

PROPOSITION 21 and its use in proving PROPOSITION 33.

According to Netz, **Proposition 21** is a “strange” subject matter, a complete break with the earlier part of *On the Sphere and Cylinder I* and with the *Elements*:

“We suddenly move to an unexpected, unconnected interlude, once again signaling a major transition in the work.” (Netz, p. 115)

That is, **Proposition 21** had no substantive or methodological link to the immediately preceding propositions 7-20 which dealt with the curved surfaces of the cone and the cylinder.

Proposition 21 deals with a planar figure (diagram) of a **4n-sided polygon inscribed in a circle**.

Netz also notes that after **Proposition 23** (which introduces objects constructed by **rotation** of a 4n-sided polygon inscribed in a great circle of a sphere – i.e., of the planar figure of Proposition 21 treated as a great circle of a sphere – and which argues via Postulate IV that the surface of the **inscribed** object obtained by rotation is **less than** the surface of the sphere),

“we move into a sequence of propositions that are felt as a sequence, leading on to the main claim of the treatise.” (Netz, p. 130)

We find that **Proposition 21** is essential for two key propositions (Props. 25 and 30) used in the proof of one of the two “main” propositions of the work, namely, Proposition 33 giving the surface area of the sphere.

I will use the statements of the propositions as given in Dijksterhuis.

Proposition 25:

“The surface of the solid *inscribed* in the sphere, which is contained by the conical surfaces, is *less than four times* the greatest circle of the sphere.”

Proposition 30:

“The surface of the solid *circumscribed* about the sphere is *greater than four times* the greatest circle of the sphere.”

Proposition 33: “The surface of any sphere is *equal to four times* that of the greatest circle.”

Also key to Archimedes’ proofs, as Dijksterhuis observed, is that:

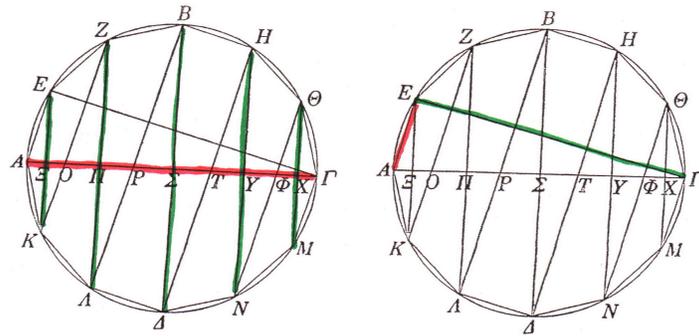
“To denote **area**, the Greek geometer had to introduce a **plane figure whose area was equal to that of the figure under construction.**” Dijksterhuis, p. 155.

In **Proposition 24**, Archimedes establishes the requisite plane figure for the surface area of solid objects inscribed in a sphere in terms of the diagram of Proposition 21.

Hence, by means of **Proposition 21**, Archimedes establishes geometric relationships within the planar diagram (of a 4n-sided polygon inscribed in a circle) which he will use to establish the requisite “**plane figure**” whose area is equal to the surface area of the inscribed [alternatively, circumscribed] solid obtained by revolution of such a polygon and, hence, the “**plane figure**” whose area is equal to the surface area of the sphere itself.

PROOF OF PROPOSITION 21.

Proposition 21: “If in a circle a polygon be inscribed with an even number of equal sides and straight lines be drawn which join the angular points of the polygon so that they are parallel to any one of [the straight lines] which subtend two sides of the polygon, all the connecting lines [together] have to the diameter of the circle the same ratio as the straight line subtending half of the sides but one has to the side of the polygon.” [Figure 68 reproduced twice below.]



[Note: “any one” implies that the choice of EK was arbitrary (Netz, p.113).]
 [As Netz (p.115) claims, the proposition “practically does not argue. Its arguments are implicit” – e.g., the lines are parallel because the alternate interior angles $\angle AEK$ and $\angle EKZ$ (formed by the transversal EK crossing the two lines AE and KZ) measure equal chords and hence are equal. (Euclid III.27, 28; I.27)]

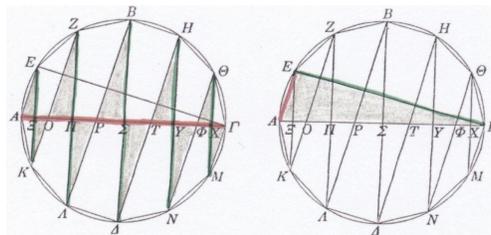
[Note: In the figure for Prop. 21 in Netz (based on Codex A and the Palimpsest), “A new, striking diagrammatic practice appears for the first time: the sides of the polygon are represented by curved, concave lines” (p.114-115). (I use Netz’s figure for Prop. 25 which likewise has the curved lines.)]

