

## Archimedes and the Archimedean Tradition

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The topic is Proposition 4 in Book II of Archimedes' "On the Sphere and the Cylinder" (SC), an important result in the Greek and Arabic Archimedean tradition,<sup>1</sup> as well as the Renaissance.<sup>2</sup>

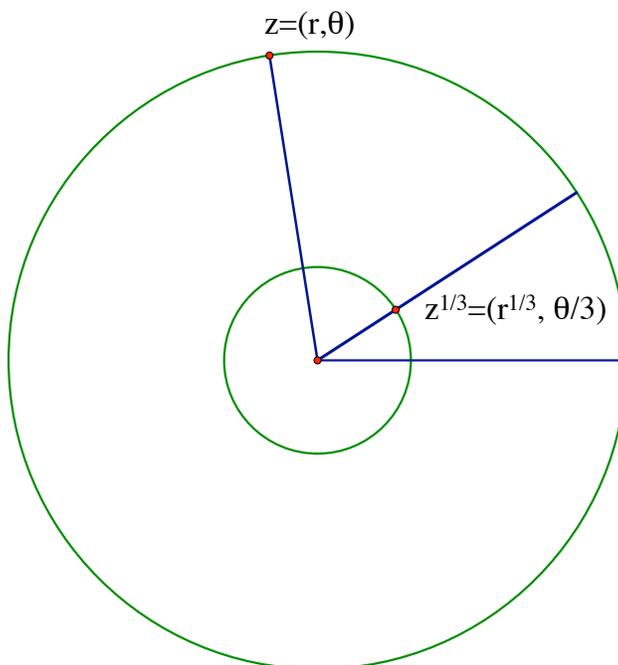
### 1. The Three Problems

All of SC is rooted in the Three Problems of Antiquity:

- *To construct a square equal in area to a given circle.*
- *To construct a cube of volume in given ratio to a given cube.*
- *To construct an angle equal to one third of a given angle.*

In modern terms, the first problem asks for a geometric construction of the number  $\pi$ . All of Archimedes' theorems on measurement in SC are reductions to this problem, which Archimedes considers elsewhere. The second and third problems are unified in modern terms as

*To construct the cube root of a given complex number.*



This is (even today) the essential problem for solving cubic equations. The ancients found geometric solutions to both aspects of the cubic problem. The later "cubic formulas" of Cardano and Viète are reductions of the solution of a general cubic equation to this problem of computing cube roots.

The inverse problems in SC II request geometric constructions of the roots of fairly general cubic polynomials. Archimedes and his successors, both near-contemporary Greek (Dionysiodorus, Diocles) and later Persian (al-Khayyam, al-Tusi), found these roots as the intersection of two conics.<sup>3</sup>

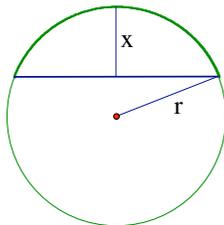
<sup>1</sup>R. Netz, "The Transformation of Mathematics in the Early Mediterranean World", C.U.P, 2004.

<sup>2</sup>See Cardano's *Ars Magna* and Viète's *Ars Analytica*.

<sup>3</sup>This is always possible: any cubic can be put in the form  $x^3 = ax^2 + b$ , whose roots are the intersection of the hyperbola  $y(x - a) = b$  with the parabola  $y = x^2$ .

## 2. Measurement in SC: A sphere and its parts

Consider a segment  $S$  of height  $x$  in a sphere of radius  $r$ :



**I.42-3:** *The surface of  $S$  (excluding the base) equals the circle whose radius is the line from the vertex of  $S$  to the perimeter.*

In our terms: <sup>4</sup>

$$\text{Surface of } S = 2\pi r x.$$

**II.2:** *The volume of  $S$  equals the cone on the same base whose height is to  $x$  as  $x' + r$  is to  $x'$ , where  $x'$  is the height of the remaining segment.*

In our terms:

$$\text{Volume of } S = \frac{\pi}{3} x^2 (3r - x).$$

## 2. Inverse problems in SC II

In his introduction to SC II, Archimedes highlights four problems, all asking for spheres or parts of spheres with given measurements or proportions. In effect, he asks how volume or area is distributed throughout a sphere.

