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STRUCTURE of ("dependencies in" [Netz, p.21]) the Propositions required to prove
PROPOSITION 33: The surface of any sphere is equal to four times that of its greatest circle.
(proof as presented by Dijksterhuis; my annotations noted by brackets and/or by indented smaller font)

Let A be [the area of] a circle whose radius is equal to the diameter [2R] of the sphere.
[Hence A is equal to 4 times the area of the greatest circle of the sphere.]

Let E be the surface area of the sphere.
Must prove: $E = A$.

If this is not true, then either (I) $A < E$ or (II) $A > E$.
[proof by contradiction ("proof by a double absurdity, manipulating proportions" (Netz, p. 147)]

> Let $E(C_n)$ = [surface area of rotated "n-gon circumscribed around sphere," where n is a multiple of 4]
> Let $E(I_n)$ = [surface area of rotated "n-gon inscribed in sphere," where n is a multiple of 4]

Case I. Assume $A < E$.

Construct two line segments B and Γ ($B > \Gamma$) such that
by Proposition 2

> ["Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the lesser a ratio less than the greater magnitude has to the lesser."]

$$\frac{\mathrm{B}}{\Gamma} < \frac{\mathrm{E}}{\mathrm{A}}$$

and **a line segment Δ [the mean proportional between B and Γ, i,e., $\Delta = \sqrt{B \cdot \Gamma}$] such that**,

$$\frac{B}{\Delta} = \frac{\Delta}{\Gamma} \ .$$

Find n so that if Z_n and z_n be the sides of the circumscribed and inscribed polygons, **then**
by Proposition 3

> ["Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to circumscribe another about it so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than the greater magnitude has to the less."]

[we can obtain the ratio inequality]

$$\frac{Z_n}{z_n} < \frac{B}{\Delta}$$

> [NOTE: By including the "mean proportional" construction within the body of the proof, Archimedes in effect incorporated the construction used (after Proposition 3) in the proof of Proposition 5 which involves the ratio of areas of the polygons vis-à-vis ratio of the sides, allowing Prop. 23 and Prop. 32 to be used in a more direct manner.

Proposition 5.

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another in it, so that the circumscribed polygon may have to the inscribed polygon a ratio less than the greater magnitude has to the less.

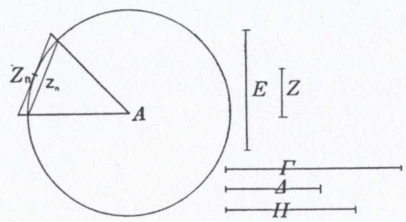

Fig. 55.

[NOTE: Archimedes at this stage of the book also **assumes Proposition 23**, after which, as Netz (p.130) notes, "a sequence of propositions … [lead] on to the main claim of the treatise." It was an extended explanation rather than a theorem statement. Its proof involved **Postulate IV** "applied successively to the surfaces of a zone (or segment) of a sphere and a truncated cone (or cone) which have the same circle as their boundary" (Dijksterhuis, p.174).

Proposition 23 [which introduces objects constructed by rotation of an inscribed 4k-sided polygon, and argues that the surface of the <u>inscribed</u> object is <u>less than</u> the surface of the sphere]

- "The solid obtained by the revolution of the inscribed [**regular 4k-sided**] polygon I_n is included between portions of the curved surfaces of cones the bounding circles of which lie in parallel planes on the surface of the sphere." (paraphrase, Dijksterhuis, p.170).
- "The surface of the sphere is greater than the surface described by the revolution of the [inscribed regular 4n-sided] polygon about the diameter of the great circle." (Heath, 31).
- ". . . . . . . The whole surface of the figure in the sphere, too, is smaller than the surface of the sphere." (Netz, p.118-119).]

Postulate IV

"[I assume] IV. That of the other surfaces which have the same boundaries, if the boundaries are in one plane, two are unequal whenever both are concave in the same direction and moreover one of them is either wholly included between the other and the plane which as the same boundaries, or is partly included by and partly coincides with the other; and that the surface which is included is the lesser."

