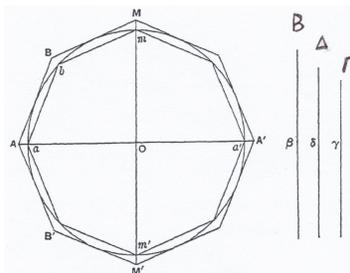


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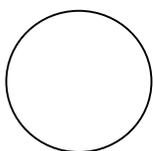
Explanation of Proof of Proposition 33



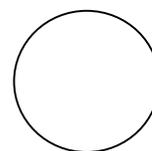
GIVEN: sphere of radius R. **In the proof**, the “assumption” (Case I or II) compares the areas of circles “A” and “E” and then, given the assumption, the “argument” compares “A” (via Prop. 25 or 30) and “E” (via Prop. 28 or 23) with circles representing the surface areas of the circumscribed and inscribed solids:



The “great circle” of the sphere of radius R



Circle “A” of radius 2R



The “goal” circle “E” (of unknown radius) equal to “surface area of sphere”

Area of great circle = πR^2

A = area of “A” = $\pi(2R)^2 = 4\pi R^2$

E = Area of “E” is unknown

Case I. Assume **A < E** (i.e., that surface of the sphere is larger than circle “A”)

- **construct** two line segments *B* and Γ ($B > \Gamma$) such that $\frac{B}{\Gamma} < \frac{E}{A}$ (Prop. 2) and the “mean proportional” line segment Δ such that $\frac{B}{\Delta} = \frac{\Delta}{\Gamma}$.
- **circumscribe** about the sphere, and **inscribe** within it, solids obtained by rotating similar regular polygons of sides Z_n and z_n respectively.

We can find *n* such that

$\frac{\text{(side } Z_n \text{ of outer polygon)}}{\text{(side } z_n \text{ of inner polygon)}} < \frac{\text{(greater line segment } B)}{\text{(mean proportional line segment } \Delta)}}$ (Prop. 3), and

$\frac{\text{(surface area } E(C_n) \text{ of circumscribed solid)}}{\text{(surface area } E(I_n) \text{ of inscribed solid)}} = \frac{\text{(side } Z_n \text{ of outer polygon)}^2}{\text{(side } z_n \text{ of inner polygon)}^2}$ (Prop. 32)

This gives the chain of ratios,

$\frac{E(C_n)}{E(I_n)} = \frac{Z_n^2}{z_n^2} < \frac{B^2}{\Delta^2} = \frac{B^2}{B \cdot \Gamma} = \frac{B}{\Gamma}$ (by, respectively, Prop. 32, Prop. 3, and “mean proportional”).

Since (by construction) we have established

$$\frac{\text{(greater line segment } B)}{\text{(smaller line segment } \Gamma)} < \frac{\text{postulated surface area } E \text{ of sphere}}{\text{area } A} \quad (\text{Prop. 2})$$

we also have

$$\frac{B}{\Gamma} < \frac{E}{A} \quad (\text{by the construction and Prop. 2}).$$

Therefore (completing the chain),

$$\frac{E(C_n)}{E(I_n)} < \frac{E}{A}. \quad (\text{this inequality is identified as } (\alpha))$$

This inequality that completes the chain can be rewritten as:

$$\frac{E(C_n)}{E} < \frac{E(I_n)}{A}. \quad (\text{rewritten inequality } (\alpha))$$

We also know from Prop. 25 that

$$\text{(surface area } E(I_n) \text{ of inscribed solid)} < \text{(4 times the area of the great circle)} \quad (\text{Prop. 25), i.e.,}$$

$$E(I_n) < A,$$

so that the “rewritten inequality (α)” is less than 1, requiring that

$$E(C_n) < E.$$

But we know further from Prop. 28 that

$$\text{(surface area } E(C_n) \text{ of circumscribed solid)} > \text{(surface area } E \text{ of sphere)} \quad (\text{Prop. 28), i.e.,}$$

$$E(C_n) > E, \text{ which gives a ratio greater than 1.}$$

Thus the inequalities obtained from Propositions 25 and 28 lead to a contradiction with the “rewritten inequality (α)” obtained from Propositions 2, 3, and 32:

$$1 < \frac{E(C_n)}{E} < \frac{E(I_n)}{A} < 1.$$

Hence, it cannot be the case that $A < E$.

So it must be the case that $A \geq E$.

Case II. Assume $A > E$ (i.e., that the surface of the sphere is smaller than circle A)

The proof is exactly the same as in Case I except that A and E are reversed, so that the chain of inequalities leads to the inequality

$$\frac{E(C_n)}{E(I_n)} < \frac{A}{E} \quad \text{[this inequality is identified as } (\beta)\text{]}$$

which can be rewritten as:

$$\frac{E(C_n)}{A} < \frac{E(I_n)}{E} \quad \text{[rewritten inequality } (\beta)\text{]}$$

We also know from Prop. 30 that

(surface $E(C_n)$ of **circumscribed** solid) $>$ **(4 times the area of the great circle)** (Prop. 30), i.e.,

$$E(C_n) > A,$$

so that the “rewritten inequality (β) ” is greater than 1, requiring that

$$E(I_n) > E.$$

But we know further from Prop. 23 that

(surface area $E(I_n)$ of **inscribed** solid) $<$ **(surface area E of sphere)** (Prop. 23), i.e.,

$$E(I_n) < E, \text{ which gives a ratio less than 1.}$$

Thus the inequalities obtained from Propositions 30 and 23 lead to a contradiction with the “rewritten inequality (β) ” obtained from Propositions 2, 3, and 32:

$$1 < \frac{E(C_n)}{A} < \frac{E(I_n)}{E} < 1.$$

Hence, it cannot be the case that $A > E$.

Thus, since it is not the case that $A < E$ (Case I), nor the case that $A > E$ (Case II),

it must be true that $A = E$,

that is, that the surface of the sphere is **equal to** 4 times the area of its greatest circle.