It is natural to begin by supposing the problem solved (*analysis*, as in setting  $y = f(x)$  and solving for  $x$ ) so as to invoke the results of SC I. After deducing the properties of a putative solution, Archimedes clears the table and gives a direct solution (*synthesis*, as in showing that  $f^{-1}(f(x)) = x$ , thus certifying that the inverse function has been found). The synthesis stands on its own as a proof of the fact, though it would be completely unmotivated without the earlier analysis.

**II.1:** *To find a sphere equal to a given cone or cylinder.*

As Tom has shown, this may be viewed as the problem of taking a cube root of a real number (the second problem of antiquity), and this was so well-studied at the time that Archimedes felt no need to say anything more, once he had reduced to the problem of real cube roots.

**II.3:** *To cut a given sphere with a plane so that the segments have their surfaces in a given ratio.*

A rare case where surface area is easier than volume! This is almost trivial, since the result in I.42-3 is so easy to invert. This is revealed by the analysis, from which the synthesis is a formality.

**II.4:** *To cut a given sphere with a plane so that the segments have their volumes in a given ratio.*

In view of II.2, the problem stated in modern terms is to solve

$$\frac{(2r - x)^2(r + x)}{x^2(3r - x)} = \rho \tag{1}$$

<sup>4</sup>Note how the modern formulas in their concision obscure the geometry of the situation. While regrettable, this lack of meaning makes the modern formulas more efficient for further computation: less freighted with specific geometric context, they are more portable. In fact, Archimedes himself applies this principle of “judicious forgetfulness” to solve his cubic problem in II.4

for the height  $x$  of the greater segment, where  $\rho$  is the ratio of the smaller segment to the greater. That is, II.2 gives  $\rho = \rho(x)$  as a rational function of  $x$  and the problem in II.4 is to find the inverse function  $x = x(\rho)$ . The function  $x(\rho)$  generally takes three values for each  $\rho$  (as a square-root takes two values). However, only one of these values will satisfy the constraint  $r \leq x(\rho) \leq 2r$ , and this value  $x(\rho)$  is our slicing point.

After simplifying, equation (1) can be viewed as a cubic equation

$$x^2(3r - x) = \frac{4r^2}{1 + \rho}, \tag{2}$$

which shows that II.4 is related to the cubic problems of Antiquity. In fact, we will see that II.4 is a problem about angle trisection.

### 3. Archimedes' derivation of his sphere-cutting cubic

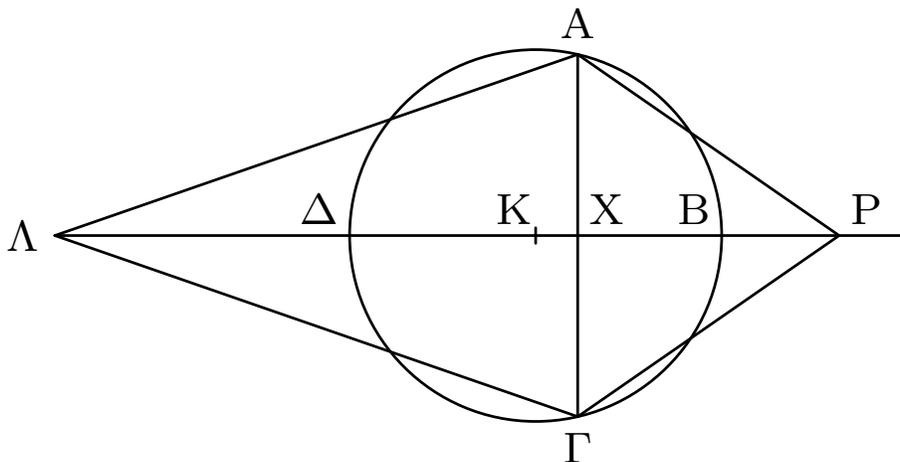
Let  $AB\Gamma\Delta$  be a sphere with diameter  $B\Delta$  and center  $K$ . Suppose the problem is solved by cutting the sphere with a plane through  $A\Gamma$  meeting  $B\Delta$  perpendicularly at  $X$ , with  $\Delta X$  the height of the greater segment. Thus,

