Math 55b, Assignment #6, March 27, 2006
(due April 6, 2006)

Problem 1 (Fourier Transform of the Gaussian Distribution).

(a) *(Changing the Order of Integration for Finite Intervals of Integration).* Let $-\infty < a < b < \infty$ and $-\infty < \alpha < \beta < \infty$. Let $f(x,y)$ be a real-valued continuous function on $[a,b] \times [\alpha, \beta]$. Prove that $\int_{y=\alpha}^{\beta} f(x,y)dy$ is a continuous function of $x \in [a,b]$ and $\int_{x=a}^{b} f(x,y)dx$ is a continuous function of $y \in [\alpha, \beta]$ and

$$\int_{x=a}^{b} \left(\int_{y=\alpha}^{\beta} f(x,y)dy \right) dx = \int_{y=\alpha}^{\beta} \left(\int_{x=a}^{b} f(x,y)dx \right) dy,$$

where all the integrals are in the sense of Riemann integration.

Hint. Choose partitions

$$a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b,$$
$$\alpha = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = \beta.$$

Let $m_{\mu,\nu}$ be the infimum of $f(x,y)$ on $[x_{\mu-1},\, x_\mu] \times [y_{\nu-1},\, y_\nu]$ and $M_{\mu,\nu}$ be the supremum of $f(x,y)$ on $[x_{\mu-1},\, x_\mu] \times [y_{\nu-1},\, y_\nu]$. Write

$$\int_{x=a}^{b} \left(\int_{y=\alpha}^{\beta} f(x,y)dy \right) dx = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \int_{x=x_{\mu-1}}^{x_\mu} \left(\int_{y=y_{\nu-1}}^{y_\nu} f(x,y)dy \right) dx,$$
$$\int_{y=\alpha}^{\beta} \left(\int_{x=a}^{b} f(x,y)dx \right) dy = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \int_{y=y_{\nu-1}}^{y_\nu} \left(\int_{x=x_{\mu-1}}^{x_\mu} f(x,y)dx \right) dy$$

and compare them with

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} m_{\mu,\nu} \left(x_\mu - x_{\mu-1}\right) \left(y_\nu - y_{\nu-1}\right),$$
$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} M_{\mu,\nu} \left(x_\mu - x_{\mu-1}\right) \left(y_\nu - y_{\nu-1}\right).$$

(b) *(Changing the Order of Integration for Infinite Intervals of Integration).* For a real-valued continuous function $g(x)$ on $\mathbb{R}$ define $\int_{-\infty}^{\infty} g(x)dx$ as the limit L of

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b g(x)dx$$

if such a limit L exists. In other words, given $\varepsilon > 0$ there exists $A > 0$ such that $\left|\int_a^b g(x)dx - L\right| < \varepsilon$ for all $a < -A$ and $b > A$. More generally, the same definition applies to a function $g(x)$ which is Lebesgue integrable on any finite interval in $\mathbb{R}$.

Let $f(x,y)$ be a real-valued continuous function on $\mathbb{R}^2$. Assume that either

$$\int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty}|f(x,y)|\,dy\right)dx < \infty$$

or

$$\int_{y=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty}|f(x,y)|\,dx\right)dy < \infty.$$

Here, for the first integral $\int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty}|f(x,y)|\,dy\right)dx$, because we do not know whether the function $F(x)$ defined by

$$x \mapsto \int_{y=-\infty}^{\infty}|f(x,y)|\,dy$$

is a continuous function of x, we interpret $F(x)$ as a Lebesgue measurable function of x and the first integral $\int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty}|f(x,y)|\,dy\right)dx$ as the Lebesgue integral of $F(x)$ over $(-\infty,\infty)$ with respect to x. For the second integral $\int_{y=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty}|f(x,y)|\,dx\right)dy$ we use a similar interpretation.

Verify that

$$\int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty}f(x,y)dy\right)dx = \int_{y=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty}f(x,y)dx\right)dy,$$

where the interpretation in terms of Lebesgue integration is used for the integration with respect to x on the left-hand side and for the integration with respect to y on the right-hand side.

(c) *(Differentiation of an Integral with Respect to a Parameter in the Integrand).* Let $-\infty < a < b < \infty$ and $g(x,y)$ be a continuous function on $[a,b]\times(-\infty,\infty)$. We say that $\int_{y=\alpha}^{\beta} g(x,y)dy$ is convergent to $\int_{y=-\infty}^{\infty} g(x,y)dy$ uniformly in $x\in[a,b]$ as $\alpha\to-\infty$ and $\beta\to\infty$ if for every $\varepsilon>0$ there exists $A>0$ such that

$$\left|\int_{y=\alpha}^{\beta} g(x,y)dy - \int_{y=-\infty}^{\infty} g(x,y)dy\right| < \varepsilon$$

for all $a < -A$ and $b > A$ and $x\in[a,b]$.

