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in the Middle Ages

VOLUME I

THE ARABO-LATIN TRADITION

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Chapter four

The *Verba filorum* of the Banū Mūsā

1. Content and Authors

We have seen that Gerard of Cremona was responsible for the most accurate and popular translation of the *De mensura circuli* which circulated in the Latin West. He was also responsible for the first knowledge of some of the conclusions of Archimedes' *On the Sphere and the Cylinder*. This came through his translation of a short but important treatise entitled in Latin *Verba filiorum Moysi filii Sekir, i.e. Maumeti, Hameti, Hasen.*[1] This Arabic treatise on the measurement of the areas and volumes of certain plane and solid curved figures was composed by the celebrated Arabic mathematicians of the ninth century, the Banū Mūsā ibn Shākir, three brothers whose contribution to Islamic mathematics was of great significance.

The importance of this work for Western geometry is twofold: it gave the formulas for the area of the circle, the area and volumes of a sphere, and so on (as did certain practical manuals available to the medieval Schoolmen); and it also presented demonstrations of an Archimedean character of these formulas. In fact, we can single out the following particular contributions of this treatise: (1) A proof of Proposition I of the *De mensura circuli* somewhat different from the proof of Archimedes but still fundamentally based on the so-called exhaustion principle. (2) A determination of the value of π drawn from Proposition III of Archimedes' *De mensura*

[1] Listed in the medieval list of Gerard's translations as *Liber trium fratrum* (see F. Wüstenfeld, "Die Übersetzungen arabischer Werke in das Lateinische," *Abhandlungen der K. Gesellschaft der Wissenschaften zu Göttingen*, vol. 22 (1877), p. 59.

circuli but with further calculations similar to those found in Eutocius' commentary on the *De mensura.* (3) Hero's theorem for the area of a triangle in terms of the sides (that is, $A^2 = s(s-a)(s-b)(s-c)$, where A is the area, s the semiperimeter, and a, b, and c are the sides) with the first demonstration of that proposition appearing in Latin. (4) Theorems for the surface area and volume of a cone, again with geometrical demonstrations. (5) Theorems for the area and volume of a sphere with demonstrations of an Archimedean character. (6) A use of a form of the formula for the area of a circle equivalent to $A = \pi r^2$ in addition to the more common Archimedean form $A = 1/2\, cr$. Instead of the modern symbol π the authors use the expression "the quantity which when multiplied by the diameter produces the circumference." (7) The first introduction into the West of the problem of finding two mean proportionals between two given quantities. In this treatise we find two solutions presented: (a) the solution attributed by the Banū Mūsā to Menelaus and by Eutocius to Archytas, and (b) the solution presented by the Banū Mūsā as their own but attributed by Eutocius to Plato. (8) The first solution in Latin of the famous Greek problem of trisecting an angle, a solution to some extent reminiscent of the one found in the so-called *Lemmata* (or *Liber assumptorum*) attributed to Archimedes. (9) A method of approximating cube roots to any desired limit.

As I have said in the first chapter, the *Verba filiorum* presents an impressively rich fare when compared with the geometric diet of the centuries immediately preceding it, that is, with the geometry produced by a Gerbert or one familiar to Schoolmen like Hugh of St. Victor. I think it just to say that this tract played an important role in the gradual spread of the knowledge of Archimedean geometry. In the first place, the famous mathematician Leonardo Fibonacci of Pisa in his revised *Practica geometrie* (1220) borrows heavily (and often in verbatim fashion)[2] from the *Verba filiorum* in his propositions relating to the area of a circle, Hero's formula for the area of a triangle, the area and volume of a cone, the area and volume of a sphere, and the finding of two continually proportional means between two given quantities. And, of course, Leonardo's work was itself quite well known down to the time of the Renaissance.

Leonardo's contemporary, the equally important Jordanus de Nemore, in his *De triangulis* takes from the *Verba filiorum* one solution of the problem of finding two means (see Appendix V) and the solution of the problem of trisecting an angle (see Appendix VI). It was possibly from

[2] B. Boncompagni, ed., *Scritti di Leonardo Pisano* (*Practica geometrie*), vol. 2 (Rome, 1862), pp. 40–42, 87–91, 153–58, 178–7[illegible]

Jordanus' treatment of the trisection problem that the solution was further altered and joined to Campanus of Novara's version of the *Elements* (see Appendix VI).

