x|\geq\eta}} F_n(x)\, dx$$

approaches 0 as $n \to \infty$.

(d) Use Part(c) to show that $\sigma_n(x)$ approaches to $f(x)$ uniformly in $-\pi \leq x \leq \pi$ as $n \to \infty$.

(e) *(Weierstrass Approximation Theorem)* Use Part(d) and the power series expansions of $\sin x$ and $\cos x$ to show that any continuous function on $[-\pi, \pi]$ can be uniformly approximated by polynomials.

Hint: First show that for any given continuous function on $[-\pi, \pi]$ it is possible to add an appropriate polynomial of degree ≤ 1 to get a continuous function on $[-\pi, \pi]$ vanishing at the two end-points of the interval.

Problem 2 (Poisson Summation Formula as a Result of Periodization and the Pointwise Convergence of the Fourier Series of Functions with Bounded Lipschitz Norm). Let $f(x)$ be a function belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$. Denote by $F(x)$ the following *periodization* of $f(x)$ for period 2π defined by

$$F(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{x}{2\pi} + n\right).$$

(a) Verify that the function $F(x)$ on $\mathbb{R}$ has bounded Lipschitz norm in the sense that

$$\sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|F(x) - F(y)|}{|x-y|} < \infty.$$

(b) Express the n-th Fourier coefficient

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-inx} dx$$

of F in terms of the value $\hat{f}(n)$ of the Fourier transform

$$\hat{f}(\xi) = \int_{x=-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx.$$

of f at n.

(c) Use the pointwise convergence of the Fourier series of F to F (according to Theorem 8.14 on page 189 of Rudin's book) to prove the following *Poisson Summation Formula* for any function f in the Schwarz space $\mathcal{S}(\mathbb{R})$.

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x} \quad \text{for all } x \in \mathbb{R}$$

and, in particular,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=\infty}^{\infty} \hat{f}(n).$$

Problem 3 (Application of Parseval's Identity to Series Summation).

(a) By applying Parseval's identity to the function $f(x) = |x|$ on $[-\pi, \pi]$, verify

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) By applying Parseval's identity to the function $f(x) = x\,(\pi - |x|)$ on $[-\pi, \pi]$, verify

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Problem 4 (Féjer's Example of a Continuous Periodic Function Whose Fourier Series Fails to Have Pointwise Convergence).

(a) Verify that

$$\sup_{x>0} \left| \int_0^x \frac{\sin t}{t}\, dt \right| < \infty.$$

Hint: For $0 < \alpha < \beta$ consider the imaginary part of $\int_\alpha^\beta \frac{e^{it}}{t} dt$ whose absolute value can be estimated, uniformly in β, by integration by parts.

(b) Verify that

$$\sup_{\substack{x\in\mathbb{R}\\ n\in\mathbb{N}}} \left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| < \infty.$$

Hint: Compare the sum with the integral in Part(a).

(c) For $n \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ let

$$(*) \qquad \varphi_{n,r}(x) = \sum_{k=1}^{n} \frac{\cos(r+n-k+1)x}{2k-1} - \sum_{k=1}^{n} \frac{\cos(r+n+k)x}{2k-1}.$$

Show that

$$\sup_{\substack{r\in\mathbb{N}\cup\{0\}\\ n\in\mathbb{N},x\in\mathbb{R}}} |\varphi_{n,r}(x)| < \infty.$$

Hint: Use trigonometric identities to show that

$$\varphi_{n,r}(x) = 2\sin\left(\left(r+n+\frac{1}{2}\right)x\right) \sum_{k=1}^{n} \frac{\sin\left(k-\frac{1}{2}\right)x}{2k-1}$$

$$= 2\sin\left(\left(r+n+\frac{1}{2}\right)x\right) \left(\sum_{k=1}^{2n} \frac{\sin \frac{k}{2}x}{k} - \frac{1}{2} \sum_{k=1}^{n} \frac{\sin kx}{k} \right)$$

and use Part(b).

(d) For every $\ell \in \mathbb{N}$ let λ_ℓ be a positive integer such that $\lambda_{\ell+1} > \lambda_\ell$ for $\ell \in \mathbb{N}$. Let $\mu_\ell = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell-1})$ for $\ell \geq 2$. Let $\mu_1 = 0$. Use Part(c) to show that the series

$$(\dagger) \qquad \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \varphi_{\lambda_\ell,\mu_\ell}(x)$$

converges uniformly to a function $f(x)$ on all of $\mathbb{R}$. (Note that the function $f(x)$ depends on the choice of the sequence $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.)

(e) Verify that the series obtained from ($\dagger$) by replacing $\varphi_{\mu_\ell,\lambda_\ell}(x)$ by an $\mathbb{R}$-linear combination of 1, $\cos kx$, and $\sin kx$ for $k \in \mathbb{N}$ according to ($*$) is the Fourier series of $f(x)$.

(f) Let

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

be the n-th partial sum of the Fourier series of $f(x)$. Show that

$$s_{\mu_\ell+\lambda_\ell}(0) = \frac{1}{\ell^2} \sum_{j=1}^{\lambda_\ell} \frac{1}{2j-1}.$$

Hence, if $\lambda_\ell = \ell^{\ell^2}$, then $s_{\mu_\ell+\lambda_\ell}(0) \to \infty$ as $\ell \to \infty$ and the Fourier series of the function $f(x)$ is not convergent at the point $x = 0$ though the function $f(x)$ is uniformly continuous on $\mathbb{R}$ with period 2π.

