**Math 55b, Assignment #3, February 24, 2006**
**(due March 2, 2006)**

*Notations.* $\mathbb{N}$ = the set of all positive integers. $\mathbb{R}$ = the set of all real numbers. For a subset $E$ of $\mathbb{R}$, $\mu^*(E)$ denotes its Lebesgue outer measure. For a measurable subset $E$ of $\mathbb{R}$, $\mu(E)$ denotes its Lebesgue measure. The symbol $\mathcal{R}(\alpha)$ denotes the set of all functions which are Riemann-Stieltjes integrable with respect to the nondecreasing function $\alpha$.

*Problem 1 (Young's Inequality and Hölder's Inequality).*

(a) *Young's Inequality.* Let $\varphi(x)$ be a continuous and strictly increasing function for $x \geq 0$ with $\phi(0) = 0$. Let $\psi(x)$ be the inverse function of $\varphi(x)$. For $a, b \geq 0$, show that

$$ab \leq \int_0^a \varphi(x)\,dx + \int_0^b \psi(x)\,dx.$$

Equality holds if and only if $b = \varphi(a)$.

*Hint:* Compare the area of the rectangle $[0,a]\times[0,b]$ with the sum of the areas of the two regions $\{0 \leq x \leq a,\ 0 \leq y \leq \varphi(x)\}$ and $\{0 \leq y \leq b,\ 0 \leq x \leq \psi(y)\}$.

For Parts (b) to (e) of this problem, $p$ and $q$ denote positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$ and $-\infty < a < b < \infty$ and $\alpha$ is a nondecreasing function on $[a, b]$.

(b) If $u \geq 0$ and $v \geq 0$, show that $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$. Equality holds if and only if $u^p = v^q$.

*Hint:* Apply Part (a) with appropriate functions $\varphi(x)$ and $\psi(x)$.

(c) If $f$ and $g$ are functions on $[a, b]$ with $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p\,d\alpha = 1 = \int_a^b g^q\,d\alpha,$$

show that

$$\int_a^b fg\,d\alpha \leq 1.$$

(d) *Hölder's Inequality.* If $f$ and $g$ are complex-valued functions on $[a, b]$ which are in $\mathcal{R}(\alpha)$, show that

$$\left|\int_a^b fg\,d\alpha\right| \leq \left(\int_a^b |f|^p\,d\alpha\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\,d\alpha\right)^{\frac{1}{q}}.$$

(e) *Hölder Norm from $L^p$ Norm of the Derivative.* Let $f(x)$ be a complex-valued function on $[a,b]$ such that $|f'(x)|^p$ is Riemann-integrable on $[a,b]$, where $f'(x)$ is the first-order derivative of $f(x)$. Let

$$C = \left(\int_a^b |f'|^p \, dx\right)^{\frac{1}{p}}.$$

For $0 < \gamma < 1$ the Hölder norm $H_\gamma(f)$ of $f$ of order $\gamma$ on $[a,b]$ is defined by

$$\sup_{a \leq x_1 \neq x_2 \leq b} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\gamma}.$$

Show that $H_{\frac{1}{q}}(f) \leq C$.

*Problem 2 (Approximation in $L^p$ Norm of Riemann-Stieltjes Integrable Functions by Continuous Functions).* Let $-\infty < a < b < \infty$. Let $\alpha$ be a nondecreasing function in $[a,b]$ and $f(x)$ be a bounded real-valued function on $[a,b]$ with $f \in \mathbb{R}(\alpha)$. Let $p \in \mathbb{N}$ and $\varepsilon > 0$. Show that there exists a continuous real-valued function $g(x)$ on $[a,b]$ such that

$$\left(\int_a^b |f(x) - g(x)|^p\right)^{\frac{1}{p}} < \varepsilon.$$

*Hint:* Let $P = \{x_0, \cdots, x_n\}$ be a suitable partition of $[a,b]$, define

$$g(t) = \frac{x_j - t}{x_j - x_{j-1}} f(x_{j-1}) + \frac{t - x_{j-1}}{x_j - x_{j-1}} f(x_j)$$

for $x_{j-1} \leq t \leq x_j$.

