Math 55b, Assignment #5, March 15, 2006
(due March 22, 2006)

Problem 1 (Expansions of Four Trigonometric Functions in Partial Fractions Derived From the Double Angle Formula for the Tangent and Cotangent Functions). Please precisely specify the meaning of convergence for the series in your answer to this problem and rigorously justify your arguments involving the manipulation and the convergence of series.

(a) Verify the following double angle formula for the cotangent function

$$2\cot 2x = \cot x - \tan x$$

from the double angle formulae for the sine and cosine functions.

(b) Iterate n times the following variation

$$\cot \pi x = \frac{1}{2}\left(\cot\frac{\pi x}{2} + \cot\frac{\pi(x-1)}{2}\right),$$
$$\cot \pi x = \frac{1}{2}\left(\cot\frac{\pi x}{2} + \cot\frac{\pi(x+1)}{2}\right)$$

of the formula of Part (a) to derive

$$\pi x \cot \pi x = \frac{\pi x}{2^n}\left(\cot\frac{\pi x}{2^n} + \sum_{k=1}^{2^{n-1}-1}\left(\cot\frac{\pi(x+k)}{2^n} + \cot\frac{\pi(x-k)}{2^n}\right) - \tan\frac{\pi x}{2^n}\right).$$

(c) Use

$$\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{\xi}{2^n} = \frac{1}{\xi}$$

and Part (b) to derive the following expansion in partial fractions of the cotangent function

$$\pi\cot\pi x = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n} + \frac{1}{x-n}\right) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}.$$

(d) Use Part (c) and

$$\pi\cot\frac{\pi x}{2} - 2\pi\cot\pi x = \pi\tan\frac{\pi x}{2}$$

to derive the following expansion in partial fractions of the tangent function

$$\pi \tan \frac{\pi x}{2} = \sum_{n=0}^{\infty} \frac{4x}{(2n+1)^2 - x^2}.$$

(e) Use Part (d) and

$$\cot x + \tan \frac{x}{2} = \frac{1}{\sin x}$$

to derive the following expansion in partial fractions of the cosecant function

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - x^2}.$$

(f) Use Part (e) and

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to derive the following expansion in partial fractions of the secant function

$$\frac{\pi}{\cos \pi x} = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n-1}{(2n-1)^2 - 4x^2}.$$

Problem 2 (Euler's Reflection Formula for the Gamma Function Derived From the Expansion in Partial Fractions of the Cotangent Function). Use the expansion in partial fractions of the cotangent function and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and the comparison of

$$\frac{d}{dx} \log\left(\Gamma(x)\Gamma\left(1-x\right)\right)$$

and

$$\frac{d}{dx} \log \frac{\pi}{\sin \pi x}$$

to verify the following reflection formula of Euler for the Gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Problem 3 (Infinite Product Expansion of the Sine Function). Use the expansion in partial fractions of the cotangent function to show

$$\sin \pi x = \pi x \prod_{n \in \mathbb{N}} \left(1 - \frac{x^2}{n^2}\right)$$

for $x \in \mathbb{R}$.

Problem 4 (Values of Riemann's Zeta Function at Positive Even Integers Derived From the Power Series Expansion and the Expansion in Partial Fractions of the Cotangent Function). The Bernoulli's numbers B_k are defined by the formula

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

They satisfy the recurrent relation

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-1} B_{n-1} = 0,$$

or the symbolic equation

$$(B+1)^n - B^n = 0$$

when B^k is interpreted as B_k. Use

$$\cot x = i\,\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$

and the expansion in partial fractions of the cotangent function to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{B_{2k} (2\pi)^{2k}}{2(2k)!}$$

for any positive integer k.

Problem 5 (Gauss's Multiplication Theorem for the Gamma Function). Prove that

$$\Gamma(x)\,\Gamma\left(x + \frac{1}{n}\right)\Gamma\left(x + \frac{2}{n}\right)\cdots\Gamma\left(x + \frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2} - nx}\,\Gamma(nx)$$

for $x > 0$.

Hint: Use Stirling's formula

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x\, e^{-x} \sqrt{2\pi x}} = 1.$$

and the Euler formula

$$\Gamma(y) = \lim_{m\to\infty} \frac{(m-1)!\, m^y}{y(y+1)\cdots(y+m-1)}$$

for $y = x + \frac{k}{n}$ $(0 \leq k \leq n-1)$ and with m replaced by nm for $y = nx$ to verify that the quotient

$$\frac{n^{nx}\,\Gamma\left(x\right)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right)}{\Gamma\left(nx\right)}$$

is independent of x.

Problem 6 (Evaluation of Definite Integrals by Using the Beta Function and Euler's Reflection Formula for the Gamma Function). Use the formula

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

for the Beta function and Euler's Reflection Formula for the Gamma function to show that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}$$

for $0 < a < 1$.