Solution of Problems in

MATH 55B QUIZ

March 20, 2006, Monday, 10 - 11 a.m.
Science Center 310

Problem 1 (Second Mean-Value Theorem). Let $a < b$ be two real numbers. Let $f(x)$ and $g(x)$ be two real-valued continuous functions on $[a, b]$. Assume that $g(x)$ is nonincreasing on $[a, b]$. Prove that there exists some $a < \xi < b$ such that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\xi f(x)\,dx + g(b)\int_\xi^b f(x)\,dx.$$

Hint: First reduce to the special case $g(b) = 0$. Let $F(x) = \int_a^x f(t)\,dt$. Let σ be the minimum of $F(x)$ on $[a, b]$ and τ be the maximum of $F(x)$ on $[a, b]$. Use the following integration-by-parts for Riemann-Stieltjes integrals

$$\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) + \int_a^b F(x)\,d\left(-g(x)\right)$$

(where the last term on the right-hand side is a Riemann-Stieltjes integral) to show that

$$\sigma g(a) \leq \int_a^b f(x)g(x)\,dx \leq \tau g(a)$$

(when $g(b) = 0$). Finally apply the intermediate-value theorem for continuous functions.

Solution. *Step One.* By replacing $g(x)$ by $g(x) - g(b)$ we can reduce the general case to the special case where $g(b) = 0$, because if

$$(\%) \qquad \int_a^b f(x)g(x)\,dx = g(a)\int_a^\xi f(x)\,dx$$

holds under the additional assumption that $g(b) = 0$, then

$$\int_a^b f(x)\left(g(x) - g(b)\right)\,dx = \left(g(a) - g(b)\right)\int_a^\xi f(x)\,dx$$

for the general case, which means that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\xi f(x)\,dx + g(b)\int_\xi^b f(x)\,dx.$$

We now can assume without loss of generality that $g(b) = 0$. We can also assume without loss of generality that $g(a) > 0$, otherwise $g(x)$ is identically zero on $[a, b]$ and we can choose ξ to be any number in (a, b).

Step Two. In this step we prove the weaker statement that there exists some $\xi \in [a, b]$ (with the possibility that ξ may be one of the two end-points a, b of the closed interval $[a, b]$) such that (%) holds. Then in Step Three we will prove that there exists some $\xi \in (a, b)$ such that (%) holds. Step Two can be skipped. It is included here just to make the idea of the solution more transparent.

Let $F(x) = \int_a^x f(t)\,dt$. Let σ be the minimum of $F(x)$ on $[a, b]$ and τ be the maximum of $F(x)$ on $[a, b]$. We now use the following integration-by-parts for Riemann-Stieltjes integrals

$$(*) \qquad \int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) + \int_a^b F(x)\,d\left(-g(x)\right),$$

where the last term on the right-hand side is a Riemann-Stieltjes integral. Since $F(a) = \int_a^a f(t)\,dt = 0$ and $g(b) = 0$, it follows from $(*)$ that

$$\int_a^b f(x)g(x)\,dx = \int_a^b F(x)\,d\left(-g(x)\right).$$

In the Riemann-Stieltjes integral $\int_a^b F(x)\,d\left(-g(x)\right)$, when we replace $F(x)$ by the minimum σ of $F(x)$ on $[a, b]$ we get a lower bound $\sigma g(a)$ for the integral, and when we replace $F(x)$ by the maximum τ of $F(x)$ on $[a, b]$ we get an upper bound $\tau g(a)$ for the integral. Thus

$$(\dagger) \qquad \sigma g(a) \le \int_a^b F(x)\,d\left(-g(x)\right) \le \tau g(a).$$

By applying the intermediate-value theorem to the continuous function $F(x)$ on $[a, b]$, we conclude that the value

$$(\$) \qquad \frac{1}{g(a)}\int_a^b f(x)g(x)\,dx = \frac{1}{g(a)}\int_a^b F(x)\,d\left(-g(x)\right)$$

which, according to (†), is no less than the minimum σ of $F(x)$ on $[a,b]$ and no greater than the maximum τ of $F(x)$ on $[a,b]$ is achieved by $F(x)$ at some $\xi \in [a,b]$. Thus for that particular ξ we have (%).

