Math 55b, Assignment #2, February 11, 2006
(due February 23, 2006)

Notations. $\mathbb{N}$ = all positive integers.
$\mathbb{R}$ = all real numbers.
$\mathbb{C}$ = all complex numbers.

Problem 1 (Inverse Mapping Theorem for Analytic Funcions). Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$-valued function $f(z_1, \cdots, z_n)$ on some open subset D of $\mathbb{F}^n$ is said to be *analytic* if for every given point $(z_1^*, \cdots, z_n^*) \in D$ the value of $f(z_1, \cdots, z_n)$ is equal to the value of a convergent power series

$$\sum_{\nu_1, \cdots, \nu_n = 0}^{\infty} c_{\nu_1, \cdots, \nu_n} (z_1 - z_1^*)^{\nu_1} \cdots (z_n - z_n^*)^{\nu_n}$$

(with some $c_{\nu_1, \cdots, \nu_n} \in \mathbb{F}$) on $\sum_{j=1}^n \left| z_j - z_j^* \right|^2 < r^2$ for some $r > 0$ (which may depend on z^*). A vector-valued function is said to be analytic if every one of its components is analytic. To be more precise, the function $f(z_1, \cdots, z_n)$ (either scalar-valued or vector-valued) is said to be *complex-analytic* or *holomorphic* when $\mathbb{F} = \mathbb{C}$ and is said to be *real-analytic* when $\mathbb{F} = \mathbb{R}$.

(a) Prove the following *Inverse Mapping Theorem* for the case of analytic functions. Let E be an open subset of $\mathbb{F}^n$ and $\mathbf{f}$ be an analytic map from E to $\mathbb{F}^n$. If $\mathbf{a} \in E$ and the derivative $\mathbf{f}'(\mathbf{a})$ of $\mathbf{f}$ at $\mathbf{a}$ is invertible as an $\mathbb{F}$-linear transformation of $\mathbb{F}^n$, then there exist an open neighborhood U of $\mathbf{a}$ in E and an open neighborhood V of $\mathbf{f}(\mathbf{a})$ in $\mathbb{F}^n$ such that $\mathbf{f}$ maps U one-one onto V and its inverse map $\mathbf{f}^{-1}$ is analytic on V.

(b) Prove the following *Implicit Function Theorem* for the case of analytic functions. Let E be an open subset of $\mathbb{F}^{m+n}$ and $\mathbf{f}$ be an analytic map from E to $\mathbb{F}^n$. Let $(\mathbf{a}, \mathbf{b}) \in E$ with $\mathbf{a} \in \mathbb{F}^m$ and $\mathbf{b} \in \mathbb{F}^n$ and $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. If the derivative of the map $\mathbf{y} \mapsto \mathbf{f}(\mathbf{a}, \mathbf{y})$ at $\mathbf{b}$ is invertible as an $\mathbb{F}$-linear transformation of $\mathbb{F}^n$, then there exit an open neighborhood U of $\mathbf{a}$ in $\mathbb{F}^m$ and an open neighborhood W of $(\mathbf{a}, \mathbf{b})$ in $\mathbb{F}^{m+n}$ and an analytic map $\mathbf{x} \mapsto \mathbf{g}(\mathbf{x})$ from U to $\mathbb{F}^n$ such that for every $\mathbf{x} \in U$ the equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ has a unique solution $\mathbf{y}$ with $(\mathbf{x}, \mathbf{y}) \in W$ and, moreover, the unique solution is precisely $\mathbf{g}(\mathbf{x})$.

Hint: After applying the Inverse Mapping Theorem and the Inverse Function Theorem for the $\mathcal{C}'$ case (*i.e.,* the case of all first-order partial derivatives being continuous), use the

Chain Rule (or the equivalent power series expansion) to estimate the absolute value of the higher-order derivatives of the function whose analyticity is to be verified and finally apply Taylor's Theorem with derivative remainders.

