Math 55b, Assignment #1, February 3, 2006
(due February 10, 2006)

Notations. $\mathbb{N}$ = all positive integers.
$\mathbb{R}$ = all real numbers.
$\mathbb{C}$ = all complex numbers.

Problem 1 (Cauchy-Riemann equations). Let $m, n \in \mathbb{N}$. Let U an open subset of $\mathbb{C}^n$ and W an open subset of $\mathbb{C}^m$ and $f : U \to W$ be a map. Let $(z_1, \cdots, z_n)$ with $z_j = x_j + \sqrt{-1}\, y_j$ be the complex coordinates of $\mathbb{C}^n$ which contains U and let $(w_1, \cdots, w_m)$ with $w_j = u_j + \sqrt{-1}\, v_j$ be the complex coordinates of $\mathbb{C}^m$ which contains W so that

$$u_j = u_j\left(x_1, \cdots, x_n, y_1, \cdots, y_n\right), \quad v_j = v_j\left(x_1, \cdots, x_n, y_1, \cdots, y_n\right)$$

for $1 \leq j \leq m$ represent the map f. Let $P_0 \in U$ and $g : \mathbb{C}^n \to \mathbb{C}^m$ be a map which is $\mathbb{R}$-linear. Assume that g approximates f at P_0 to an order higher than the first in the sense that

$$\lim_{P \to P_0} \frac{\left\|(f(P) - f\left(P_0\right)) - g\left(P - P_0\right)\right\|}{\left\|P - P_0\right\|} = 0,$$

where the difference $P - P_0$ of the two points P_0 and P_1 is considered naturally as a vector in $\mathbb{C}^n$ going from P_0 to P_1 and $\|\cdot\|$ means the Euclidean norm in the vector space $\mathbb{C}^n$ or $\mathbb{C}^m$. Show that the map $g : \mathbb{C}^n \to \mathbb{C}^m$ is linear over $\mathbb{C}$ if and only if the following Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$ are satisfied at the point P_0.

Problem 2 (Weierstrass nowhere differentiable function). Let $0 < b < 1$ and a be an *odd* positive integer such that $ab > 1 + \frac{3\pi}{2}$. Let

$$f(x) = \sum_{n=0}^{\infty} b^n \cos\left(a^n \pi x\right).$$

Verify that the continuous function $f(x)$ on $\mathbb{R}$ is nowhere differentiable.

Hint: Fix $x \in \mathbb{R}$. For any $k \in \mathbb{N}$ write $a^k x = n_k + y_k$ uniquely with $-\frac{1}{2} \leq y_k < \frac{1}{2}$ and $n_k \in \mathbb{N}$. Let $h_k = \frac{1-y_k}{a^k}$. Then

$$\left| \sum_{n=1}^{k-1} \frac{b^n \cos\left(a^n \pi \left(x + h_k\right)\right) - b^n \cos\left(a^n \pi x\right)}{h_k} \right| \leq \frac{\pi \left(ab\right)^k}{ab - 1}$$

and the absolute value of the series

$$\sum_{n=k}^{\infty} \frac{b^n \cos\left(a^n \pi \left(x + h_k\right)\right) - b^n \cos\left(a^n \pi x\right)}{h_k}$$

is no less than the absolute value of its first term which is at least $\frac{2}{3} \left(ab\right)^k$.

Problem 3 (Differentiation term-by-term). Let $-\infty < a < b < \infty$ and $f_k : (a, b) \to \mathbb{R}$ for $k \in \mathbb{N}$. Assume that the second-order derivative $f_k{}''(x)$ of f_k at x exists for every $x \in (a, b)$ and $k \in \mathbb{N}$. Let C be a positive number and assume that $\sum_{k \in \mathbb{N}} |f_k{}''(x)| \leq C$ for $x \in (a, b)$. Let $x_0 \in (a, b)$ and assume that the two series $\sum_{k \in \mathbb{N}} f_k\left(x_0\right)$ and $\sum_{k \in \mathbb{N}} f_k{}'\left(x_0\right)$ both converge, where $f_k{}'(x)$ means the first-order derivative of f_k at x. Show that the series $\sum_{k \in \mathbb{N}} f_k(x)$ converges at every $x \in (a, b)$ and can be differentiated term-by-term in the sense that the first-order derivative of the function $\sum_{k \in \mathbb{N}} f_k(x)$ at the point $x \in (a, b)$ is equal to $\sum_{k \in \mathbb{N}} f_k{}'(x)$. Verify that the same conclusion holds when $f_k : (a, b) \to \mathbb{R}$ is replaced by a vector-valued function $f_k : (a, b) \to \mathbb{R}^n$ (where $n \in \mathbb{N}$ is the same for all k) and the absolute value $|\cdot|$ is replaced by the norm $\|\cdot\|$.