Step Three. We now show that there exists some $\xi \in (a,b)$ such that (%) holds. Since σ is the minimum of $F(x)$ on $[a,b]$, if $\sigma < F(x)$ for every $x \in (a,b)$, then

$$\sigma g(a) < \int_a^b F(x)\, d\left(-g(x)\right),$$

because $g(x)$ is continuous on $[a,b]$. Thus either there exists some $a' \in (a,b)$ such that $F(a') = \sigma$ or

$$\sigma < \frac{1}{g(a)} \int_a^b F(x)\, d\left(-g(x)\right). \tag{$\flat$}$$

In either case there exist $a < a_1 < b_1 < b$ such that

$$\frac{1}{g(a)} \int_a^b F(x)\, d\left(-g(x)\right)$$

is no less than the minimum of $F(x)$ on $[a_1, b_1]$, because when $F(a') = \sigma$ we can choose $a < a_1 \le a' \le b_1 < b$ and when ($\flat$) holds we can choose $a < a_1 < b_1 < b$ such that the difference between σ and the minimum of $F(x)$ on $[a_1, b_1]$ is no more than

$$-\sigma + \frac{1}{g(a)} \int_a^b F(x)\, d\left(-g(x)\right).$$

Likewise, since τ is the maximum of $F(x)$ on $[a,b]$, if $F(x) < \tau$ for every $x \in (a,b)$, then

$$\int_a^b F(x)\, d\left(-g(x)\right) < \tau g(a),$$

because $g(x)$ is continuous on $[a,b]$. Thus either there exists some $b' \in (a,b)$ such that $F(b') = \tau$ or

$$\frac{1}{g(a)} \int_a^b F(x)\, d\left(-g(x)\right) < \tau. \tag{$\sharp$}$$

In either case there exist $a < a_2 < a_1 < b_1 < b_2 < b$ such that

$$\frac{1}{g(a)} \int_a^b F(x)\, d\left(-g(x)\right)$$

is no greater than the maximum of $F(x)$ on $[a_2, b_2]$, because when $F(b') = \tau$ we can choose $a < a_2 \leq b' \leq b_2 < b$ and $a < a_2 < a_1 < b_1 < b_2 < b$ and when ($\sharp$) holds we can choose $a < a_2 < a_1 < b_1 < b_2 < b$ such that the difference between τ and the maximum of $F(x)$ on $[a_2, b_2]$ is no more than

$$\tau - \frac{1}{g(a)} \int_a^b F(x)\, d\,(-g(x))\,.$$

Thus the value (\$) is no less than the minimum of $F(x)$ on $[a_2, b_2]$ and is no greater than the maximum of $F(x)$ on $[a_2, b_2]$. By applying the intermediate-value theorem to the continuous function $F(x)$ on $[a_2, b_2]$, we conclude that the value (\$) is achieved by $F(x)$ at some $\xi \in [a_2, b_2]$. Thus for that particular ξ which is a point of (a, b) we have (%).

Problem 2 (Term-by-Term Differentiation of a Series of Nondecreasing Functions). Let $a < b$ be two real numbers. For every positive integer n let $f_n(x)$ be a real-valued nondecreasing function on $[a, b]$. Assume that for every $a \leq x \leq b$ the series $\sum_{n=1}^{\infty} f_n(x)$ converges to a real number $s(x)$. Prove that $s'(x) = \sum_{n=1}^{\infty} {f_n}'(x)$ almost everywhere for $x \in (a, b)$, where $s'(x)$ means the derivative of the function $s(x)$ and ${f_n}'(x)$ means the derivative of the function $f_n(x)$.

The following two statements can be used in your answer without proofs.

(i) A nondecreasing function is differentiable almost everywhere (Homework Assignment #3, Problem 5(b)).

(ii) (*Vitali's Covering Argument*) Let E be a subset of (a, b) with positive outer Lebesgue measure $\mu^*(E) > 0$. Let $x \mapsto \eta_x > 0$ (for $x \in E$) be a positive-valued function on E. Let $0 < \alpha < 1$. Then there exist a finite number of points $x_1, \cdots, x_N$ in E with $x_{j+1} \geq x_j + \eta_{x_j}$ for $1 \leq j < N$ such that

$$\mu^* \left(E \cap \bigcup_{j=1}^{N} \left(x_j, x_j + \eta_{x_j}\right) \right) \geq \alpha\, \mu^*(E).$$

(Homework Assignment #3, Problem 5(a)).

Hint: Assume the contrary. First show that there exist two rational numbers $\alpha < \beta$ with the property that the set E of all $x \in (a, b)$ such that

$\sum_{n=1}^{\infty} f_n{}'(x) < \alpha$ and $\beta < s'(x)$ has positive outer Lebesgue measure. Let $s_m = \sum_{n=1}^{m} f_n$. For every fixed m and every $x \in E$ define $\eta_{x,m} > 0$ such that

$$\frac{s_m\left(x+\eta_{x,m}\right)-s_m(x)}{\eta_{x,m}} < \alpha$$

and

$$\beta < \frac{s\left(x+\eta_{x,m}\right)-s(x)}{\eta_{x,m}}.$$

For every fixed m, apply Vitali's covering argument to E and $x \mapsto \eta_{x,m}$ to arrive at a conclusion which contradicts $s(a) = \sum_{n=1}^{\infty} f_n(a)$ and $s(b) = \sum_{n=1}^{\infty} f_n(b)$.

Solution. First we make the following simple observation.