Then

by Proposition 32 [involves **2 solids, one inscribed in, the other circumscribed about, a single sphere – establishes the ratio of "surface area of circumscribed solid to surface area of inscribed solid" and ratio of "volume of circumscribed solid to volume of inscribed solid" as equal, respectively, to the "duplicate ratio" and the "triplicate ratio" of the sides of circumscribed to inscribed polygon]**

["If in a sphere a solid be inscribed and another be circumscribed about it, and the two solids have been generated as above by similar polygons, the surface of the circumscribed solid is to the surface of the inscribed solid in <u>the duplicate ratio</u> of the side of the polygon circumscribed about the greatest circle to the side of the polygon inscribed in the same circle, and the [circumscribed] solid itself is to the [inscribed] solid in <u>the triplicate ratio</u> of the same sides."]

NOTE: The proof of Prop. 33 makes use of <u>the duplicate ratio</u> of the sides of circumscribed to inscribed polygon as in proof of Prop. 5 (using here the label from Figure 75 of Prop. 32, not the labels in Prop. 33):

$$\frac{E(C_n)}{E(I_n)} = \frac{E\Lambda[\Lambda M + Z\Theta + ...]}{AK[KN + B\Delta + ...]} = \left(\frac{E\Lambda}{AK}\right)^2$$

NOTE: Proof of Proposition 32 is based on constructions and theorems developed in **Proposition 31** (itself based on **Proposition 26**, and hence on **Props. 18, 19, and 20** from the section on the "Curved Surface of Cylinders and Cones" which relate surface area and volume) and on **Proposition 30** (itself based on both **Props. 21** and **29**).

Proposition 30 [the polygon circumscribed around sphere > 4 times greatest circle]

"The surface of the solid circumscribed about the sphere is *greater than four times* the greatest circle of the sphere."

Proposition 21

Proposition 23

Proposition 29 [derives expression for surface area of circumscribed 4k-sided regular polygon]

"The surface of the solid circumscribed about the sphere is equal to a circle the square on whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to [the sum of] all the lines joining the angular points of the polygon, parallel to one of the lines subtending two sides of the polygon."

Since Proposition 29 reduces to Proposition 24 in its proof, the resulting inequality is based as well on the result expressed in Proposition 24:

Proposition 24 [derives expression for surface area of inscribed 4k-sided regular polygon]

"The surface of the solid inscribed in the sphere is equal to a circle the square on whose radius is equal to the rectangle contained by the side of the (revolving) figure and a straight line equal to [the sum of] all the connecting lines which area parallel to the straight line subtending two sides of the polygon."

Proposition 31 – re 1 solid constructed between 2 concentric spheres (circumscribed about the smaller sphere & inscribed in the larger sphere) – <u>this proposition relevant for volume of sphere (Prop.34)</u>.

"The solid circumscribed about the smallest sphere is equal to a cone whose base is a circle which is equal to the surface of the solid and whose height is equal to the radius of the sphere."

Proposition 26

"The solid inscribed in the sphere, which is contained by the conical surfaces, is equal to a cone whose base is a circle equal to the surface of the solid inscribed in the sphere and whose height is equal to the perpendicular drawn from the center of the sphere to one side of the polygon."

Proposition 18

"Any solid rhombus consisting of isosceles cones is equal to a cone which has a base equal to the surface of one of the cones composing the rhombus and a height equal to the perpendicular drawn from the vertex of the other cone to one side of the first cone."

Proposition 20

"If one of the isosceles cones forming a rhombus be cut by a plane parallel to the base, if on the resulting circle a cone be described which has the same vertex as the other cone, and if the rhombus so formed be taken away from the whole rhombus, the part remaining will be equal to a cone which has a base equal to the surface of the first cone between the parallel planes, and a height equal to the perpendicular drawn from the vertex of the second cone to the side of the first cone."

Proposition 18 (as above)

Proposition 19

"If an isosceles cone be cut by a plane parallel to the base, if on the resulting circle a cone be described which has the centre of the base for its vertex, and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to a cone which has a base equal to the surface of the cone between the parallel planes, and a height equal to the perpendicular drawn from the centre of the base to one side of the cone."

Proposition 17

"If in two isosceles cones the surface of one cone be equal to the base of the other, while the perpendicular from the centre of the base on the side of the (first) cone be equal to the height (of the second), the cones will be equal."