Problem 4 (Chain rule for second-order derivatives). Let U and V be finite-dimensional $\mathbb{R}$-vector spaces (not necessarily of the same dimension), endowed with norms $\|\cdot\|$, and D be an open subset of U and G be an open subset of V. Let $f : D \to G$ and $P_0 \in D$. We say that f is twice differentiable at P_0 if there exist an $\mathbb{R}$-linear map $A : U \to V$ and an $\mathbb{R}$-bilinear map $B : U \times U \to V$ such that f is approximated at P_0 by $f\left(P_0\right) + A\left(P - P_0\right) + \frac{1}{2} B\left(P - P_0, P - P_0\right)$ to an order higher than the second in the sense that

$$\lim_{P \to P_0} \frac{\left\| f(P) - \left(f\left(P_0\right) + A\left(P - P_0\right) + \frac{1}{2} B\left(P - P_0, P - P_0\right)\right)\right\|}{\left\|P - P_0\right\|^2} = 0,$$

where $P - P_0$ is naturally regarded as an element of U. We call the $\mathbb{R}$-linear map $A : U \to V$ the first-order derivative of f at P_0 and call the $\mathbb{R}$-bilinear map $B : U \times U \to V$ the second-order derivative of f at P_0. Suppose W is a finite-dimensional $\mathbb{R}$-vector space endowed with a norm $\|\cdot\|$ and H is an open subset of W and $g : G \to H$ is a map. Let $Q_0 = f(P_0)$. Assume that $g : D \to H$ is twice differentiable at Q_0 with first-order derivative $S : V \to W$ at Q_0 and second-order derivative $T : V \times V \to W$ at Q_0. Let $h = g \circ f$ so that h maps the open subset D of U to the open subset H of W. Denote by $X : U \to W$ the $\mathbb{R}$-linear map $S \circ A$ and denote by $Y : U \times U \to W$ the $\mathbb{R}$-bilinear map $S \circ B + T \circ (A \times A)$, where $A \times A : U \times U \to V \times V$ is defined by

$$U \times U \ni (P_1, P_2) \mapsto (A(P_1), A(P_2)) \in V \times V.$$

Show that h is twice differentiable at P_0 with X as its first-order derivative at P_0 and with Y as its second-order derivative at P_0.

Problem 5 (Uniqueness of solutions of ordinary differential equations).

(a) Let $k \in \mathbb{N}$ and $-\infty < \alpha_j < \beta_j < \infty$ for $1 \leq j \leq k$ and $-\infty < a < b < \infty$. Let $\vec{\phi}(x, y_1, \cdots, y_k)$ be an $\mathbb{R}^k$-valued function for $a \leq x \leq b$ and $\alpha_j \leq y_j \leq \beta_j$ for $1 \leq j \leq k$. Let $\vec{c} = (c_1, \cdots, c_k)$ with $\alpha_j \leq c_k \leq \beta_j$ for $1 \leq j \leq k$. Let $\vec{y} = (y_1, \cdots, y_k)$. A *solution* of the initial-value problem

$$\vec{y}' = \vec{\phi}(x, \vec{y}), \quad \vec{y}(a) = \vec{c}$$

is by definition a differentiable vector-valued function $\vec{f}(x) = (f_1(x), \cdots, f_k(x))$ for $a \leq x \leq b$ such that $\vec{f}(a) = \vec{c}$ and $\vec{f}'(x) = \vec{\phi}\left(x, \vec{f}(x)\right)$ for $a \leq x \leq b$. Show that such a problem has at most one solution if there exists $A \in \mathbb{R}$ such that

$$\left\|\vec{\phi}(x, \vec{y}) - \vec{\phi}(x, \vec{z})\right\| \leq A \|\vec{y} - \vec{z}\|$$

for $\vec{y} = (y_1, \cdots, y_k)$ and $\vec{z} = (z_1, \cdots, z_k)$ with $\alpha_j \leq y_j \leq \beta_j$ and $\alpha_j \leq z_j \leq \beta_j$ for $1 \leq j \leq k$.

(b) Specialize Part (a) by considering the system

$$y_j{}' = y_{j+1} \quad (y = 1, \cdots, k-1),$$
$$y_k{}' = f(x) - \sum_{j=1}^{k} g_j(x) y_j,$$

where $f, g_1, \cdots, g_k$ are continuous $\mathbb{R}$-valued functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_2, \quad \cdots, \quad y^{(k-1)}(a) = c_k.$$

Hint: This problem is from #28 and #29 of page 119 of Rudin's book. See the hints given there.