Must prove:

$$\frac{EK + ZA + B\Delta + \dots + \Theta M}{A\Gamma} = \frac{E\Gamma}{AE}$$

Proof:

By similarity of triangles (Euclid, see Netz, p.114, notes 216-219),
 [here, by similarity of the shaded triangles in the **left diagram**],



$$\frac{E\Xi}{EA} = \frac{K\Xi}{EO} = \dots = \frac{MX}{X\Gamma}$$

[long green leg of a given “right triangle”/short red leg of that “right triangle”]

so

$$\frac{EΞ + KΞ + \dots + MΞ}{ΞA + ΞO + \dots + XΓ} = \frac{EΞ}{ΞA} \quad \text{[by summing the antecedents and consequents, respectively]}$$

which simplifies to

$$\frac{EK + ZΛ + \dots + ΘM}{AΓ} = \frac{EΞ}{ΞA} \quad \text{[since consequents (red) sum to the diameter]}$$

Also, by similarity of triangles [here, by similarity of the far left small shaded triangle $EΞA$ (in **left diagram**) and the large shaded triangle $ΓEA$ (in **right diagram**) – since in the “right triangle” $ΓEA$ of the **right diagram**, the line $EΞ$ is a perpendicular drawn from the vertex of the right angle at E to the hypotenuse (diameter $AΓ$)],

$$\frac{EΞ}{ΞA} = \frac{EΓ}{AE} \quad \text{[long green side of the “right triangle”/short red side of that “right triangle”]}$$

Thus,

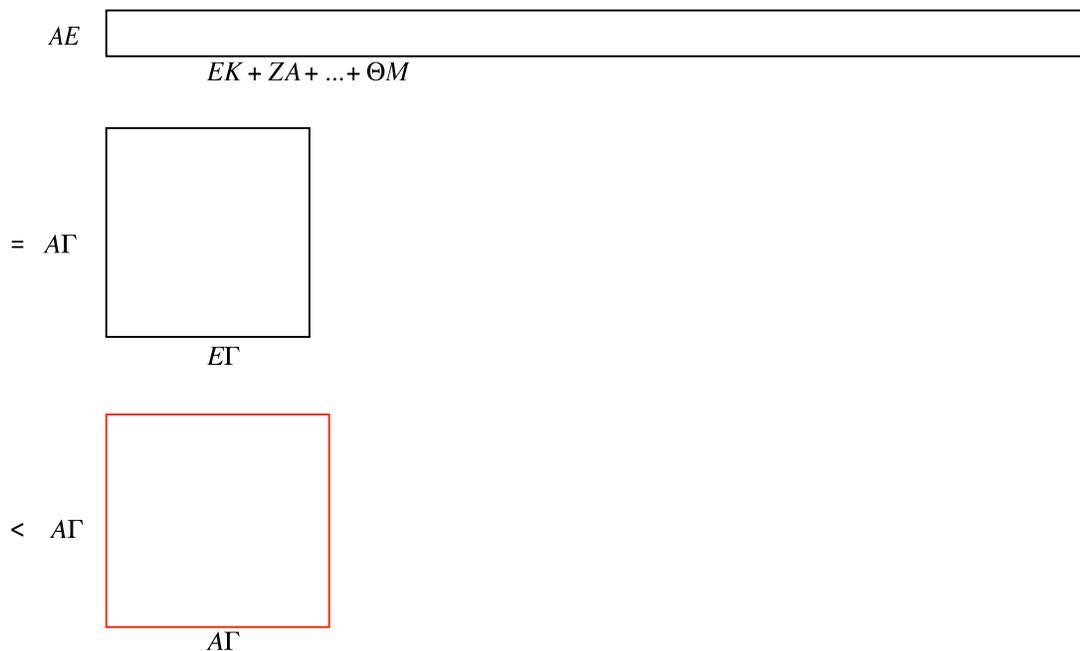
$$\frac{EK + ZΛ + \dots + ΘM}{AΓ} = \frac{EΓ}{AE}$$

Proof of Proposition 21 completed.

Observe that, by the operations of proportions, the equal products form “equal rectangles” (area),

$$AE \cdot (EK + ZΛ + \dots + ΘM) = AΓ \cdot EΓ, \text{ which area is } < (AΓ)^2 \text{ since } EΓ < AΓ,$$

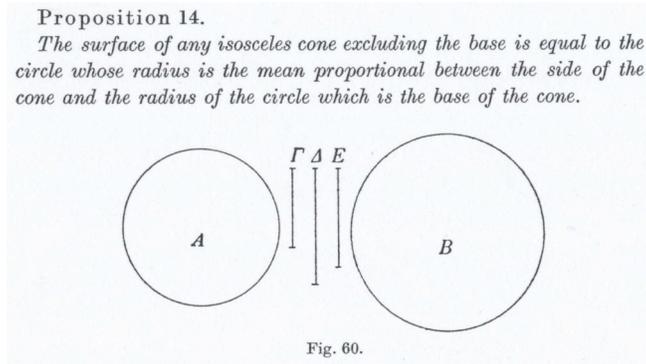
where $AΓ$ is equal to the diameter (i.e., is equal to $2R$), so that the left terms are $< 4R^2$.