$$\frac{\text{seg. } AB\Gamma}{\text{seg. } A\Delta\Gamma} = \rho,$$

the given ratio. As in II.4, choose points  $\Lambda$  and  $P$  on  $B\Delta$  extended so that

$$\text{cone } A\Lambda\Gamma = \text{seg. } A\Delta\Gamma \quad \text{and} \quad \text{cone } AP\Gamma = \text{seg. } AB\Gamma.$$

Here is a metrically accurate diagram for  $\rho = 1/2$ :



Since cones on the same base are to each other as their heights, the problem is now one dimensional: we have three unknown points  $P, \Lambda, X$  satisfying the relation

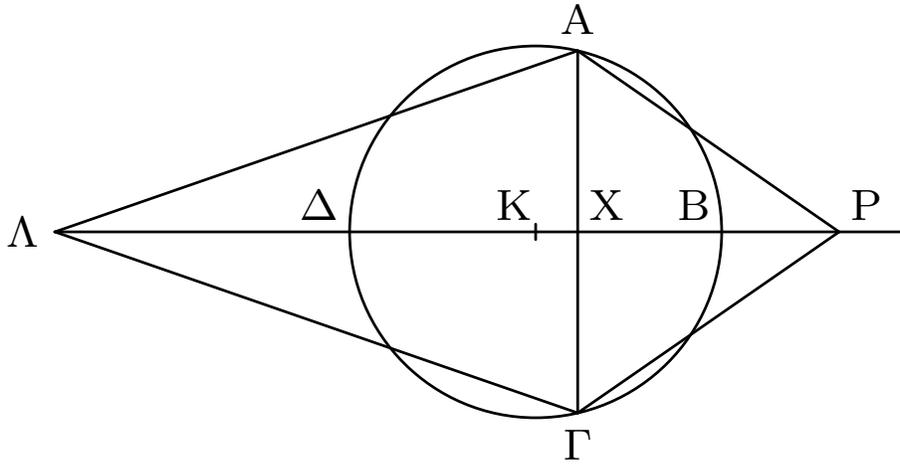
$$\frac{P\Lambda}{X\Lambda} = \frac{\text{both cones}}{\text{larger cone}} = \rho + 1. \tag{3}$$

From equation (3) we want to eliminate  $P$  and  $\Lambda$ , to arrive at an equation for  $X$  in terms of given points. The sphere does not appear explicitly in (3), so the first step is to include the point  $\Delta$  from the sphere:

$$\frac{P\Lambda}{\Lambda\Delta} \cdot \frac{\Lambda\Delta}{X\Lambda} = \rho + 1. \tag{4}$$

We first eliminate  $P, \Lambda$  in  $P\Lambda/\Lambda\Delta$ . The cone  $A\Lambda\Gamma$  is defined by the proportion

$$\frac{KB + BX}{BX} = \frac{\Lambda X}{\Delta X}, \tag{5}$$



or

$$\frac{KB}{BX} = \frac{\Lambda\Delta}{\Delta X}. \quad (6)$$

Likewise, the cone  $AP\Gamma$  is defined by the proportion

$$\frac{K\Delta}{\Delta X} = \frac{PB}{BX}. \quad (7)$$

Since  $K\Delta = KB$ , we can cross-multiply to make  $\Delta X/BX$  appear in both equations (6) and (7), and get

$$\frac{\Lambda\Delta}{K\Delta} = \frac{\Delta X}{BX} = \frac{KB}{PB}. \quad (8)$$

