Solution of Problem 3 of Assignment #1
of Math 55b Assigned on February 3, 2006

Problem 3 (Differentiation term-by-term). Let $-\infty < a < b < \infty$ and $f_k : (a,b) \to \mathbb{R}$ for $k \in \mathbb{N}$. Assume that the second-order derivative ${f_k}''(x)$ of f_k at x exists for every $x \in (a,b)$ and $k \in \mathbb{N}$. Let C be a positive number and assume that $\sum_{k\in\mathbb{N}} |{f_k}''(x)| \leq C$ for $x \in (a,b)$. Let $x_0 \in (a,b)$ and assume that the two series $\sum_{k\in\mathbb{N}} f_k(x_0)$ and $\sum_{k\in\mathbb{N}} {f_k}'(x_0)$ both converge, where ${f_k}'(x)$ means the first-order derivative of f_k at x. Show that the series $\sum_{k\in\mathbb{N}} f_k(x)$ converges at every $x \in (a,b)$ and can be differentiated term-by-term in the sense that the first-order derivative of the function $\sum_{k\in\mathbb{N}} f_k(x)$ at the point $x \in (a,b)$ is equal to $\sum_{k\in\mathbb{N}} {f_k}'(x)$. Verify that the same conclusion holds when $f_k : (a,b) \to \mathbb{R}$ is replaced by a vector-valued function $f_k : (a,b) \to \mathbb{R}^n$ (where $n \in \mathbb{N}$ is the same for all k) and the absolute value $|\cdot|$ is replaced by the norm $\|\cdot\|$.

Solution. First we would like to remark that without loss of generality we can assume that the assumptions hold when (a,b) is replaced by some finite interval $(\tilde{a},\tilde{b})$ with $[a,b] \subset (\tilde{a},\tilde{b})$. The vector-valued version can be done in the same way as the scalar-valued version. So we will only confine ourselves to the scalar-valued version.

Step One *(Reduction to uniform estimates on the tail part of the difference between the difference quotient of the infinite series of functions and the infinite series of the derivatives of the functions).* First we do one simple reduction to get our conclusion after stating some assumptions which will be proved in later steps. We assume the following statements.

(1) Both $\sum_{k=1}^{\infty} f_k(x)$ and $\sum_{k=1}^{\infty} f_k'(x)$ exist for $x \in (a,b)$.

(2) For fixed $x \in (a,b)$, given any $\varepsilon > 0$ there exists $N = N(x,\varepsilon) \in \mathbb{N}$ such that for any $0 < h < b - x$

$$\left| \frac{\sum_{k=n}^{\infty} f_k(x+h) - \sum_{k=n}^{\infty} f_k(x)}{h} - \sum_{k=n}^{\infty} f_k'(x) \right| < \varepsilon \quad \text{for all } \ n \geq N.$$

Then we conclude that

$$\frac{d}{dx} \sum_{k=1}^{\infty} f(x) = \sum_{k=1}^{\infty} f_k'(x) \quad \text{for all } \ x \in (a,b).$$

The reason is as follows. For any given ε and any $x \in (a,b)$,

$$\left|\frac{\sum_{k=1}^{\infty} f_k(x+h) - \sum_{k=1}^{\infty} f_k(x)}{h} - \sum_{k=1}^{\infty} f_k'(x)\right|$$
$$\leq \left|\frac{\sum_{k=1}^{n-1} f_k(x+h) - \sum_{k=1}^{n-1} f_k(x)}{h} - \sum_{k=1}^{n-1} f_k'(x)\right|$$
$$+ \left|\frac{\sum_{k=n}^{\infty} f_k(x+h) - \sum_{k=n}^{\infty} f_k(x)}{h} - \sum_{k=n}^{\infty} f_k'(x)\right| < 2\varepsilon$$

when we first *fix* some $n \geq N$ independent of h from Property (2) so that

$$\left|\frac{\sum_{k=n}^{\infty} f_k(x+h) - \sum_{k=1}^{\infty} f_k(x)}{h} - \sum_{k=n}^{\infty} f_k'(x)\right| < \varepsilon$$

and then choose $h > 0$ sufficiently small so that

$$\left|\frac{\sum_{k=1}^{n-1} f_k(x+h) - \sum_{k=1}^{n-1} f_k(x)}{h} - \sum_{k=1}^{n-1} f_k'(x)\right| < \varepsilon.$$

STEP TWO *(Use of some specially defined entity which is motivated by the integral of the absolute value of the derivative of a function over an interval).* For a function g on an interval $[\alpha, \beta]$ whose derivative is uniformly bounded on $[\alpha, \beta]$ we are going to define a nonnegative number $I(\alpha, \beta, g)$ satisfying the following three properties:

(i) $|g(\beta) - g(\alpha)| \leq I(\alpha, \beta, g)$.