As a hint on how to keep track of the growth order in the estimate of high-order derivatives, we sketch how it is done in the simple case of inverting a power series of a single variable. Given $g(x) = \sum_{\nu=1}^{\infty} b_\nu x^x$ with $b_\nu \in \mathbb{R}$ for $\nu \in \mathbb{N}$ and $b_1 \neq 0$ and the convergence of $\sum_{\nu=1}^{\infty} b_\nu x_0^x$ for some $x_0 \neq 0$. We would like to invert $y = g(x)$ at the origin to get a convergent power series $f(y) = \sum_{\nu=1}^{\infty} a_\nu y^\nu$ with $a_\nu \in \mathbb{R}$ for $\nu \in \mathbb{N}$ such that $f(g(x)) \equiv x$.

We use the equivalent power series expansion instead of the Chain Rule to solve for a_ν for $\nu \in \mathbb{N}$. The power series expansion of $f(g(x)) \equiv x$ yields

$$(*) \qquad \sum_{\mu=1}^{\infty} a_\mu \left(\sum_{\nu=1}^{\infty} b_\nu x^\nu \right)^\mu \equiv x.$$

We solve for a_k by induction on $k \in \mathbb{N}$ after equating the coefficients of x^k in $(*)$. It is the same as getting $a_1 = \frac{1}{b_1}$ and then, for $k \geq 2$, equating the coefficients of x_k in

$$(**) \qquad a_k \left(b_1 x\right)^k = -\sum_{\mu=1}^{k-1} a_\mu \left(\sum_{\nu=1}^{k} b_\nu x^\nu \right)^\mu \quad \text{modulo } x^{k+1}.$$

The main point is to keep track of the growth order of $|a_k|$ as a function of k. From the convergence of $\sum_{\nu=1}^{\infty} b_\nu x_0^x$ we conclude that $|b_\nu| \leq \frac{M}{|x_0|^\nu}$ for some $M > 0$ and all $\nu \in \mathbb{N}$. By changing first the scale of the variable y and then the scale of the variable x, we can assume without loss of generality that $|b_1| = 2$ and $|b_\nu| \leq \frac{1}{2^{\nu-1}}$ for $\nu \geq 2$. By comparing the equating the coefficients of x^k with the process of setting $x = 1$ in $(**)$, we conclude that

$$a_k \, 2^k = -\sum_{\mu=1}^{k-1} a_\mu \left(2 + \sum_{\nu=2}^{k} |b_\nu| \right)^\mu,$$

which implies that

$$|a_\nu| \leq \frac{3}{2^k} \sum_{\nu=1}^{k-1} |a_\nu|,$$

because $\sum_{\nu=2}^{\infty} |b_\nu| \leq \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Inductively we conclude that $|a_k| \leq 1$ for all k, because $|a_k| \leq \frac{3(k-1)}{2^k}$ for $k \geq 2$ and $a_1 = \frac{1}{b_1} = \frac{1}{2}$.

Problem 2 (A Generalization of the Implicit Function Theorem for Smooth Functions Adapted from Michael Artin's 1968 Inventiones Mathematicae Paper). Let a and b be positive numbers and $f(x, y)$ be an $\mathbb{R}$-valued infinitely differentiable function on $\{|x| < a, \, |y| < b\}$ with $f(0,0) = 0$ and $f_y(0,0) = 0$, where f_y is the first-order partial derivative of $f(x, y)$ with respect to y. Let

$y = g(x)$ be an $\mathbb{R}$-valued infinitely differentiable function on $|x| < a$ with $g(0) = 0$ and $\sup_{|x|<a} |g(x)| < b$. Assume that

$$\frac{f(x, g(x))}{(f_y(x, g(x)))^2}$$

is equal to an $\mathbb{R}$-valued infinitely differentiable function $v(x)$ on $|x| < a$ with $v(0) = 0$ in the sense that

$$f(x, g(x)) = v(x)\ (f_y(x, g(x)))^2$$

on $|x| < a$. Prove that there exists an $\mathbb{R}$-valued infinitely differentiable function $h(x)$ on $|x| < \eta$ for some positive number η such that $h(0) = 0$ and the function

$$q(x) := g(x) + f_y(x, g(x))\, h(x)$$

satisfies $f(x, q(x)) \equiv 0$ on $|x| < \eta$.