($\natural$) Suppose $\varphi(x)$ and $\psi(x)$ are two real-valued functions on $[a,b]$ such that $\varphi(x)-\psi(x)$ is nondecreasing on $[a,b]$. Let $\alpha < \beta$ and $L > 0$. If $a \leq \gamma < \delta < b$ such that

$$\frac{\psi(\delta)-\psi(\gamma)}{\delta-\gamma} < \alpha,$$
$$\beta < \frac{\varphi(\delta)-\varphi(\gamma)}{\delta-\gamma},$$

then

$$(\varphi-\psi)(\delta)-(\varphi-\psi)(\gamma) > (\beta-\alpha)(\delta-\gamma),$$

because we can subtract the first inequality from the second inequality and multiply the result by $\delta-\gamma$. If

$$a \leq \gamma_1 < \delta_1 \leq \gamma_2 < \delta_2 \leq \cdots \leq \gamma_N < \delta_N \leq b$$

and

$$\frac{\psi(\delta_j)-\psi(\gamma_j)}{\delta_j-\gamma_j} < \alpha,$$
$$\beta < \frac{\varphi(\delta_j)-\varphi(\gamma_i)}{\delta_j-\gamma_j}$$

for $1 \leq j \leq N$ and if

$$\sum_{j=1}^{N} (\delta_j-\gamma_j) \geq L,$$

then

$$\sum_{j=1}^{N}\left(\left(\varphi-\psi\right)\left(\delta_j\right)-\left(\varphi-\psi\right)\left(\gamma_j\right)\right)>\left(\beta-\alpha\right)\sum_{j=1}^{N}\left(\delta_j-\gamma_j\right)$$

and as a consequence

$$\left(\varphi-\psi\right)(b)-\left(\varphi-\psi\right)(a)>\left(\beta-\alpha\right)\,L$$

by the nondecreasing property of $\varphi(x)-\psi(x)$.

We now give a solution of the problem. Since every nondecreasing function is differentiable almost everywhere, there exists a set $Z\subset(a,b)$ of measure zero such that both $s'(x)$ and $f_n'(x)$ exist for $x\in(a,b)-Z$ for every n. Let $s_m=\sum_{n=1}^{m}f_n$. Since each $f_n(x)$ is nondecreasing for every n, it follows that $s(x)-s_m(x)$ is nondecreasing and $s'(x)\geq s_m'(x)$ for every m and $x\in(a,b)-Z$ and as a consequence $s'(x)\geq\sum_{n=1}^{\infty}f_n'(x)$ for $x\in(a,b)-Z$.

Assume that it is not true that $s'(x)=\sum_{n=1}^{\infty}f_n'(x)$ almost everywhere. We are going to derive a contradiction. Since the set of all rational numbers is countable and is dense in the set of all real numbers and since a countable union of measure-zero sets is again of measure zero, for some rational numbers $\alpha<\beta$ the the set E of all $x\in(a,b)-Z$ such that $\sum_{n=1}^{\infty}{f_n}'(x)<\alpha$ and $\beta<s'(x)$ has positive outer Lebesgue measure. In particular, for any fixed m we have $\sum_{n=1}^{m}{f_n}'(x)<\alpha$ and $\beta<s'(x)$ for every $x\in E$, because each $f_n(x)$ is nondecreasing.

By the definition of the derivative as the limit of a difference quotient, we conclude that for every fixed m and every $x\in E$ we can find $\eta_{x,m}>0$ such that

$$\frac{s_m\left(x+\eta_{x,m}\right)-s_m(x)}{\eta_{x,m}}<\alpha$$

and

$$\beta<\frac{s\left(x+\eta_{x,m}\right)-s(x)}{\eta_{x,m}}.$$

Fix any $0<\theta<1$. For every fixed m, apply Vitali's covering argument to E to find a finite number of points $x_{1,m},\cdots,x_{N_m,m}$ in E with $x_{m,j+1}\geq x_{j,m}+\eta_{x_{j,m},m}$ for $1\leq j<N_m$ such that

$$\mu^*\left(E\cap\bigcup_{j=1}^{N_m}\left(x_{j,m},\,x_{j,m}+\eta_{x_{j,m},m}\right)\right)\geq\theta\,\mu^*(E).$$

We now apply Observation ($\natural$) to

$$\begin{gathered}\varphi(x) = s(x), \quad \psi(x) = s_m(x), \quad L = \theta\, \mu^*(E), \\ \gamma_j = x_{j,m}, \quad \delta_j = x_{j,m} + \eta_{x_{j,m},m}, \quad N = N_m,\end{gathered}$$

to conclude that

$$(s - s_m)\,(b) - (s - s_m)\,(a) \geq (\beta - \alpha)\,\theta\,\mu^*(E) \tag{$\ddagger$}$$

for every m. Now let $m \to \infty$ in ($\ddagger$) to get the limit 0 on the left-hand side of ($\ddagger$), which gives the contradiction that the positive number $(\beta - \alpha)\,\theta\,\mu^*(E)$ on the right-hand side of ($\ddagger$) is no more than 0.