Remark. The main idea of Féjer's example is as follows. When the terms in the Fourier series are grouped together in a special way to form a new series of functions, each term in the new series of functions is dominated uniformly and absolutely by the corresponding term in a convergent series of numbers. However, some partial sequence of the Fourier series at the origin diverges.

Problem 5 (Gaussian Distribution Function as Limit of Probability Distribution of Tossing a Number of Coins When the Number of Coins Goes to Infinity — REVISITED*).*

A *random variable* X is a variable whose value is generated by a process of chance which assigns a probability p_j for X to assume a value x_j. For example, when the random variable X is the face value of a cube-shaped die marked with values from 1 to 6 and which is randomly thrown, the probability for X to assume the value 1 is $\frac{1}{6}$, and the probability for X to assume the value 2 is $\frac{1}{6}$, and so forth. The *characteristic function* $\varphi_X(t)$ for X in this case is defined as the Fourier series

$$\varphi_X(t) = \sum_{k=1}^{6} \frac{1}{6} e^{ikt}.$$

In general, given a discrete set J, when the probability for the random variable X to assume the value x_j is p_j for $j \in J$, the characteristic function $\varphi_X(t)$ for X is defined as

$$\varphi_X(t) = \sum_{j \in J} p_j e^{ix_j t}.$$

When the range of values to be assumed by a random variable X is all of $\mathbb{R}$ (instead of by a discrete set J), the characteristic function $\varphi_X(t)$ of X is

$$\varphi_X(t) = \int_{x=-\infty}^{\infty} p(x) e^{ixt} dx,$$

where $p(x)$ is the probability *density* for the random variable X in the sense that the probability for X to assume a value in a subset E of $\mathbb{R}$ is given by $\int_E p(x)dx$. The case of the random variable X assuming only discrete values x_j with probability p_j corresponds to replacing $p(x)dx$ by $d\alpha(x)$ to form a Riemann-Stieltjes integral, where $\alpha(x)$ is a nondecreasing function of x which is constant outside the discrete set $\{x_j\}_{j \in J}$ with a jump of magnitude p_j at the point x_j.

When the random variables X and Y are *independent*, the characteristic function $\varphi_{X+Y}(t)$ for $X + Y$ is equal to $\varphi_X(t)\varphi_Y(t)$.

Now let F be the random variable of flipping a coin which assigns the value $\frac{1}{2}$ to the head and the value $\frac{-1}{2}$ to the tail. Let $F_1, \cdots, F_n$ be n independent random variables, each of which has precisely the same value-set and the same probabilities for its values as the random variable F. Let G_n be the random variable

$$\frac{1}{\sqrt{\frac{n}{4}}} \left(F_1 + \cdots + F_n\right),$$

which means the random variable $F_1 + \cdots + F_n$ with its value multiplied by the factor $\frac{1}{\sqrt{\frac{n}{4}}}$ but with all the probabilities unchanged. Let G be the random variable whose probability density $p(x)$ is given by

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}.$$

(a) Verify that the characteristic function for the random variable G_n is

$$\varphi_{G_n}(t) = \frac{1}{2^n}\left(e^{-\frac{it}{\sqrt{n}}} + e^{\frac{it}{\sqrt{n}}}\right)^n.$$

(b) Verify that the characteristic function for the random variable G is

$$\varphi_G(t) = e^{-\frac{t^2}{2}}.$$

(c) Verify that for any $A > 0$ the characteristic function $\varphi_{G_n}(t)$ for the random variable G_n converges to the characteristic function $\varphi_G(t)$ for the random variable G uniformly for $-A \le t \le A$.

(d) For $n \in \mathbb{N}$ let $\beta_n(x)$ be the nonnegative function on $\mathbb{R}$ which is continuous from the right at every point of $\mathbb{R}$ such that the domain under its graph is the union of the $n+1$ rectangles with vertices

$$\left(\frac{k - \frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, 0\right), \quad \left(\frac{k + 1 - \frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, 0\right),$$
$$\left(\frac{k - \frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, \frac{\sqrt{\frac{n}{4}}\binom{n}{k}}{2^n}\right), \quad \left(\frac{k + 1 - \frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, \frac{\sqrt{\frac{n}{4}}\binom{n}{k}}{2^n}\right)$$

for $0 \le k \le n$. Use Part(c) to show that for any $a < b$ in $\mathbb{R}$,

$$\lim_{n\to\infty} \int_a^b \beta_n(x)dx = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} dx.$$

Hint: Let $\alpha_n(x)$ be a nondecreasing function on $\mathbb{R}$ with $\lim_{x\to\infty} \alpha_n(x) = 1$ and $\lim_{x\to-\infty} \alpha_n(x) = 0$ such that

$$\varphi_{G_n}(t) = \int_{x=-\infty}^{\infty} e^{ixt} d\alpha_n(x).$$

Use the analog of the Fourier Inversion (Problem 3(c) of Assignment #6) for Riemann-Stieltjes integrals and use Part(c) to show that

$$\lim_{n\to\infty} (\alpha_n(b) - \alpha_n(a)) = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, dx.$$

Finally compare $\int_a^b \beta_n(x)dx$ with $\alpha_n(b) - \alpha_n(a)$.