I will now show how Archimedes used Proposition 21 in Proposition 25 (and Proposition 30) to transform the result of Proposition 24.

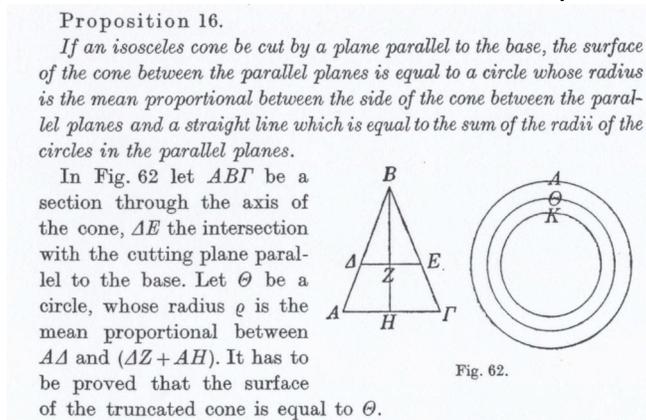
As Dijksterhuis explains (p. 170): In Proposition 24, an expression for “the surface area of the solid obtained by rotation of an inscribed $4k$ -sided polygon” – that is, $E(I_n)$ -- is obtained by a process of summation of surface areas of segments of the sphere, with the aid of:

- **Proposition 14 (area of curved surface of the cone)**



That is, the lateral surface of a cone is equal to a circle the square of whose radius is equal to the rectangle formed by the side of the cone and the radius of the base of the cone.

- **Proposition 16 (area of the curved surface of a truncated cone).**



Proposition 16 establishes the relation between the lateral surface area of a truncated cone and a planar figure (circle) whose radius is the mean proportional between “the side of an inscribed (or circumscribed) polygon” and “the average of the diameters of the circles bounding the truncated cone.”

The sphere can be “sliced” at each of the green parallel lines in Figure 68 (left), creating successive truncated cones of different radii (pictured in Figure 70, associated with Proposition 24). This relationship invites the use of Proposition 21.

In Proposition 25 (and 30), Archimedes uses the “planimetric lemma” in Proposition 21 to “reduce” that expression for $E(I_n)$ found in Proposition 24 to be strictly less than the surface area of the sphere.

Proposition 24:

The surface of the solid inscribed in the sphere is equal to a circle the square on whose radius is equal to the rectangle [upper rectangle on page 2] contained by the side of the (revolving) figure and a straight line equal to [the sum of] all the connecting lines which are parallel to the straight line subtending two sides of the polygon.

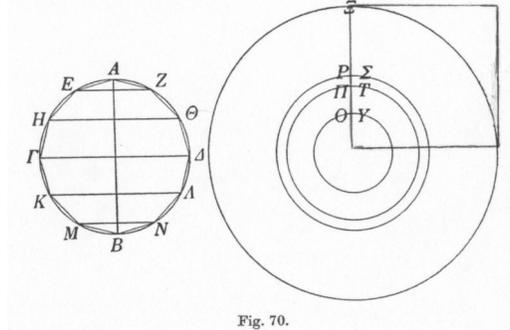


Fig. 70.

In Figure 70, let $AZ...E$ be the polygon I_n revolving about AB . If Ξ be the circle referred to in the proposition, it has to be proved that $E(I_n) = \Xi$. [Note: I inserted the square on the radius of circle Ξ .]

Proof:

Construct circles O, Π, P, Σ, T, Y such that the squares described on their radii as sides are successively equal to the rectangles

$$AE \cdot \frac{1}{2}EZ, \quad AE \cdot \frac{EZ + H\Theta}{2}, \quad \dots, \quad AE \cdot \frac{1}{2}MN,$$

[e.g., the radius of circle O is equal to $\sqrt{AE \cdot \frac{1}{2}EZ}$, the radius of circle Π is equal to $\sqrt{AE \cdot \frac{EZ + H\Theta}{2}}$, etc.]

then by **Proposition 14** and **Proposition 16** these [constructed] circles represent the surfaces described successively by AE, EH, \dots, MB [as side of the respective truncated cone].

The sum of these squares is equal (by factoring out AE) to the rectangle [as depicted on page 2]

$$AE \cdot (EZ + H\Theta + \dots + MN)$$

and consequently to the square described on the radius of Ξ as side.