The outer terms of (8) are ratios (one inverted) of the radius of the sphere to the distance from the cone vertex to the sphere. The middle term is the cut ratio on the diameter. Since we want  $P\Lambda/\Lambda\Delta$ , we invert the outer two ratios:

$$\frac{K\Delta}{\Lambda\Delta} = \frac{PB}{KB},$$

then add one to both sides and get

$$\frac{K\Delta}{\Lambda\Delta} = \frac{PK}{KB} = \frac{PK}{K\Delta} = \frac{PK + K\Delta}{K\Delta + \Lambda\Delta} = \frac{P\Lambda}{K\Lambda}.$$

The product of the first and last terms is the square of the first term:

$$\frac{P\Lambda}{\Lambda\Delta} = \frac{K\Delta^2}{\Lambda\Delta^2}.$$

Cross-multiplying (6) and remembering that  $KB = K\Delta$ , we have

$$\frac{K\Delta}{\Lambda\Delta} = \frac{K\Delta + \Delta\Lambda}{\Lambda\Delta} = \frac{BX + \Delta X}{\Delta X} = \frac{B\Delta}{\Delta X},$$

so we have eliminated  $\Lambda$  and  $P$  in our first term:

$$\frac{P\Lambda}{\Lambda\Delta} = \frac{B\Delta^2}{\Delta X^2}. \quad (9)$$

For the second term  $\Lambda\Delta/X\Lambda$ , choose  $Z$  on  $KB$  extended so that  $BZ = KB$  is the radius of the sphere. Thus  $Z$  is a given point, making  $KB + BX = ZX$ , so the equation (5) of cone  $\Lambda\Gamma$  becomes

$$\frac{ZX}{BX} = \frac{\Lambda X}{\Delta X}, \quad \text{or} \quad \frac{\Delta X}{\Lambda X} = \frac{BX}{ZX},$$

or

$$\frac{\Delta\Lambda}{\Lambda X} = \frac{BZ}{ZX}, \tag{10}$$

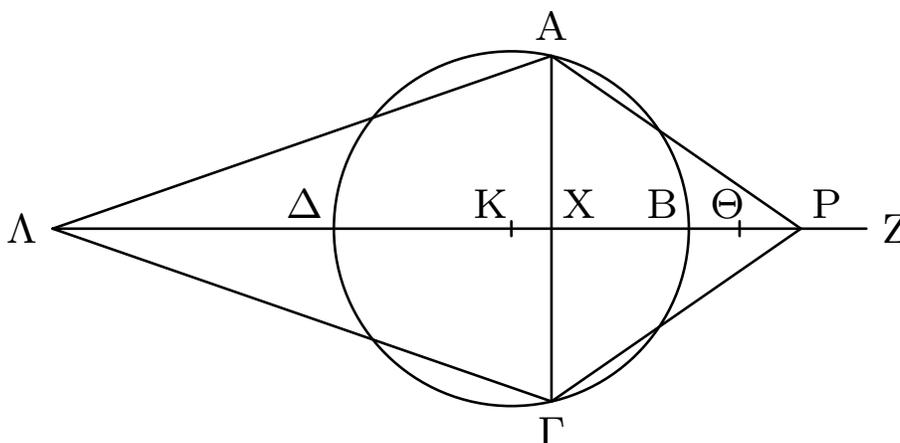
so that P, Λ have been eliminated from the second term. Combining (9) and (10) we now have

$$\frac{P\Lambda}{X\Lambda} = \frac{B\Delta^2}{\Delta X^2} \cdot \frac{BZ}{ZX} = \rho + 1.$$

Cut BZ at Θ so that  $BZ/Z\Theta = \rho + 1$  and obtain **Archimedes' sphere-cutting cubic**:

$$\frac{XZ}{Z\Theta} = \frac{B\Delta^2}{\Delta X^2}. \tag{11}$$

In equation (11), the points B, Δ and Z are given by the sphere, while Θ divides the radius BZ in the given ratio  $\rho + 1$ . The cut point X is thus the solution of an equation whose remaining ingredients are given.