(This conclusion generalizes the Implicit Function Theorem, because it tells us that, even when $f_y(0, 0) = 0$ we can still solve $f(x, q(x)) \equiv 0$ for $q(x)$, if we start out with some $g(x)$ so that $f(x, g(x))$ already vanishes to an order higher than the square of $f_y(x, g(x))$ at $x = 0$, then we can modify $g(x)$ to get $q(x)$ with the difference $q(x) - g(x)$ vanishing to an order higher than $f_y(x, g(x))$ at $x = 0$.)

Hint: First verify that for any infinitely differentiable function $F(x)$ there exists an infinitely differentiable function $G(x, u)$ such that $F(x + u) = F(x) + F'(x)u + G(x, u)u^2$. Take away a factor from $f(x, g(x)+f_y(x, g(x))h) = 0$ so that the (usual) Implicit Function Theorem can be applied to the resulting equation $H(x, h) = 0$ to obtain a function $h(x)$ satisfying $H(x, h(x)) = 0$.

Problem 3 (Michael Artin's Generalization of the Implicit Function Theorem for Analytic Functions). Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let E be an open subset of $\mathbb{F}^{m+n}$ and $\mathbf{f}$ be an analytic map from E to $\mathbb{F}^n$. Let $(\mathbf{a}, \mathbf{b}) \in E$ with $\mathbf{a} \in \mathbb{F}^m$ and $\mathbf{b} \in \mathbb{F}^n$ and $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. For $(\mathbf{x}_0, \mathbf{y}) \in E$ let $\mathbf{T}(\mathbf{x}_0, \mathbf{y})$ be the $\mathbb{F}$-linear transformation of $\mathbb{F}^n$ which is the derivative of the map $\mathbf{y} \mapsto \mathbf{f}(\mathbf{x}_0, \mathbf{y})$ at $\mathbf{y}$ when $\mathbf{x}_0$ is held constant. Let D be an open neighborhood of $\mathbf{a}$ in $\mathbb{F}^m$ and $\mathbf{g}_0 : D \to \mathbb{F}^n$ be an analytic map such that $(\mathbf{x}, \mathbf{g}_0(\mathbf{x})) \in E$ for $\mathbf{x} \in D$. Let $t(\mathbf{x})$ be the determinant of the $\mathbb{F}$-linear transformation $\mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}))$ of $\mathbb{F}^n$. Assume that there exists an analytic map $\mathbf{v} : D \to \mathbb{F}^n$ such that

$$(\natural) \qquad \mathbf{f}(\mathbf{x}, \mathbf{g}_0(\mathbf{x})) = (t(\mathbf{x}))^2\, \mathbf{v}(\mathbf{x})$$

for $\mathbf{x} \in D$. Show that there exist an open neighborhood U of $\mathbf{a}$ in D and an open neighborhood W of $(\mathbf{a}, \mathbf{b})$ in $\mathbb{F}^{m+n}$ and an analytic map $\mathbf{g} : U \to \mathbb{F}^n$ such that for every $\mathbf{x} \in U$ the equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ has a unique solution $\mathbf{y}$ with $(\mathbf{x}, \mathbf{y}) \in W$ and, moreover, the unique solution is precisely $\mathbf{g}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x}) - \mathbf{g}_0(\mathbf{x})$ is of the form $t(\mathbf{x})\, \mathbf{u}(\mathbf{x})$ for some analytic map $\mathbf{u} : U \to \mathbb{F}^n$.

Hint: By Cramer's rule there exists an analytic map $\mathbf{S}(\mathbf{x})$ from D to the space of $\mathbb{F}$-linear transformations of $\mathbb{F}^n$ such that

$$(\dagger) \qquad \mathbf{S}(\mathbf{x})\, \mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x})) = \mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}))\, \mathbf{S}(\mathbf{x}) = t(\mathbf{x})\, \mathbf{I},$$

where $\mathbf{I}$ is the identity transformation of $\mathbb{F}^n$. Note that $\mathbf{S}(\mathbf{x})$ is simply the "adjoint" matrix of the matrix $\mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}))$ constructed from the cofactors of the entries of $\mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}))$. As in Problem 2, we seek a solution $\mathbf{g}(\mathbf{x})$ of the form $\mathbf{g}(\mathbf{x}) = \mathbf{g}_0(\mathbf{x}) + t(\mathbf{x})\, \mathbf{u}(\mathbf{x})$ for the equation