Let $f(x,y)$ be a continuous function on $[a,b]\times(-\infty,\infty)$ such that $\frac{\partial f}{\partial x}$ is also continuous on $[a,b]\times(-\infty,\infty)$. Assume that

(i) for each $x\in[a,b]$ the integral $\int_{y=\alpha}^{\beta} f(x,y)dy$ is convergent to $\int_{y=-\infty}^{\infty} f(x,y)dy$ as $\alpha\to-\infty$ and $\beta\to\infty$ and

(ii) $\int_{y=\alpha}^{\beta} \frac{\partial f(x,y)}{\partial x}\,dy$ is convergent to $\int_{y=-\infty}^{\infty} \frac{\partial f(x,y)}{\partial x}\,dy$ uniformly in $x\in[a,b]$ as $\alpha\to-\infty$ and $\beta\to\infty$.

Verify that

$$\frac{d}{dx}\int_{y=-\infty}^{\infty} f(x,y)\,dy = \int_{y=-\infty}^{\infty} \frac{\partial f(x,y)}{\partial x}\,dy$$

for $x\in(a,b)$.

(d) *(Fourier Transform of the Gaussian Distribution).* The Fourier transform $\hat{f}(\xi)$ of a function $f(x)$ on $(-\infty,\,\infty)$ is defined by

$$\hat{f}(\xi)=\int_{x=-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx.$$

Prove that

$$\int_{x=0}^{\infty} e^{-x^2}\cos\left(2xy\right)\,dx=\frac{1}{2}\sqrt{\pi}\,e^{-y^2}$$

by using $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$ and by showing that the integral

$$I(y):=\int_{x=0}^{\infty} e^{-x^2}\cos\left(2xy\right)dx$$

satisfies the differential equation $\frac{d\,I(y)}{dy}=-2y\,I(y)$. Hence show that the Fourier transform of $e^{-\pi x^2}$ is equal to $e^{-\pi\xi^2}$. More generally, show that for $\delta>0$ the Fourier transform of $x\mapsto e^{-\pi\delta x^2}$ is $\xi\mapsto\frac{1}{\sqrt{\delta}}\,e^{\frac{-\pi x^2}{\delta}}$.

Problem 2 (Approximate Identity). For $\delta>0$ let $K_\delta(x)$ be a real-valued continuous function on $(-\infty,\,\infty)$. We say that the family of functions $K_\delta(x)$ is an *approximate identity* if the following three conditions are satisfied.

(i) *(Nonnegativity)* $K_\delta(x)\geq 0$ for $x\in(-\infty,\,\infty)$ and $\delta>0$.

(ii) *(Unit Integral)* $\int_{-\infty}^{\infty} K_\delta(x)\,dx=1$ for all $\delta>0$.

(iii) *(Integral Outside Any Neighborhood of the Origin Approaching 0)* For any $\eta > 0$ the integral $\int_{|x|\geq\eta} K_\delta(x)\,dx$ approaches 0 as $\delta \to 0$.

For two functions $f(x)$ and $g(x)$ on $\mathbb{R}$ the *convolution* $f * g$ of f and g is a function on $\mathbb{R}$ and is defined by

$$(f * g)(x) = \int_{t=-\infty}^{\infty} f(x-t)g(t)\,dt.$$

For a function $h(x)$ on $\mathbb{R}$ and $p > 0$ let

$$\|h\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |h(x)|^p\right)^{\frac{1}{p}}.$$

For $p = \infty$ let

$$\|h\|_{L^p(\mathbb{R})} = \sup_{x\in\mathbb{R}} |h(x)|.$$

The function $h(x)$ is said to be L^p on $\mathbb{R}$ if $\|h\|_{L^p(\mathbb{R})}$ is finite.

(a) *(Convolution of a Function by Approximate Identity Approaches the Original Function).* Let $f(x)$ be a uniformly continuous function on $\mathbb{R}$ which is also uniformly bounded. Verify that, for any family $K_\delta(x)$ of functions which is an approximate identity, the function $(f * K_\delta)(x)$ converges to $f(x)$ uniformly in $x \in (-\infty, \infty)$ as $\delta \to 0$. In general, for $1 \leq p \leq \infty$, if $f(x)$ is a uniformly continuous function on $\mathbb{R}$ which is L^p on $\mathbb{R}$, then

$$\|(f * K_\delta) - f\|_{L^p(\mathbb{R})} \to 0 \quad \text{as} \quad \delta \to 0.$$