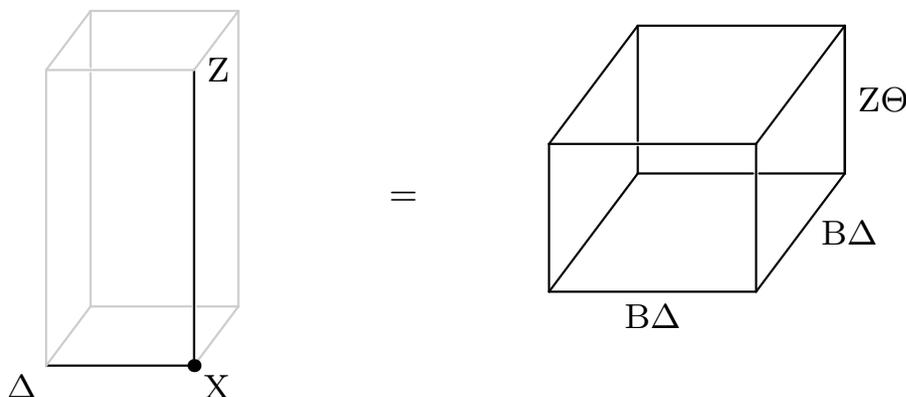


Independently of the sphere, one can visualize and remember equation (11) as follows. Put the unknown magnitudes on one side and the given magnitudes on the other side:

$$XZ \cdot \Delta X^2 = Z\Theta \cdot B\Delta^2. \tag{12}$$

Let us agree that a “box” is a rectangular parallelepiped with a square base. Thus, (12) is an equality of box volumes. Cutting a segment determines a box, where the short part of the segment is the side of the base and the long part is the height. Let us call this “bending a segment into a box”. To solve (11) is now the problem:

*To bend a given segment into a box of given volume:*



#### 4. Archimedes' solution of the cubic

As we did with the box, Archimedes now views the cubic

$$\frac{XZ}{Z\Theta} = \frac{B\Delta^2}{\Delta X^2} \tag{13}$$

independently of the sphere, so as to ignore the complexity in the point  $\Theta$ . Thus,  $B\Delta^2$  is now an arbitrary area,  $\Delta Z$  and  $Z\Theta$  are two lines, and we seek to cut  $\Delta Z$  at  $X$  so that (13) holds. As we can see from the box, these lines must satisfy some constraints in order for  $X$  to exist in the general equation (13), but in the sphere-cutting situation (11) it is clear from continuity principles that a unique point  $X$  exists.

In any case, we see that Archimedes is here forgetting some of the original geometry, and thinking more like an algebraist. Accordingly, we switch to modern notation. Setting