The *Verba filiorum* was also known in the thirteenth century to Roger Bacon, who gives it in a list of geometrical works under the title *Liber trium fratrum*.[3] Somewhat later, in the fourteenth century, the author of some geometric questions which are in a manuscript now at Paris was clearly influenced by the treatise of the Banū Mūsā and quotes it in connection with the area and volume of a sphere.[4] Furthermore, at about the same time Thomas Bradwardine (or, more probably, one of his contemporaries) was perhaps aware of Proposition IV from the *Verba filiorum* when constructing his demonstration of the general proposition relative to the area of a circle in terms of the radius and circumference of the circle. (That demonstration is published in Chapter Five under the name of Pseudo-Bradwardine.) Furthermore, from the late fourteenth or early fifteenth century stems an anonymous treatise *De inquisicione capacitatis figurarum* which cites the *Verba* as *Geometria trium fratrum*.[5]

Before passing on to the text and its construction, I should like to say something about the activity of the Banū Mūsā, who played a central role as patrons and students of Greek science in Baghdad in the ninth century.

[3] Roger Bacon, *Communia mathematica*, ed. of R. Steele (Oxford, 1940), p. 44, lines 20–25: "Et hee practice [partes geometrie] omnes traduntur in libris propriis secundum numerum earum, et notum est illis qui eas sciunt, quorum nomina sunt imposita secundum proprietatem earum, ut *liber de Ysoperimetris*, et *de Replentibus Locuum*, et *de Curvis Superficiebus* Archimenidis, et *liber Trium Fratrum*...."

[4] MS Paris, BN lat. 7377B, 37v: "In mensuratione autem spere primo per quantitatem diametri ipsius invenis quantitatem circuli maioris eius, quam postea multiplicas in 4 et provenit superficies totius spere, quod probatur in libro trium fratrum. Deinde superficiem spere multiplicas in sextam diametri et provenit soliditas totius spere, quia in libro trium fratrum probatur quod multiplicatio medietatis diametri spere in tertiam superficiei ipsius equatur soliditati totius spere." The *Liber trium fratrum* is also cited (folio 33r) for the solution of the area of a triangle in terms of its sides.

[5] *De inquisicione capacitatis figurarum*, ed. of M. Curtze in *Abhandlungen zur Geschichte der Mathematik*, 8. Heft (1898), pp. 37–38. The following propositions of the Banū Mūsā treatise are used and cited by the unknown author: Proposition IV and its corollary, Proposition VI (cited as Proposition 7 by the author), Proposition XIV (cited as Proposition 15? [or 17?] by the author; Curtze reads it as 15 and then gives proposition 15 in a note, which proposition refers to the volume of the sphere, but it is clear from the context that Proposition XIV is the required proposition). The fact that Proposition VI is cited as 7 suggests that the author was using a copy of the *Verba filiorum* in the tradition of *MaR* where VI is given as 7 (see Sigla). On the other hand, XIV is given by *MaR* as 17, not as 15 which this work seems to have (cf. another manuscript of the *De inquisicione* in Vienna, Nat.-bibl. 5277, 102r).

They were sons of Musā ibn Shākir, who is variously described as a reformed bandit and an astronomer or astrologer of the Caliph al-Maʿmūn.[6] He is said to have given his sons to the astronomer Yaḥya ibn Abī Manṣūr for instruction in mathematics. At any rate, they joined the circle of mathematicians that grew up in Baghdad at the time of al-Ma'mūn (fl. 813–33) and his successors. The names of the three brothers were Abū Jaʿfar Muḥammad, Abū 'l-Kāsim Aḥmad, and al-Ḥasan. They devoted their energies and resources to the acquisition of Greek scientific manuscripts and the propagation of their contents either by translation or by independent works.[7] They sent agents or went themselves on trips into the Byzantine provinces to search for and purchase manuscripts. It was probably on one such trip that Muḥammad (or, as some say, Aḥmad) met the mathematician and translator Thābit ibn Qurra of Ḥarran and persuaded him to come to Baghdad and join their circle. As a matter of fact, there is a distinct possibility that Thābit ibn Qurra had something to do with the preparation of the work which we are here editing, since Thābit is credited with the writing of a tract having almost the same title: *On the Measure of Plane Figures and Other Surfaces and Corporeal Figures*.[8]

While devoting their talents to science and learning, the Banū Mūsā were also involved in the political quarrels and court fighting of the day. Their hostility to the famous philosopher al-Kindī is said to have stemmed from the fact that it was he rather than they who was picked to educate the son of the Caliph al-Muʿtaṣim, and they later actively opposed the accession of that son to the caliphate.