*Problem 3 (Integration of the Gaussian Distribution Function and the Convergence of the Histogram of the Binomial Coefficient Distribution to the Normal Curve).*

(a) Show that

$$\int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

by using the following description of the Gaussian distribution $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ as the limit of the probability distribution obtained by flipping $n$ coins as $n \to \infty$. Here the meaning of the improper integral is given by

$$\int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \lim_{a \to \infty} \int_{x=-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

(b) A *random variable* $X$ is a variable whose value is generated by a process of chance which assigns a probability $p_j$ for $X$ to assume a value $x_j$. For example, when the random variable $X$ is the face value of a cube-shaped die marked with values from 1 to 6 and which is randomly thrown, the probability for $X$ to assume the value 1 is $\frac{1}{6}$, and the probability for $X$ to assume the value 2 is $\frac{1}{6}$, and so forth. The *characteristic function* $\varphi_X(t)$ for $X$ in this case is defined as the Fourier series

$$\varphi_X(t) = \sum_{k=1}^{6} \frac{1}{6}\, e^{ikt}.$$

In general, given a discrete set $J$, when the probability for the random variable $X$ to assume the value $x_j$ is $p_j$ for $j \in J$, the characteristic function $\varphi_X(t)$ for $X$ is defined as

$$\varphi_X(t) = \sum_{j\in J} p_j\, e^{ix_j t}.$$

When the range of values $x_\tau$ to be assumed by a random variable $X$ is indexed by $\tau \in \mathbb{R}$ (instead of by a discrete set $J$), the characteristic function $\varphi_X(t)$ of $X$ is

$$\varphi_X(t) = \int_{\tau=-\infty}^{\infty} p_\tau\, e^{ix_\tau t}\, d\tau,$$

where $p_\tau$ is the probability *density* for the value $x_\tau$. When the random variables $X$ and $Y$ are *independent*, the characteristic function $\varphi_{X+Y}(t)$ for $X+Y$ is equal to $\varphi_X(t)\varphi_Y(t)$.

Now let $F$ be the random variable of flipping a coin which assigns the value $\frac{1}{2}$ to the head and the value $\frac{-1}{2}$ to the tail. Let $G$ be the random variable whose probability density is given by the normal curve

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

for the value $x$. Show that the limit of the characteristic function for the random variable

$$\frac{nF}{\sqrt{\frac{n}{4}}}$$

is the characteristic function of the random variable $G$ as $n \to \infty$. (Here the expression $\sqrt{\frac{n}{4}}$ in the denominator is the standard variation of the random variable $nF$. The standard variation is the square root of the sum whose terms are the product of the probability times the square of the difference of the value of the random variable and the mean.) This means that the histogram which consists of the $n+1$ rectangles with vertices

$$\left(\frac{k-\frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, 0\right), \quad \left(\frac{k+1-\frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, 0\right),$$

$$\left(\frac{k-\frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, \frac{\sqrt{\frac{n}{4}}\binom{n}{k}}{2^n}\right), \quad \left(\frac{k+1-\frac{n+1}{2}}{\sqrt{\frac{n}{4}}}, \frac{\sqrt{\frac{n}{4}}\binom{n}{k}}{2^n}\right)$$

for $0 \leq k \leq n$ approaches the function

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

as $n \to \infty$ in the above sense of interpretation in terms of the characteristic functions.

For this problem the following Stirling's formula is assumed known.

$$n! = n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}\,(1 + a_n)$$

with $\lim_{n\to\infty} a_n = 0$. The factor $\sqrt{2\pi}$ in the denominator of $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$ comes from applying Stirling's formula to $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ which contains two factorials in the denominator versus only one factorial in the numerator. Because of the factor $\sqrt{\frac{n}{4}}$ in the denominator for the base of each rectangle in the histogram, the sum of the bases of all the rectangles in the histogram is of the order $\sqrt{n}$, which is the reason why the abscissa $x$ occurs in the form of $x^2$ in $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$.

*Problem 4* (*A Non Measurable Set Constructed by Using the Axiom of Choice.*) Let $\sim$ be the equivalence relation for the points in the interval $[-1,1]$ in $\mathbb{R}$ defined as follows. Two points $x, y \in [-1,1]$ are equivalent (in notations $x \sim y$) if and only if $x - y$ is a rational number. Let $E$ be a subset of $[-1,1]$ obtained by selecting precisely one point from each equivalence class. Show that $E$ is not Lebesgue measurable.

*Hint:* Let $\{r_k\}_{k\in\mathbb{N}}$ be the set of all rational numbers in $[-1,1]$. For $k \in \mathbb{N}$ let $A_k \subset \mathbb{R}$ be the right translate of $A$ by $r_k$, that is, $A_k$ is the set of all points $x + r_k$ with $x \in A$. For $k \neq \ell$ the two sets $A_k$ and $A_\ell$ are disjoint. Use $[-1,1] \subset \bigcup_{k\in\mathbb{N}} A_k \subset [-2,2]$ and the fact that the Lebesgue measure is countably additive and translational invariant.