(ii) $I(\alpha, \beta, g) \leq I(a, b, g)$ for $[\alpha, \beta] \subset [a, b]$ if g is defined on $[a, b]$ with uniformly bounded derivative on $[a, b]$.

(iii) $\sum_{k=1}^{\infty} I(a, b, f_k') \leq C(b-a)$.

We will define $I(\alpha, \beta, g)$ in the next step. First we assume that we have $I(\alpha, \beta, g)$ with Properties (i), (ii), and (iii) and prove Property (2) in Step One.

$$\left|\sum_{k=n}^{m} \frac{f_k(x+h) - f_k(x)}{h} - \sum_{k=n}^{m} f_k'(x)\right|$$

$$= \left| \sum_{k=n}^{m} f_k' \left(x_h \right) - \sum_{k=n}^{m} f_k'(x) \right|$$

$$\text{(by the Mean Value Theorem applied to } \sum_{k=n}^{m} f_k \left(x \right) \text{)}$$

$$\leq \sum_{k=n}^{m} \left| f_k' \left(x_h \right) - f_k'(x) \right|$$

$$\leq \sum_{k=n}^{m} I \left(x, x_h, f_k' \right)$$

(by Property (i))

$$\leq \sum_{k=n}^{m} I \left(a, b, f_k' \right)$$

(by Property (ii))

$$\leq \sum_{k=n}^{\infty} I \left(a, b, f_k' \right),$$

where x_h is between x and $x+h$. By letting $m \to \infty$ we get

$$\left| \sum_{k=n}^{\infty} \frac{f_k(x+h) - f_k(x)}{h} - \sum_{k=n}^{\infty} f_k'(x) \right| \leq \sum_{k=n}^{\infty} I \left(a, b, f_k' \right).$$

By

$$\sum_{k=1}^{\infty} I \left(a, b, f_k' \right) \leq C(b-a) < \infty$$

from Property (iii), it follows that for any given $\varepsilon > 0$ we can find N such that

$$\sum_{k=n}^{\infty} I \left(a, b, f_k' \right) < \varepsilon \quad \text{for} \quad n \geq N.$$

Thus we have Property (2) of Step One. For Property (1) see the remark at the end of Step Four.

Step Three *(A simple observation)* To prepare for the definition of $I(\alpha, \beta, g)$, we make the following simple observation. Given g_k $(1 \leq k \leq n)$ continuous on $[\alpha,\beta]$ and differentiable on (α,β) with $\sum_{k=1}^{n} |g_k'(\gamma)| \le C < \infty$ for all $\gamma \in (\alpha,\beta)$, then

$$\sum_{k=1}^{n} |g_k(\alpha) - g_k(\beta)| \le C(\beta-\alpha).$$

The verification is as follows. Choose ϵ_k for $1 \le k \le n$ such that $|\epsilon_k| = 1$ and

$$\epsilon_k \left(g_k(\alpha) - g_k(\beta)\right) = |g_k(\alpha) - g_k(\beta)| \quad \text{for} \quad 1 \le k \le n.$$

By the Mean Value Theorem applied to the function $\sum_{k=1}^{n} \epsilon_k\, g_k(x)$ we conclude that

$$\sum_{k=1}^{n} |g_k(\alpha) - g_k(\beta)| = \sum_{k=1}^{n} \epsilon_k \left(g_k(\alpha) - g_k(\beta)\right)$$

$$= \sum_{k=1}^{n} \epsilon_k\, g_k'(\gamma)(\beta-\alpha) \le \sum_{k=1}^{n} |g_k'(\gamma)|\,(\beta-\alpha) \le C(\beta-\alpha).$$

