Math 55b, Assignment #4, March 3, 2006
(due March 9, 2006)

Problem 1 (First Half of the Fundamental Theorem of Calculus in Lebesgue's Integration Theory). Let $-\infty < a < b < \infty$ and $f(x)$ be a Lebesgue integrable function on $[a, b]$.

(a) If $\int_a^x f(t)\,dt = 0$ for $a \leq x \leq b$, show that $f(x) = 0$ almost everywhere. (*Hint:* if the outer measure of the set of points where $f(x)$ is nonzero is positive, construct appropriate simple functions to derive a contradiction.)

(b) Show that $\frac{d}{dt}\int_a^x f(t)\,dt$ exists and is equal to $f(x)$ almost everywhere.

Hint: By writing $f(x)$ as the difference of two nonnegative integrable functions, we can assume without loss of generality that $f(x)$ is nonnegative. By writing $f(x)$ as the limit of $\max(f(x), n)$ as $n \to \infty$, we can assume without loss of generality that $f(x)$ is bounded. Let $F(x) = \int_a^x f(t)\,dt$ and use the fact that the derivative of a nondecreasing function exists almost everywhere. For a decreasing sequence h_n of positive numbers approaching zero, apply Lebesgue's bounded convergence theorem to the integral over $[a, \xi]$ of the sequence of functions

$$\frac{F(x+h_n) - F(x)}{h_n}$$

as $n \to \infty$ and write

$$\int_a^\xi \frac{F(x+h_n) - F(x)}{h_n}\,dx = \frac{1}{h_n}\int_\xi^{\xi+h_n} F(x)\,dx - \frac{1}{h_n}\int_a^{a+h_n} F(x)\,dx$$

which approaches $F(\xi) - F(a)$ by the continuity of $F(x)$ and yields

$$\int_a^\xi F'(x)\,dx = F(\xi) - F(a)$$

from the earlier application of Lebesgue's bounded convergence theorem. Finally apply Part (a) to the identity

$$\int_a^\xi \left(F'(x) - f(x)\right)\,dx = 0$$

for $a \leq \xi \leq b$.

Problem 2 (Second Half of the Fundamental Theorem of Calculus in Lebesgue's Integration Theory).

Definition of Absolute Continuity. A function $f(x)$ on a finite interval I in $\mathbb{R}$ (which may be open, closed or half-open) is *absolutely continuous* if it is continuous at the end-points of I and if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for any open subset G of $\mathbb{R}$ contained in I with $\mu(G)$ less than δ the inequality

$$\sum_{j=1}^{N} |f(\beta_j) - f(\alpha_j)| < \varepsilon$$

holds when G is written as a *disjoint* union $\bigcup_{j=1}^{N}(\alpha_j, \beta_j)$ of open intervals with $N \leq \infty$. Let $-\infty < a < b < \infty$.

(a) If $f(x)$ is absolutely continuous on (a, b) and $f'(x) = 0$ almost everywhere, show that $f(x)$ is constant on (a, b).

Hint: Take any $a < \xi < b$. For any positive number ε and for any point x of (a, ξ) where $f'(x) = 0$, there exists some $\eta_x > 0$ such that $|f(x + \eta_x) - f(x)| < \varepsilon\, \eta_x$. Apply Vitali's covering argument in Problem 5(a) of Homework #3 to construct $\bigcup_{j=1}^{N}\left(x_j, x_j + \eta_{x_j}\right)$ and apply the definition of absolute continuity of $f(x)$ to the open subset $(a, \xi) - \bigcup_{j=1}^{N}\left[x_j, x_j + \eta_{x_j}\right]$ to show that $|f(\xi) - f(a)| < \varepsilon(\xi - a)$.

(b) Show that any absolutely continuous function on $[a, b]$ can be written as the difference of two continuous nondecreasing functions on $[a, b]$.

(c) Let $f(x)$ be a function on $[a, b]$. Then $f(x)$ is absolutely continuous if and only if $f'(x)$ exists almost everywhere and is Lebesgue integrable on $[a, b]$ and $f(x) = \int_a^x f'(t)\, dt$ for $a \leq x \leq b$.

Hint: For the "only if" part, by Problem 5(c) of Homework #3 and Part (b) of this problem $f'(x)$ exists almost everywhere and is Lebesgue integrable on $[a, b]$. Apply Problem 1 (b) and Part (a) of this problem.

Problem 3 (A non-absolutely-continuous yet nondecreasing continuous function). For any point $0 \leq x \leq 1$, write

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

with each $a_k = 0, 1$, or 2. If no a_k is equal to 1, define

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k}{2^{k+1}}.$$

If k_0 is the smallest integer with the property that $a_{k_0} = 1$, define

$$f(x) = \frac{1}{2^{k_0}} + \sum_{k=1}^{k_0-1} \frac{a_k}{2^{k+1}}.$$

Show that $f(x)$ is continuous and nondecreasing on $[0,1]$ and that $f'(x) = 0$ almost anywhere on $[0,1]$ but $\int_0^1 f'(x)\,dx$ is not equal to $f(1) - f(0)$ and, as a consequence, $f(x)$ cannot be absolutely continuous on $[0,1]$. Show directly that $f(x)$ cannot be absolutely continuous by exhibiting a countable union of disjoint open intervals which violates the definition of absolute continuity for $f(x)$ on $[0,1]$.

Problem 4 (Lebesgue Set).

