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STRUCTURE and EXPLANATION of proof of
PROPOSITION 34:

> "Any sphere is equal to four times **the cone** whose base is equal to the greatest circle of the sphere, and whose height is equal to the radius of the sphere." (Heath, p. 41)

Corollary (called Porism in Dijksterhuis):

> "From what has been proved it follows that ***every cylinder*** *whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is 3/2 of the sphere, and its surface together with its bases is 3/2 of the surface of the sphere*." (Heath, 43)

[It is the geometric depiction of this corollary which Archimedes had put on his tombstone.[

Proposition 34, structure of propositions used (according to Dijksterhuis' proof):

Proposition 2
Proposition 3
Proposition 32 – the ratio of volumes of two solids is equal to the "triplicate ratio" of their respective sides

- Proposition 30
  - Proposition 21
  - Proposition 23
  - Proposition 29
  - Proposition 24
    - Proposition 14
    - Proposition 16
- Proposition 31
  - Proposition 26
    - Proposition 18
    - Proposition 20
      - Proposition 18
    - Proposition 19
      - Proposition 17
      - Proposition 18

Footnote proving an inequality resulting from use of 2 arithmetic means
Postulate 4 – volume of circumscribed solid is greater than volume of the sphere
and volume of inscribed solid is less than volume of the sphere

> "That of the other surfaces which have the same boundaries, if the boundaries are in one plane, two are unequal whenever both are concave in the same direction and moreover one of them is either wholly included between the other and the plane which has the same boundaries, or is partly included by and partly coincides with the other; and that the surface which is included is the lesser."

Proposition 27 – volume of solid inscribed in a sphere is less than 4 times the volume of a cone whose base = "great circle of the sphere" and whose height = "radius of the sphere"

> "The solid inscribed in the sphere, which is contained by the conical surfaces [found in **Proposition 26**], is less than four times **the cone** whose base is equal to the greatest circle of the sphere and whose height is equal to the radius of the sphere" (Dijksterhuis, p. 177).

- Proposition 26*
- Proposition 25
  - Proposition 23
  - Proposition 24*
  - Proposition 21

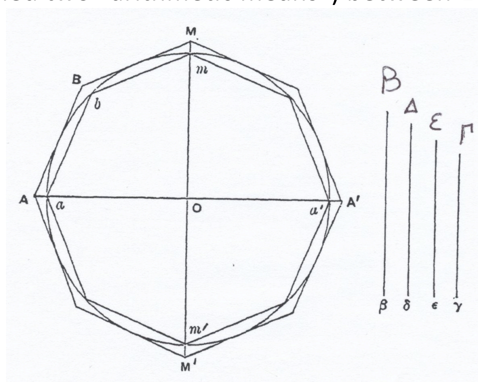
Proposition 31* Porism (together with Proposition 30*) **–**
volume of solid circumscribed in a sphere is greater than 4 times the volume of a cone whose base = "great circle of the sphere" and whose height = "radius of the sphere"

Postulate 4

Explanation of Proof of PROPOSITION 34

The logic of the proof is similar to the proof of Proposition 33 (that is, a proof by contradiction – what Netz (p.147) described as a "proof by double absurdity, manipulating proportions").

The figure is also similar to that of Proposition 33, but the proof involves the construction of two "mean proportionals" (alternatively called two "arithmetic means") between B and Γ.



The "geometrical configuration" of the Proposition involves:

a sphere, a great circle, and a cone (base = *great circle*, height=radius of sphere)

The Proof constructs a different cone (incorporating the factor 4 in the base), yielding:

a sphere, great circle, and a cone (base = 4 times the *great circle*, height=radius of sphere)

__**Proof**__(based on Heath and Dijksterhuis):

Let the sphere be that [whose *great circle* is depicted in the above figure].

If now the sphere is not equal to four times the cone described, it is either greater or less.

__Case I__. If possible, **let the sphere be __greater than__ four times the cone**.

Suppose **V** to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere.

Then, by hypothesis, the sphere is greater than **V**;
and [we can **find two line segments B and Γ ($B > \Gamma$) such that**

$$B : \Gamma < \text{(volume of sphere)} : \mathbf{V} \qquad \text{(Prop. 2)}$$

Between B and Γ place two arithmetic means Δ and E.

As usual,

- **__circumscribe__ about the sphere, and __inscribe__ within it, solids obtained by rotating similar regular polygons of sides** Z_n and z_n respectively.