$$x = \Delta Z \quad a = \Delta Z, \quad d = B\Delta, \quad c = Z\Theta,$$

equation (13) becomes

$$\frac{a - x}{c} = \frac{d^2}{x^2},$$

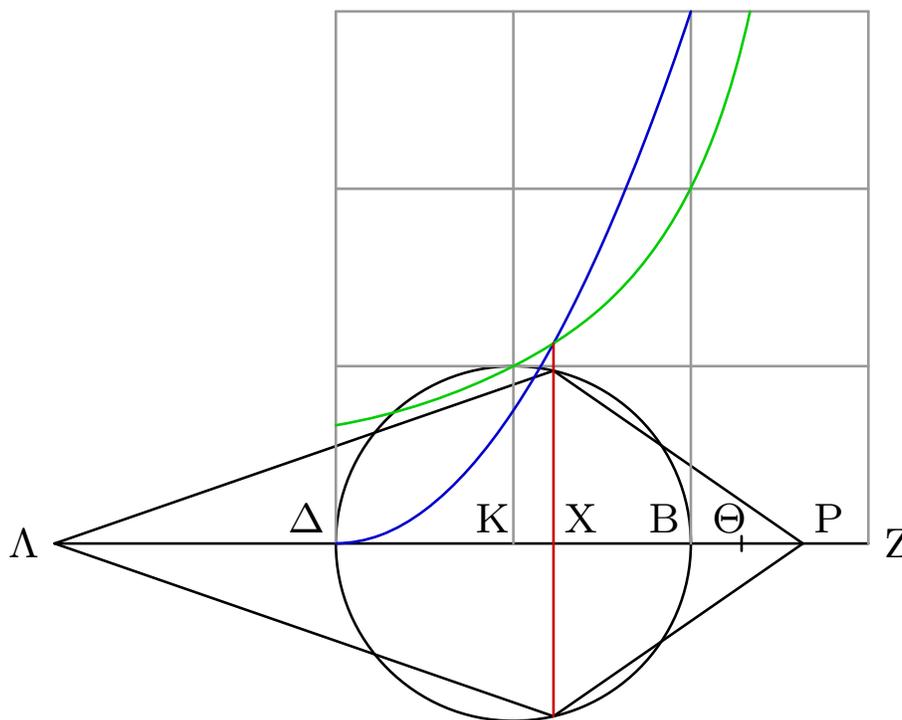
or

$$x^2(a - x) = cd^2. \tag{14}$$

The solution lies on the intersection of the parabola and hyperbola:

$$d^2y = ax^2, \quad y(a - x) = ad.$$

In the case of sphere-cutting, we have  $a = 3r$ ,  $d = 2r$  and  $c = r/(1 + \rho)$ . We illustrate with a metrically accurate picture for  $\rho = 1/2$ .



The light grid squares of side  $r$  are included, since the parabola (blue) and hyperbola (green) pass through grid points. Their intersection above the sphere gives the cutting point  $X$ , slicing down along the red line.

There are two other intersection points, to the left and right of the sphere; these solutions are irrelevant for the sphere cutting problem.

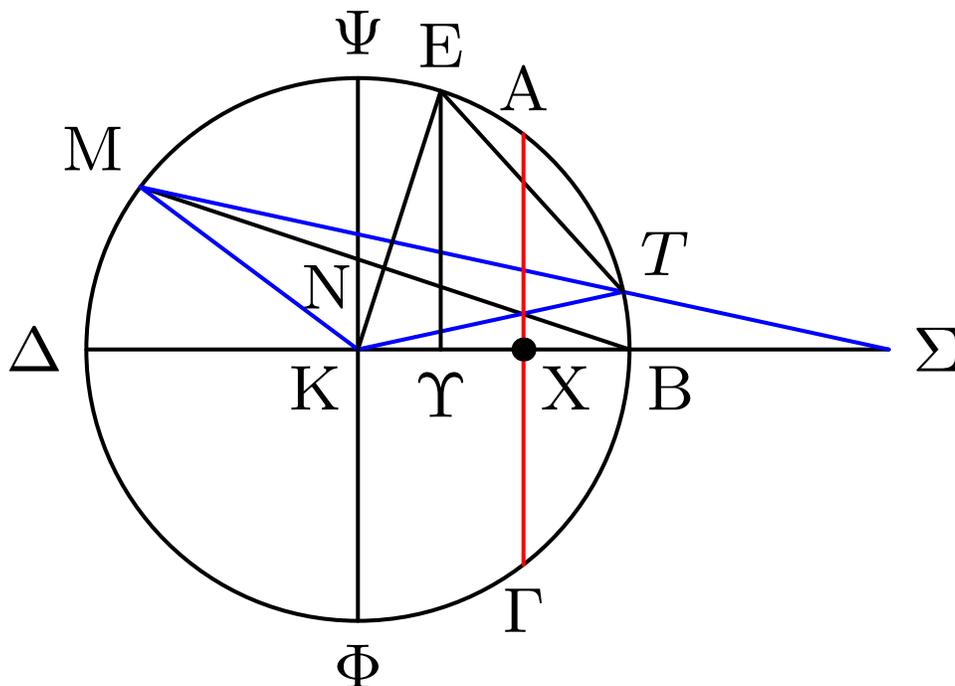
### 5. A solution in principle only?