(a) Let $-\infty < a < b < \infty$ and Y be a metric space with metric $d_Y(\cdot,\cdot)$. Let Y' be a *countable* dense subset of Y. Let $f(x,y)$ be an $\mathbb{R}$-valued function on $[a,b] \times Y$. Assume that for any $y_0 \in Y$ any $\varepsilon > 0$ there exists $\delta > 0$ (which may depend on y_0) such that $|f(x,y) - f(x,y_0)| < \varepsilon$ for all $y \in Y$ with $d_Y(y,y_0) < \delta$ and for all $a \le x \le b$. Assume that for every fixed $y_0 \in Y$ the function $f(x,y_0)$ is a Lebesgue integrable function of x on $[a,b]$. Show that there exists a set E of measure zero in $[a,b]$ such that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t,y)\,dt = f(x,y)$$

for all $x \in [a,b] - E$ and $y \in Y$.

(b) Let $-\infty < a < b < \infty$ and $g(x)$ be a Lebesgue integrable function on $[a,b]$. Show that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h |g(x+t) - g(x)|\,dt = 0$$

for almost all $x \in [a,b]$. (The set of all points x where the above limit is equal to zero is known as the *Lebesgue set* of the function $g(x)$ on $[a,b]$.)

Hint: Apply Part (a) to the case $f(x,y) = |g(x) - y|$.

Problem 5 (*A Characterization of An Almost-Everywhere Constant Function.*) Let $-\infty < a < b < \infty$ and $f(x)$ be a Lebesgue integrable function on $[a, b]$. Assume that

$$\lim_{h \to 0} \frac{1}{h} \int_a^b |f(x+h) - f(x)| \, dx = 0$$

(where inside the integral $f(x)$ is interpreted as 0 when x is not in $[a, b]$). Show that there exists a constant c such that $f(x) = c$ almost everywhere on $[a, b]$.

Problem 6 (*Almost-Everywhere Convergence, Strong Convergence in Measure, and Almost Uniform Convergence.*) Let $-\infty < a < b < \infty$. Let f_n ($n \in \mathbb{N}$) and f be measurable functions on $[a, b]$.

Definition. The sequence of functions f_n is said to *converge almost everywhere* to f if there exists a subset Z of measure zero in $[a, b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [a, b] - Z$.

Definition. The sequence of functions f_n is said to *converge almost uniformly* to f if for every $\delta > 0$ there exists a subset Z_δ of measure $< \delta$ in $[a, b]$ such that $f_n(x)$ converges uniformly to $f(x)$ on $[a, b] - Z_\delta$.

Definition. The sequence of functions f_n is said to *converge in measure* to f if for every $\epsilon > 0$ the measure of

$$\left\{ x \in [a, b] \,\middle|\, |f_n(x) - f(x)| \geq \epsilon \right\}$$

goes to zero as $n \to \infty$.

Definition. The sequence of functions f_n is said to *converge strongly in measure* to f if for every $\epsilon > 0$ the measure of

$$\bigcup_{k=n}^{\infty} \left\{ x \in [a, b] \,\middle|\, |f_k(x) - f(x)| \geq \epsilon \right\}$$

goes to zero as $n \to \infty$.

(a) Show that the following three statements are equivalent.

(i) f_n converges almost everywhere to f.

(ii) f_n converges strongly in measure to f.

(iii) f_n converges almost uniformly to f.

Remark. The implication (i) $\Rightarrow$ (iii) is known as Egoroff's Theorem.

Hint: The set $A_{n,\epsilon}$ defined by

$$A_{n,\epsilon} = \bigcup_{k=n}^{\infty} \left\{ x \in [a,b] \,\middle|\, |f_k(x) - f(x)| \geq \epsilon \right\}$$

is precisely the set of points $x \in [a,b]$ such that the statement

$$|f_k(x) - f(x)| < \epsilon \quad \text{for all } k \geq n$$

fails to hold. Use the fact that, for subsets $E_m \subset [a,b]$ with $E_{m+1} \subset E_m$, the limit of the measure of E_m as $m \to \infty$ equals to the measure of $\bigcap_{m=1}^{\infty} E_m$. For (i) $\Rightarrow$ (ii), use $\bigcap_{n=1}^{\infty} A_{n,\varepsilon} \subset Z$. Let ε_ν be a decreasing sequence of positive numbers approaching 0 as $\nu \to \infty$. For (ii) $\Rightarrow$ (iii), choose Z_δ to be $\bigcup_{\nu=1}^{\infty} A_{n_\nu,\varepsilon_\nu}$ for some suitable $n_\nu \in \mathbb{N}$. For (iii) $\Rightarrow$ (i), use $Z \subset \bigcap_{\delta>0} Z_\delta$.

(b) If f_n converges in measure to f, then there exists a subsequence f_{n_ℓ} of f_n such that f_{n_ℓ} $(\ell \in \mathbb{N})$ converges strongly in measure to f as $\ell \to \infty$.

(c) *Counter-Example.* For $1 \leq j \leq n$ and $n \in \mathbb{N}$ let $k = \frac{1}{2}n(n-1) + j$ and let $g_k(x)$ be the characteristic function of $\left[\frac{j-1}{n}, \frac{j}{n}\right)$. Show that the sequence of functions g_k $(k \in \mathbb{N})$ on $[0,1]$ converges in measure to 0, but does not converge to 0 almost everywhere.