There is little information which makes individuals of the three brothers. Some would say that Muḥammad, the eldest of the three, was the most important of the brothers.[9] He was learned in the works of Euclid, Ptolemy, and other mathematicians and astronomers. He is said by the *Fihrist* (edition of G. Flügel, p. 271) to have died in the year 259 A.H. (873 A.D.). Aḥmad is described as having been particularly interested in mechanics[10] and is named by the *Fihrist* (*loc. cit.*) as the author of a treatise on mechanics that is elsewhere attributed to all of the brothers. Al-Ḥasan's

[6] H. Suter, "Die Mathematiker and Astronomer der Araber und Ihrer Werke," *Abhandlungen zur Geschichte der Mathematik*, *10*. Heft (1900), p. 20. See also *Encyclopedia of Islam*, 1st ed., "Banu Musa" (J. Ruska). Cf. *Abhandlungen*..., *7*. Heft (1892), p. 24.

[7] M. Meyerhof, "New Light on Hunain ibn Ishaq and his Period," *Isis*, vol. *8* (1926). pp. 714–15.

[8] A. G. Kapp, "Arabische Übersetzer und Kommentatoren Euklids," *Isis*, vol. *23* (1935), p. 61.

[9] M. Casiri, *Bibliotheca Arabico-Hispana Escurialensis*, vol. *1* (Madrid, 1790), p. 418.

[10] *Ibid.*, pp. 418–19.

special interest was geometry.[11] Among the many works attributed to the brothers, we can single out, in addition to the treatise on mechanics already mentioned, a treatise on the *Ellipse* by al-Ḥasan; a new edition of or commentary on Apollonius' *Conics*, a *Book on the Movement of the First Sphere* by Muḥammad; a book by Aḥmad demonstrating that there is no movement of the ninth sphere beyond the sphere of the fixed stars, and of course the work which we are here editing and which is attributed to all three brothers. In the *Fihrist* its title is given as *The Book of the Measurement of the Sphere, the Trisection of an Angle, and the Finding of Two* Quantities Between Two Quantities Such That All of the Quantities are Continually Proportional*—which sounds more like a partial enumeration of the contents than a title. But in the extant manuscripts of the thirteenth-century revision by al-Ṭūsī the work is entitled *The Book of the Knowledge of the Measurement of Plane and Spherical Figures by the Sons of Moses: Muḥammad, and al-Ḥasan, and Aḥmad* (see Arabic variant readings below). This seems more probable as the original title. The word *Verba* used by Gerard of Cremona in his shortened title is no doubt a rendering of كتاب since, of course, *Verba* can be used in the sense of "a discourse."

2. The Text of the *Verba filiorum*

I know of eight manuscripts of Gerard of Cremona's translation of the *Verba filiorum* of the Banū Mūsā (see Sigla below). Of these eight, five are complete copies of the text: *P* (early fourteenth century), *Zm* (fourteenth century), *H* (fourteenth century), *Ma* (late fourteenth or early fifteenth century), and *R* (late fifteenth century?). Manuscripts *S* and *T*, both of the fourteenth century, are much truncated copies and are of little help for textual construction. For the fragmentary eighth copy, see page 236.

Manuscript *P* is the best manuscript,[1] although *Zm* is very good indeed.

[11] *Ibid.*

* Corrected from "one quantity" in the *Fihrist*.