*Problem 5 (Amost-Everywhere Differentiability of Nondecreasing Functions).*

(a) (*Vitali's Covering Argument*) Let $-\infty < a < b < \infty$ and let $E$ be a subset of $(a, b)$ with positive outer Lebesgue measure $\mu^*(E) > 0$. Let $x \mapsto \eta_x > 0$ (for $x \in E$) be an $\mathbb{R}$-valued function on $E$. Let $0 < \alpha < 1$. Show that there exist a finite number of points $x_1, \cdots, x_N$ in $E$ with $x_{j+1} \geq x_j + \eta_{x_j}$ for $1 \leq j < N$ such that

$$\mu^*\left(E \cap \bigcup_{j=1}^{N} (x_j, x_j + \eta_{x_j})\right) \geq \alpha\, \mu^*(E).$$

*Hint:* For $\gamma > 0$ let $E_\gamma$ be the subset of $E$ consisting of all $x \in E$ such that $\eta_x \geq \gamma$. By writing $E$ as the union of $E_{\frac{1}{n}}$ for $n \in \mathbb{N}$ and replacing $E$ by $E_{\frac{1}{n_0}}$ for some sufficiently large $n_0$, we can assume without loss of generality that $\eta_x \geq \frac{1}{n_0}$ for $x \in E$. Inductively choose $x_{\ell+1}$ no less than and very close to the infimum of $E \cap [x_\ell + \eta_{x_\ell}, \infty)$.

(b) Let $f$ be a nondecreasing function on $(a, b)$. Show that there exists a subset $Z$ of $(a, b)$ of measure zero such that the derivative $f'(x)$ of $f(x)$ exists for every point $x$ of $(a, b) - Z$.

*Hint:* Let $g(x)$ (respectively $h(x)$) be the *lim inf* (respectively *lim sup*) of the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for $h > 0$ as $h \to 0$. First show that $g(x) = h(x)$ almost everywhere. Then the almost-everywhere existence of $f'$ follows from using analogous arguments involving any two of the *lim inf* and *lim inf* of the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for $h > 0$ and for $h < 0$ as $h \to 0$. Suppose $r < s$ be any two rational numbers. Let $E$ be the set of points of $(a, b)$ where $g(x) < r < s < h(x)$. Assume $\mu^*(E) > 0$ to derive a contradiction. Let $\varepsilon > 0$. Choose an open subset $G$ of $(a, b)$ containing $E$ such that $\mu(G) < (1 + \varepsilon)\mu^*(E)$. For $x \in E$ choose $\eta_x > 0$ such that

$$\frac{f(x + \eta_x) - f(x)}{\eta_x} < r$$

and $(x, x+\eta_x) \subset G$. Find $x_1, \cdots, x_N$ from Part (a) with $\alpha = \frac{1}{1+\varepsilon}$. Let $G' = \bigcup_{j=1}^{N} \left(x_j, x_j + \eta_{x_j}\right)$ and $E' = E \cap G'$. Repeat the argument with $E$ replaced by $E'$ and $G$ by $G'$ and $\eta_x$ replaced by $\eta'_x$ with

$$\frac{f(x+\eta'_x) - f(x)}{\eta'_x} > s$$

and derive a contradiction by choosing $\varepsilon$ sufficiently small.

(c) Let $f(x)$ be a continuous nondecreasing function on $[a, b]$. Show that $f'(x)$ is Lebesgue integrable and $\int_a^b f'(x) \leq f(b) - f(a)$.

*Hint:* For a decreasing sequence $h_n$ of positive numbers approaching zero, apply Fatou's theorem (page 320 of Rudin's book) to the integral over $[a, \xi]$ of the sequence of nonnegative functions

$$\frac{f(x+h_n) - f(x)}{h_n}$$

as $n \to \infty$ and write

$$\int_a^{\xi} \frac{f(x+h_n) - f(x)}{h_n}\, dx = \frac{1}{h_n} \int_{\xi}^{\xi+h_n} f(x)\, dx - \frac{1}{h_n} \int_a^{a+h_n} f(x)\, dx$$

which approaches $f(\xi) - f(a)$ by the continuity of $f(x)$.

(d) Show that in Part (c) the assumption of the continuity of $f(x)$ is unnecessary. (*Hint:* First prove the statement when $[a, b]$ is replaced by $[a', b'] \subset [a, b]$ with $f(x)$ continuous at both $a'$ and $b'$. Then use Part (b).)