Hint. For the case $p = \infty$, write

$$\begin{aligned}&(f * K_\delta)(x) - f(x)\\ &= \int_{|t|<\eta} K_\delta(t)\,(f(x-t) - f(x))\,dt + \int_{|t|\geq\eta} K_\delta(t)\,(f(x-t) - f(x))\,dt\end{aligned}$$

and estimate the first term on the right-hand side by

$$\left(\int_{|t|<\eta} K_\delta(t)dt\right) \sup_{|t|<\eta} |f(x-t) - f(x)|$$

and the second term on the right-hand side by

$$\left(\int_{|t|\geq\eta} K_\delta(t)dt\right)\left(\sup_{|t|\geq\eta} |f(x-t)| + |f(x)|\right).$$

(b) *(Approximate Identity from the Gaussian Distribution).* Let

$$K_\delta(x) = \frac{1}{\sqrt{\delta}}\, e^{-\frac{\pi x^2}{\delta}} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad \delta > 0.$$

Verify that the family of functions $K_\delta(x)$ is an approximate identity.

Problem 3 (Schwartz Space, Multiplication Formula, Fourier Inversion, and Plancherel Formula). The Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ is defined as consisting of all complex-valued functions $f(x)$ on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |x|^k \left| \frac{d^\ell f(x)}{dx^\ell} \right| < \infty \quad \text{for all nonnegative integers} \quad k \quad \text{and} \quad \ell.$$

(a) *(Schwartz Space Closed Under Fourier Transform).* Verify that for $f \in \mathcal{S}(\mathbb{R})$ the Fourier transform $\hat{f}$ of f also belongs to $\mathcal{S}(\mathbb{R})$.

(b) *(Multiplication Formula).* Use Problem 1(b) to show that

$$\int_{x=-\infty}^{\infty} f(x)\, \hat{g}(x)\, dx = \int_{y=-\infty}^{\infty} \hat{f}(y)\, g(y)\, dy$$

for $f,\, g \in \mathcal{S}(\mathbb{R})$, where $\hat{f}$ is the Fourier transform of f and $\hat{g}$ is the Fourier transform of g.

(c) *(Fourier Inversion).* For $f \in \mathcal{S}(\mathbb{R})$ verify that

$$(\dagger) \qquad f(0) = \int_{\xi=-\infty}^{\infty} \hat{f}(\xi)\, d\xi$$

(where $\hat{f}$ is the Fourier transform of f) by using Part(b) with $g(x) = e^{-\pi\delta x^2}$ and $\hat{g}(x) = \frac{1}{\sqrt{\delta}}\, e^{-\frac{\pi x^2}{\delta}}$ and letting $\delta \to 0+$ and using Problem 2(b). Hence derive the Fourier inversion formula

$$f(x) = \int_{\xi=-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

by using the function $y \mapsto f(x+y)$ in $(\dagger)$ whose value at $y = 0$ is $f(x)$.

(d) *(Plancherel Formula).* For $f \in \mathcal{S}(\mathbb{R})$, prove the Plancherel formula

$$\int_{x=-\infty}^{\infty} |f(x)|^2\, dx = \int_{\xi=-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi$$

(where $\hat{f}$ is the Fourier transform of f) by using the following steps. Define $g(x) = \overline{f(-x)}$ and let $h = f * g$ be the convolution of f and g. Verify that $\hat{h}(\xi) = \left|\hat{f}(\xi)\right|^2$ (where $\hat{h}$ is the Fourier transform of h) and that $h(0) = \int_{x=-\infty}^{\infty} |f(x)|^2 \, dx$. Then apply (†) to the function $h(x)$.

Problem 4 (Definite Integrals Evaluated by Using the Beta Function).

(a) If $\alpha > 0$ and $\beta > 0$ and $x > y$, show that

$$\int_{t=y}^{x} (x-t)^{\alpha-1} (t-y)^{\beta-1} \, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-y)^{\alpha+\beta-1}.$$

Remark. The formula is useful in estimating the relation between the composition of the two convolutions, one by $\frac{1}{|x|^{1-\alpha}}$ and the other by $\frac{1}{|x|^{1-\beta}}$, and the single convolution by $\frac{1}{|x|^{1-\alpha-\beta}}$. In the case of $x \in \mathbb{R}^3$, an appropriate constant times the convolution of a function f by $\frac{1}{|x|}$ over $\mathbb{R}^3$ is equal to the solution of the Laplace equation whose right-hand side is f.

(b) If $\alpha > 0$ and $\beta > 0$ and $x > y$ and either $\lambda < y$ or $\lambda > x$, show that

$$\int_{t=y}^{x} \frac{(x-t)^{\alpha-1} (t-y)^{\beta-1}}{|t-\lambda|^{\alpha+\beta}} \, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{(x-y)^{\alpha+\beta-1}}{|x-\lambda|^{\beta} \, |y-\lambda|^{\alpha}}.$$

Hint. For Part(a) apply an appropriate change of variables to the following relation between the beta function and the gamma function

$$\int_{s=0}^{1} (1-s)^{\alpha-1} s^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

For Part(b) use an appropriate change of variables to reduce it to Part(a).