From this it follows that

$$\Xi = O + \Pi + P + \Sigma + T + Y$$

or

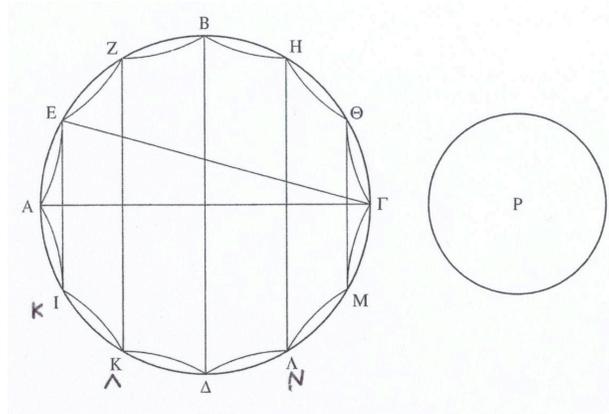
$$E(I_n) = \Xi.$$

Proposition 25:

The surface of the solid inscribed in the sphere, which is contained by the conical surfaces, is less than four times the greatest circle of the sphere.

Proof:

In Figure 68 of Proposition 21 [and in Netz' version¹ of the figure for Proposition 25 shown here, with three vertices re-labeled to conform to that of the figure for Proposition 21],



let P be a circle [which Netz drew much smaller than to scale] the square on whose radius is equal to

$$AE \cdot (EK + Z\Lambda + \dots + \Theta M), \quad [\text{again, as depicted on pages 2 and 5}]$$

and K a circle on $A\Gamma$ as diameter [i.e., K is the area of the great circle of the sphere],

then by **Proposition 24**

$$P = E(I_n).$$

By **Proposition 21**, however, we have

$$\frac{EK + Z\Lambda + \dots + \Theta M}{A\Gamma} = \frac{E\Gamma}{AE}$$

and using Euclid VI.16, we have

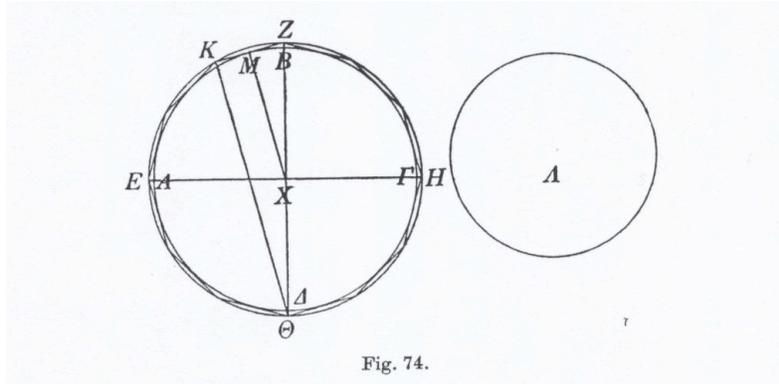
$$AE \cdot (EK + Z\Lambda + \dots + \Theta M) = A\Gamma \cdot E\Gamma < (A\Gamma)^2 \quad [\text{since } E\Gamma < A\Gamma]$$

therefore $P < 4 \cdot K$.

¹ Note the use of curved, concave lines for the sides of the polygon (as explained on p. 2).

Proposition 30:

The surface of the solid circumscribed about the sphere is greater than four times the greatest circle of the sphere.



Observe: XM is radius of inner circle (hence of the great circle of the sphere), and $B\Delta$ is its diameter.
 Observe: The regular polygon is circumscribed about the sphere, and then a larger sphere of diameter $Z\Theta$ [with the same center as the great circle of the smaller sphere] is circumscribed about that polygon (by Proposition 28), so that **Proposition 24** applies to this circumscribed polygon treated as inscribed in a larger sphere (**Proposition 29**).

Proof:

In Fig. 74, let the surface under consideration [the surface of the circumscribed solid] be equal to that of the circle Δ .

In entirely the same way as in **Proposition 25** [i.e., using the result of Proposition 24], it then follows from **Propositions 21** and **29** that the square on the radius of Δ is equal to $Z\Theta \cdot \Theta K$.

Therefore the radius of Δ is greater than ΘK .

But $\Theta K = B\Delta$ [the diameter of the inner circle, and hence of the sphere] since $\Theta K = 2XM$.

Thence the radius of Δ is greater than the diameter of the sphere.

Thus

$$\Delta > 4 \cdot (\text{surface of a great circle of the sphere}).$$

Netz notes (p. 147-148) the mathematical elegance of the proof of Proposition 33 in contrast to the “seemingly haphazard structure” of the earlier propositions.

By building up in advance the multiply interconnected propositions required to establish the seven Propositions called upon in the proof of Prop. 33 (namely, Props. 2 and 3, 25 and 30, 23 and 28, and 32), Archimedes gives a concise and straight-forward argument (that may appear deceptively simple).