Proposition 18 (as above)

$$\frac{E(C_n)}{E(I_n)} < \Delta\Lambda \frac{B}{\Delta} = \frac{\mathrm{B}}{\Gamma}$$ [i.e., in standard notation, $\frac{E(C_n)}{E(I_n)} = \frac{Z_n{}^2}{z_n{}^2} < \frac{B^2}{\Delta^2} = \frac{B^2}{B \cdot \Gamma} = \frac{B}{\Gamma}$]

[i.e., the ratio of surface areas $E(C_n)$ to $E(I_n)$ is the duplicate ratio of the sides Z_n and z_n of the polygons C_n and I_n (as in proof of Prop. 5)]

therefore,

$$\frac{E(C_n)}{E(I_n)} < \frac{E}{A} .$$ [this inequality is identified as (α)]

However
by Proposition 28 [**which introduces objects constructed by rotation of a circumscribed polygon, and argues that the surface of the <u>circumscribed</u> object is <u>greater than</u> the surface of the sphere]**

"Upon revolution of the circumscribed polygon C_n a similar solid is formed, the bounding portions of the conical surfaces, however, in this case touching the sphere in circles lying in parallel planes, while the whole of the solid, in the manner of Prop. 23, is inscribed in a sphere which is concentric with the given sphere and has a greater radius than the latter." (paraphrase, Dijksterhuis, p.170).

$$E(C_n) > E$$

and
by Proposition 25 [the polygon inscribed in sphere < 4 times the greatest circle of the sphere]

["The surface of solid inscribed in the sphere, which is contained by the conical surfaces, is *less than four times* the greatest circle of the sphere."]
NOTE: Proof of Proposition 25 (as above with Prop. 30) depends on Proposition 21, but in this case on its use in combination with Propositions 23 and 24.

Proposition 23 (see above)

Postulate IV

Proposition 24 [derives expression for surface area of inscribed 4k-sided regular polygon]

(see statement of proposition above)

Proposition 14 [area of the curved surface of a cone]

"The surface of any isosceles cone excluding the base is equal to the circle whose radius is the mean proportional between the side of the cone and the radius of the circle which is the base of the cone."

Proposition 16 [area of the curved surface of a truncated cone]

"If an isosceles cone be cut by a plane parallel to the base, the surface of the cone between the parallel planes is equal to a circle whose radium is the mean proportional between the side of the cone between the parallel planes and a straight line which is equal to the sum of the radii of the circles in the parallel planes."

Proposition 21 [see attached statement and proof]

$$E(I_n) < A ,$$

which is <u>contrary to</u> the inequality (α) based on Proposition 32, namely, $\frac{E(C_n)}{E(I_n)} < \frac{E}{A}$,

which itself may be written

$$\frac{E(C_n)}{E} < \frac{E(I_n)}{A},$$

and since $E(I_n) < A$ (the inequality drawn from Proposition 25),

a fortiori it would have to be that $E(C_n) < E$,

in contradiction with the inequality $E(C_n) > E$ drawn from Proposition 28.

Hence, it cannot be the case that $A < E$.
So it must be the case that $A \geq E$.

Case II. Assume $A > E$.

Proceeding as above, we **find n such that**

$$\frac{E(C_n)}{E(I_n)} < \frac{A}{E}.$$ [this inequality is identified as (β)]

However,
by Proposition 30 [the polygon circumscribed around sphere > 4 times greatest circle of the sphere]

> "The surface of the solid circumscribed about the sphere is *greater than four times* the greatest circle of the sphere."
> NOTE: The proof of this proposition uses **Proposition 23**, **Proposition 21**, and **Proposition 29**.

$$E(C_n) > A$$

and
by Proposition 23 [which introduces objects constructed by rotation of an inscribed 4k-sided polygon, and argues that the surface of the <u>inscribed</u> solid is <u>less than</u> the surface of the sphere]

$$E(I_n) < E,$$

leading, as above (by rewriting the inequality (β)), to a proof of the impossibility of the inequality (β).

Hence, it cannot be the case that $A > E$.

Thus, since it is not the case that $A < E$ (Case I) nor the case that $A > E$ (Case II),
it must be true that $A = E$.