We can find n such that

$$\frac{\text{(side } Z_n \text{ of outer polygon)}}{\text{(side } z_n \text{ of inner polygon)}} < \frac{\text{(greater line segment } B\text{)}}{\text{(mean proportional line segment } \Delta\text{)}}$$ (Prop. 3), and

$$\frac{\text{(volume of outer solid)}}{\text{(volume of inner solid)}} = \frac{\text{(side } Z_n \text{ of outer polygon)}^3}{\text{(side } z_n \text{ of inner polygon)}^3}$$ (Prop. 32)

This gives the chain of ratios,

(vol. of outer solid) : (vol. of inner solid) < $B^3 : \Delta^3$ (by hypothesis)

< $B : \Gamma$ (*a fortiori*, since $B:\Gamma > B^3:\Delta^3$)

< **(volume of sphere) : V** (*a fortiori*)

Now the resulting inequality

$$\frac{\textbf{(volume of outer solid)}}{\textbf{(volume of inner solid)}} < \frac{\textbf{(volume of sphere)}}{\textbf{V}}$$ (inequality (α))

can be rewritten as

$$\frac{\textbf{(volume of outer solid)}}{\textbf{(volume of sphere)}} < \frac{\textbf{(volume of inner solid)}}{\textbf{V}}$$ (rewritten inequality (α))

We also know from **Prop. 27** that

(volume of the inner solid) < V (Prop. 27),

so that the "rewritten inequality (α)" is less than 1, requiring that

(volume of outer solid) < (volume of sphere).

But we know further from Postulate 4 (or Prop. 28) that

(volume of outer solid) > (volume of sphere) (Post. 4 or Prop. 28),

which gives a ratio greater than 1.

Thus the inequalities obtained from Postulate 4 and Propositions 27 (and 28) lead to a contradiction with the "rewritten inequality (α)" obtained from Propositions 2, 3, and 32:

$$1 < \frac{\textbf{(volume of outer solid)}}{\textbf{(volume of sphere)}} < \frac{\textbf{(volume of inner solid)}}{\textbf{V}} < 1$$

Hence, it **cannot be the case** that the volume of the sphere is <u>greater than **V**</u>.
So it **must be the case** that the volume of the sphere is <u>less than or equal to **V**</u>.

Case II. If possible, **let the volume of the sphere be <u>less than</u> four times the cone.**

The first part of the proof is exactly the same as in Case I except that **V** and **(volume of sphere)** are reversed, so that the chain of inequalities leads to the inequality

$$\frac{\textbf{(volume of outer solid)}}{\textbf{(volume of inner solid)}} < \frac{\textbf{V}}{\textbf{(volume of sphere)}} \qquad \text{(inequality } (\beta))$$

which can be rewritten as

$$\frac{\textbf{(volume of outer solid)}}{\textbf{V}} < \frac{\textbf{(volume of inner solid)}}{\textbf{(volume of sphere)}} \qquad \text{[rewritten inequality } (\beta)]$$

We also know from Proposition 31 Porism that

(volume of outer solid) > V (Prop. 31 Porism),

so that the "rewritten inequality (β)" is greater than 1, requiring that

(volume of inner solid) > (volume of sphere).

But we know further from Postulate 4 that

(volume of inscribed solid) < (volume of sphere) (Post. 4)

which gives a ratio less than 1.

Thus the inequalities obtained from Proposition 31 and Postulate 4 lead to a contradiction with the "rewritten inequality (β)" obtained from Propositions 2, 3, and 32:

$$\textbf{1} < \frac{\textbf{(volume of outer solid)}}{\textbf{V}} < \frac{\textbf{(volume of inner solid)}}{\textbf{(volume of sphere)}} < \textbf{1}$$

Hence, it **cannot be the case** that **(volume of sphere) < V**.

Thus,
since it is **not** the case that **(volume of sphere) > V** (Case I),
nor the case that **(volume of sphere) < V** (Case II),
it **must be true** that **(volume of sphere) = V,**

that is, that the volume of the sphere is **equal to** four times the volume of a cone whose base is equal to the great circle of the sphere and whose height is equal to its radius.

Note: To prove the Corollary to Prop. 34, Archimedes uses in addition to Propositions 33 and 34 only one further proposition, **Proposition 13** (the "lateral surface area" of a right cylinder is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of its base). The cylinder of the Corollary has diameter and height 2R (diameter of an inscribed sphere). Archimedes' proof was as follows:

<u>Volume</u> of that cylinder = 3 cones of ht 2R = 6 cones of ht R = 3/2 (4 cones of height R) = **3/2 (volume sphere)**

[alternatively, "area of base" x "height" = $2R(\pi R^2) = 2\pi R^3 = 3/2\ (4/3\ \pi R^3) = 3/2$ (volume sphere)]

<u>Surface area</u> of that cylinder (incl. bases): $2(\pi R^2) + \pi(2R)^2 = 6\pi R^2 = 3/2\ (4\pi R^2) =$ **3/2 (area sphere)**