Problem II.4 asks for the point  $X$  to cut the sphere in a given ratio. Archimedes finds  $X$  as the solution of a cubic equation  $x^2(a - x) = cd^2$ . He then ignores the information in the constant  $c$  (involving the radius and the given ratio) and gives the solution of this more abstract equation as an intersection of two conics.

One can question whether the problem is solved to the same satisfaction as problems in the *Elements* are solved. In a practical sense, it is difficult to draw, or even imagine an accurate picture of any conic other than a circle. In a deeper sense, it is almost tautological that the solutions of a cubic equation are obtained as the intersection of a line and a cubic. Does this differ from an intersection of two conics only because a cubic is not constructible within the ancient framework? If so, is it only the convention of his time that permits Archimedes to claim that he has solved the cubic in any sense beyond the tautological?

In fact there is a direct construction of the solution to Archimedes' sphere-slicing problem, using Euclidean methods plus one angle trisection from Archimedes' Book of Lemmas Prop. 8.

1. Let  $B\Psi\Delta\Phi$  be a great circle of the sphere with center  $K$  and orthogonal diameters  $B\Delta$ ,  $\Psi\Phi$ .
2. Cut  $K\Psi$  extended (if necessary) at  $N$ , where  $KN^2 : K\Psi^2 = \rho$  is the given ratio.
3. Produce  $BN$ , cutting the circle at  $M$ .
4. Cut the arc  $B\Psi M$  at  $T$ , such that the arc  $BT$  is one third of the arc  $\Delta M$  (see blue lines).
6. On the base  $TK$  construct the equilateral triangle  $TKE$  such that  $E$  lies on the arc  $B\Psi$ .
7. Draw the perpendicular from  $E$  to  $B\Delta$ , meeting  $B\Delta$  at  $\Upsilon$ .
8. Cut  $KB$  at  $X$  such that  $K\Upsilon = \Upsilon X$ .



Then if we slice the sphere with the plane orthogonal to  $B\Delta$  at  $X$ , the segment containing  $B$  to the segment containing  $\Delta$  will be as the given ratio  $\rho$ .

As with efficient constructions of the pentagon, it is easiest to verify this with trigonometry. In fact the construction above was obtained by geometrizing the trigonometric solution: Let  $t = r + x$ , to place the

origin at  $K$ . Let  $\tau = t/r$ . Then equation (11) becomes

$$\tau^3 - 3\tau = 2 \left( \frac{\rho - 1}{\rho + 1} \right).$$

From the triple angle identity, the equation  $\tau^3 - 3\tau = 2 \cos \theta$  has the solutions

$$\tau = 2 \cos \frac{\theta}{3}, \quad 2 \cos \frac{2\pi - \theta}{3}, \quad 2 \cos \frac{2\pi + \theta}{3}.$$

We choose  $\theta \in (\pi/2, \pi)$  such that  $\cos \theta = \frac{\rho-1}{\rho+1}$ . The solution with  $\tau \in (0, 1)$  is

$$\tau = 2 \cos \frac{2\pi - \theta}{3}, \quad \text{or} \quad t = 2r \cos \frac{2\pi - \theta}{3},$$

which is what the construction above produces. Indeed,

$$\begin{aligned} M &= r(\cos \theta + i \sin \theta), & T &= r\left(\cos \frac{\pi - \theta}{3} + i \sin \frac{\pi - \theta}{3}\right), & E &= r\left(\cos \frac{2\pi - \theta}{3} + i \sin \frac{2\pi - \theta}{3}\right), \\ \Upsilon &= r \cos \frac{2\pi - \theta}{3}, & X &= 2r \cos \frac{2\pi - \theta}{3}. \end{aligned}$$

This shows that SC II.4 is essentially an angle trisection problem.