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}) + t(\mathbf{x})\; \mathbf{u}(x)) \\ &= \mathbf{f}(\mathbf{x}, \mathbf{g}_0(\mathbf{x})) + t(\mathbf{x})\; \mathbf{T}(\mathbf{x}, \mathbf{g}_0(\mathbf{x}))\; \mathbf{u}(x) + (t(\mathbf{x}))^2\, \mathbf{Z}(\mathbf{x}, \mathbf{u}(x))\, (\mathbf{u}(x), \mathbf{u}(x)), \end{aligned}$$

where $\mathbf{Z}(\mathbf{x}, \mathbf{u}(x))(\cdot, \cdot)$ is a bilinear form acting on $(\mathbf{u}(x), \mathbf{u}(x))$ like a second-order derivative. Left-multiply the equation by $\mathbf{S}(\mathbf{x})$ and use ($\natural$) and ($\dagger$) to factor out $(t(\mathbf{x}))^2$ and then apply the Implicit Function Theorem to solve for $\mathbf{u}(\mathbf{x})$.

Problem 4 (Map of Fatou-Bieberbach). The Implicit Function Theorem makes it possible to define a function $\mathbf{y} = \mathbf{g}(\mathbf{x})$ by an equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Some important useful functions are defined by using the more general process of introducing another set of given functions $\mathbf{h}_1(\mathbf{x}), \cdots, \mathbf{h}_k(\mathbf{x})$ to define the function $\mathbf{y} = \mathbf{g}(\mathbf{z})$ from the equation

$$\mathbf{f}(\mathbf{x}, (\mathbf{g} \circ \mathbf{h}_1)(\mathbf{x}), \cdots, (\mathbf{g} \circ \mathbf{h}_k)(\mathbf{x})) = \mathbf{0}$$

when the function $\mathbf{f}(\mathbf{x}, \mathbf{y}_1, \cdots, \mathbf{y}_k)$ is given. This homework problem deals with the map of Fatou-Bieberbach which is defined by such a more general process. Let $a, b \in \mathbb{C}$. Consider the unknown $\mathbb{C}$-valued function $\varphi(x, y)$ defined by the equation

$$(\ddagger) \qquad \varphi(4x, 4y) - 4\varphi(x, y) = a\varphi(2x, -2y)^2 + b\varphi(2x, -2y)^5$$

for the variables x, y with values in $\mathbb{C}$. Here the function $\mathbf{g}$ to be defined is the scalar-valued function $\varphi(x, y)$ of the two variables x, y; the additional set of given functions is

$$\mathbf{h}_1(x, y) = (x, y), \quad \mathbf{h}_2(x, y) = (2x, -2y), \quad \mathbf{h}_3(x, y) = (4x, 4y);$$

and the given function $\mathbf{f}$ is the scalar-valued function

$$f(\mathbf{x}, y_1, y_2, y_3) = y_3 - 4y_1 - a{y_2}^2 - b{y_2}^5.$$

(a) Show that there exists a unique analytic solution φ of (‡) the form

$$\varphi(x, y) = x + y + \sum_{p+q\geq 2} c_{pq} x^p y^q$$

with $c_{pq} \in \mathbb{C}$ on some open neighborhood U of the origin $(0,0)$ in $\mathbb{C}^2$.

Hint: Use the Implicit Function Theorem for analytic functions to show that there exists a convergent series

$$\Psi(t) = \sum_{n=2}^{\infty} \alpha_n t^n$$

satisfying the equation

$$\Psi(t) = a\left(t + \Psi(t)\right)^2 + b\left(t + \Psi(t)\right)^5.$$

The function $\psi(x, y) := \Psi(x + y)$ satisfies the equation

$$(\flat) \qquad \psi(2x, 2y) = a\left(2(x + y) + \psi(2x, 2y)\right)^2 + b\left(2(x + y) + \psi(2x, 2y)\right)^5.$$