Step Four $\big($*Definition of* $I(\alpha,\beta,g)$ $\big)$. For any $\nu \in \mathbb{N}$ let $\left\{x_0^{(\nu)}, x_1^{(\nu)}, \cdots, x_{\ell_\nu}^{(\nu)}\right\}$ the set of *all* rational numbers of the form $\frac{p}{\nu}$ with $p \in \mathbb{Z}$ such that

$$\alpha \le x_0^{(\nu)} < x_1^{(\nu)} < \cdots < x_{\ell_\nu}^{(\nu)} \le \beta.$$

Note that we do not require $\alpha = x_0^{(\nu)}$ and do not require $x_{\ell_\nu}^{(\nu)} = \beta$. We require only that $\alpha \le x_0^{(\nu)}$ and $x_{\ell_\nu}^{(\nu)} \le \beta$. We now define

$$I\left(\alpha,\beta,g\right) = \liminf_{\nu\to\infty} \sum_{j=1}^{\ell_\nu} \left| g\left(x_j^{(\nu)}\right) - g\left(x_{j-1}^{(\nu)}\right)\right|.$$

For the verification of Property (iii) we use

$$\liminf_{\nu\to\infty} A_\nu + \liminf_{\nu\to\infty} B_\nu \le \liminf_{\nu\to\infty} \left(A_\nu + B_\nu\right)$$

and the technique of Step Three (using ϵ_k). Property (ii) follows from the triangle inequality and the technique of Step Three (using ϵ_k) and the fact that we require only that $\alpha \le x_0^{(\nu)}$ and $x_{\ell_\nu}^{(\nu)} \le \beta$. Property (i) follows from

$$|g(\beta) - g(\alpha)| \le \left|g(x_0^{(\nu)}) - g(\alpha)\right| + \sum_{j=1}^{\ell_\nu} \left| g\left(x_j^{(\nu)}\right) - g\left(x_{j-1}^{(\nu)}\right)\right| + \left|g(\beta) - g(x_{\ell_\nu}^{(\nu)})\right|$$

and $\left|x_0^{(\nu)} - \alpha\right| \le \frac{1}{\nu}$ and $\left|\beta - x_{\ell_\nu}^{(\nu)}\right| \le \frac{1}{\nu}$ and the Mean Value Theorem and the technique of Step Three (using ϵ_k).

Finally we observe that the the technique of Step Three (using ϵ_k) implies

$$\sum_{k=1}^{\infty} |f_k'(x)| \leq \sum_{k=1}^{\infty} |f_k'(x_0)| \leq C\, |x - x_0|\,,$$

giving the convergence of $\sum_{k=1}^{\infty} f_k'(x)$ for $x \in (a,b)$. Similarly we get the convergence of $\sum_{k=1}^{\infty} f_k(x)$ for $x \in (a,b)$.

Remarks.

Remark on the Motivation for the Techniques of the Proof. The proof is motivated by the use of integrals in the sense of Lebesgue, but the proof itself does not use such integrals. The introduction of $I(\alpha, \beta, g)$ is motivated by $\int_\alpha^\beta |g'|$. If we define $I(\alpha, \beta, g)$ by $\int_\alpha^\beta |g'|$, then Properties (i), (ii), (iii) hold. For Property (iii) we need

$$\sum_{k=1}^{\infty} \int_a^b |f_k''| = \int_a^b \sum_{k=1}^{\infty} |f_k''|$$

which follows from Theorem 11.3 on p.320 of Rudin's book and the fact that $f_k''(x)$ as the limit

$$\lim_{n \to \infty} \frac{f_k'\left(x + \frac{1}{n}\right) - f_k'(x)}{\frac{1}{n}}$$

is measurable by Corollary (b) on p.312 of Rudin's book because of the continuity of $f_k'(x)$ due to the existence of $f_k''(x)$ and hence the measurability of $f_k'(x)$.

Remark on the Purpose of the Problem. The purpose of the problem is to provide an opportunity to think about the concept of integrals from the first principle and in terms of the need for estimates and the commutation of taking limits. It is for preparation for the presentation of Riemann-Stieltjes and Lebesgue integrals in the course. No hints for any specific directions are given in the problem in order to shift the focus of the problem from filling in the details in the hints to exploring different possible ways of looking at the questions of the definitions of integrals.