[1] The reader will remember that the best tradition of Gerard's translation of the *De mensura circuli* is also contained in *P* (see Chapter Two, Section 2). Its excellence for other texts of Gerard's translations has been noted by P. Tannery, A. A. Björnbo, and H. Suter. Tannery, "Sur le 'liber augmenti et diminutionis' etc.," *Bibliotheca Mathematica*, 3. Folge. vol. 2 (1901), p. 45, says: "le Lat. 9335 [= *P*] semble bien voisin de l'an 1300. Ce manuscrit, in-fol., qui a appartenu à Ismaël Boulliau, mais où je n'ai trouvé aucun indice sur les

The Discourse of the Sons of Mūsā Ibn Shākir: Muḥammad, Aḥmad, and Ḥasan

[Preface]

Because we have seen (1) that there is fitting need for the knowledge of the measure of surface figures and of the volume of bodies, and we have seen (2) that there are some things, a knowledge of which is necessary for this field of learning but which—as it appears to us—no one up to our time understands, and [(3) that] there are some things we have pursued because certain of the ancients who lived in the past had sought understanding of them and yet knowledge has not come down to us, nor does any one of those we have examined understand, and [(4) that] there are some things which some of the early savants understood and wrote about in their books but knowledge of which, although coming down to us, is not common in our time—for all these reasons it has seemed to us that we ought to compose a book in which we demonstrate the necessary part of this knowledge that has become evident to us.

quod nobis manifestum est de hac scientia. Et si viderimus aliquid eorum, que posuerunt antiqui et quorum scientia publicata est in hominibus nostri temporis, quo indigeamus ad testificandum super aliquod eorum que ponemus in libro nostro, dicemus illud rememorando tantum; et non erit nobis necessarium narrare illud in libro nostro, cum sit eius scientia publicata; propterea quia querimus abbreviationem. Et si viderimus aliquid eorum que posuerunt antiqui de illis, quorum rememoratio non est famosa, et non est exquisita eius scientia, cuius narratione indigeamus in hoc nostro libro, ponemus illud in eo, et proportionabimus illud eius auctori. Et declarabitur ex eo quod narrabimus de compositione huius nostri libri quod oportet ei qui vult legere et intelligere ipsum, ut sit bene instructus in libris geometrie publicatis in hominibus nostri temporis.

Proprietas communis omnis superficiei est quod est habens longitudinem et latitudinem tantum. Sed proprietas in figura corporea est quod est habens longitudinem et latitudinem et altitudinem. Et longitudo et latitudo et altitudo sunt quantitates que terminant magnitudinem omnis corporis. Et longitudo est prima quantitatum que terminant illud. Et est illud quod extenditur secundum rectitudinem

17–18 viderimus...que: videmus aliquem illorum qui (que *Ma*) *MaR*
18 est *om MaR*
19 nostris (*!*) *Ma*
20 que ponemus: quod ponimus *MaR*
22 cum: tamen *R* (*?*) *Ma* / sit *om. MaR* / publica *Zm*
24 quarum *H* / est¹ *Ma om. R* / exquesita *H*
26 declarabitur: declinabitur *H*
27 nostri libri *tr. H*
28 ei: illum *MaR* / et intelligere *om. MaR*
29 temporis: operis *H*
30 omni *MaR*
31 in figura: infinita *H*
32 et¹ *om. H* / altitudinem et latitudinem *MaR*
35 Et: Longitudo *S* (*cf. T*) / extenditur: *MaR*
35–95 Et....volumus: Longitudo est illud quod extenditur secundum rectitudinem in duas partes simul tantum. Sed cum hec (*corr. ex* huius) extenditur latitudinaliter scilicet preter (*corr. ex* partem) in partem suam [et] suam rectitudinem dicitur latitudo ista extensio. Unde patet error dicentis latitudinem esse lineam continentem superficiem in parte alia a longitudine sua. Altitudo [est] extensio superficiei in partem aliam a longitudine et latitudine sua, scilicet in altum. Unde altitudo non est linearum. Scientia amplitudinis et magnitudinis in mensurando est per unum superficiale quo ad superficierum mensurationem, cuius longitudo est una et latitudo est una, cuius anguli sunt recti, et per unum corporale quo ad magnitudines quod est corpus cuius longitudo est una, latitudo una, et cuius altitudo est una, et elevatio superficierun eius quarundam super alias est super angulos rectos. Cuius ratio quia oportet quod quantitas qua mensurentur super-

35 illud¹: الأشكال (*figures*)

And if we consider some of those things which the ancients posed and the knowledge of which has become public among men of our time but which we need for the proof of something we pose in our book, we shall merely call it to mind and it will not be necessary for us in our book to describe it [in detail], since knowledge of it is common; for this reason we seek only a brief statement. On the other hand, if we consider something which the ancients posed and which is not well remembered nor excellently known but the explanation of which we need in our book, then we shall put it in our book, relating it to its author. It will be evident from what we shall recount concerning the composition of our book that one who wishes to read and understand it must be well instructed in the books of geometry in common usage among men of our time.