Use the comparison test as follows to conclude the convergence of the power series solution $\varphi(x, y)$ of (‡) from the convergence of the power series $\Psi(t) = \sum_{n=2}^{\infty} \alpha_n t^n$. The Chain Rule (or equivalently the power series expansion) applied to (‡) yields for every $(p, q) \in \mathbb{N}^2$ with $p + q \geq 2$ a polynomial $\sigma_{p,q}\left(a, b, \{c_{\mu\nu}\}_{\mu+\nu<m+n}\right)$ of the variables a and b and $c_{\mu\nu}$, for $\mu + \nu < m + n$, such that

$$c_{p,q} = \frac{\sigma_{p,q}\left(a, b, \{c_{\mu\nu}\}_{\mu+\nu<p+q}\right)}{4^{p+q} - 4}.$$

Let $\psi(x, y) = \sum_{p+q\geq 2} e_{pq} x^p y^q$. The Chain Rule (or equivalently the power series expansion) applied to $(\flat)$ yields for every $(p, q) \in \mathbb{N}^2$ with $p+q \geq 2$ a polynomial $\tau_{p,q}\left(a, b, \{e_{pq}\}_{\mu+\nu<p+q}\right)$ of the variables a and b and e_{pq}, for $\mu + \nu < p + q$, such that

$$e_{pq} = \frac{\tau_{p,q}\left(a, b, \{e_{\mu\nu}\}_{\mu+\nu<p+q}\right)}{2^{p+q}}.$$

By comparing the equation (‡) and the equation $(\flat)$, verify that the following domination relation between the polynomials $\sigma_{p,q}$ and $\tau_{p,q}$ for all values of the complex variables $\xi_{\mu\nu}$ for $\mu + \nu \leq p + q$.

$$\left|\sigma_{p,q}\left(a, b, \{\xi_{\mu\nu}\}_{\mu+\nu<m+n}\right)\right| \leq \tau_{p,q}\left(|a|, |b|, \{|\xi_{\mu\nu}|\}_{\mu+\nu<m+n}\right).$$

(b) Use the functional equation (‡) to show that the power series solution $\varphi(x, y)$ in Part (a) converges everywhere on $\mathbb{C}^2$.

(c) Set $a = -5$ and $b = 2$. Let $u(x, y) = \varphi(2x, -2y)$ and $v(x, y) = \varphi(4x, 4y)$. Verify that the determinant $J(x, y)$ of the 2×2 matrix

$$\begin{pmatrix} \frac{\partial u(x,y)}{\partial x} & \frac{\partial u(x,y)}{\partial y} \\ \\ \frac{\partial v(x,y)}{\partial x} & \frac{\partial v(x,y)}{\partial y} \end{pmatrix}$$

is equal the constant -4 on all of $\mathbb{C}^2$.

Hint: Compute explicitly $J(0, 0)$ and use the identity $J(x, y) = J\left(\frac{x}{2^n}, \frac{y}{(-2)^n}\right)$ for any $x, y \in \mathbb{C}$ and any $n \in \mathbb{N}$.

(d) Show that the map

$$\Phi : (x, y) \mapsto (u(x, y), v(x, y))$$

maps $\mathbb{C}^2$ injectively into $\mathbb{C}^2$. The map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ is known as *the map of Fatou-Bieberbach.*

Hint: Use the Inverse Mapping Theorem to show first that the map Φ is injective on some open neighborhood of the origin in $\mathbb{C}^2$. Show that if Φ maps (x_1, y_1) and (x_2, y_2) to the same point, then it maps also $\left(\frac{x_1}{2^n}, \frac{y_1}{(-2)^n}\right)$ and $\left(\frac{x_2}{2^n}, \frac{y_2}{(-2)^n}\right)$ to the same point for any $n \in \mathbb{N}$.

(e) Show that the image of the map Φ of Fatou-Bieberbach is disjoint from some open neighborhood of $(1, 1)$ in $\mathbb{C}^2$.

Hint: Show that for some sufficiently small $\varepsilon > 0$ the inequalities $|u(x, y) - 1| < \varepsilon$ and $|v(x, y) - 1| < \varepsilon$ imply the inequalities $\left|u\left(\frac{x}{2^n}, \frac{y}{(-2)^n}\right) - 1\right| < \varepsilon$ and $\left|v\left(\frac{x}{2^n}, \frac{y}{(-2)^n}\right) - 1\right| < \varepsilon$ for any $n \in \mathbb{N}$.