The common property of every surface is the possession of length and breadth alone, while the property of a corporeal figure is the possession of length, breadth, and height. Length, breadth, and height are quantities which delimit the magnitude of every body. Length is the first of the quantities which delimit the body and it is that which is extended in a straight line in both directions simultaneously. For nothing except length alone arises from it. When length is extended latitudinally, that is, in

55v c. 1 in duas / partes simul. Nam non fit ex eo nisi longitudo tantum. Et cum extenditur longitudo latitudinaliter, scilicet preter in partem suam et suam rectitudinem, tunc illa extensio est latitudo. Et tunc provenit superficies. Et latitudo quidem non est sicut estimant plures hominum, scilicet quod est linea que continet superficiem in parte alia a longitudine sua. Et si esset illud sicut dicunt, non esset superficies habens longitudinem et latitudinem tantum, et esset tunc latitudo longitudo etiam. Et illud est quoniam latitudo in eorum estimationibus est linea et linea est longitudo. Et Euclides quidem iam sapienter dixit illud ubi dixit quod linea est longitudo tantum, et superficies est habens longitudinem et latitudinem. Altitudo vero est extensio superficiei in partem que est preter longitudinem et latitudinem, scilicet extensio eius in altum. Et illi quidem qui estimaverunt quod latitudo est linea estimaverunt iterum quod altitudo est linea. Et declaratio erroris eorum in illo est equalis.

Iam ergo ostensum est quid sit longitudo et quid latitudo et quid altitudo. Et declaratur cum hoc quod iste tres quantitates, scilicet longitudo, latitudo et altitudo, determinant magnitudinem omnis corporis et extensionem omnis superficiei. Et declaratur iterum quod non est aliquid corporum indigens quantitate alia quarta qua eius

ficies et corpora sit talis ut cum duplantur continuentur ad invicem taliter ne dimittat in vacuitatibus aliquod de superficie et corpore mensuratis super quod non veniat. Et est necessarium cum hoc ut sit illud super quod venit mensuratio de superficie aut corpore facile dum non prohibetur eius mensuratio. Et illud non est repertum nisi in quadrato et in tali figura quia quando duplatur alteratur eius quantitas sed remanet eius quadratura et iterum continuatio unius cum altero quando duplatur est continuatio non dimittens in vacuitatibus huius quod mensuratur super (*corr. ex* sicud) quod non veniat, et illud velocius sit in corporibus et superficiebus per quadratum orthogonium quia ipsum est maius aliis. Igitur et cetera. *T*

36 fit: sit *MaR*
38 est: dicitur *S* (*cf. T*) / pervenit *HMaR*
39 quidem: que *MaR* / extimant *MaRZm*
40 scilicet *om. MaRS* / a: in *H*
41 illud *om. H* / dicunt: illi dicunt *H*
42 latitudo *om. MaR*
43 Et...estimationibus: quia latitudo ut dicunt *S* / extimationibus *MaRZm*
44 Et[2]...illud: sicut dixit euclides *MaR* / quidem *om. S* / iam: tam (?) H / sapienter: sapiens *H*
46 Altitudo vero: et eadem ratione altitudo non est linea sed *S* / vera *MaR*
47 parte *MaR* / longitudine et latitudine *MaR*
48 quidem *om. MaR*
48–61 Et....nisi: illa mensuratio corporum et superficierum est *S*
48, 49 extimaverunt *MaRZm*
49 erroris eorum *tr. MaR* / eorum *om. H*
52 declaratur cum: declaratum est *H*
53 *post* longitudo *add. PZm* et; *sed om. HMaR*
54 *de* extensionem *scr. P mg. et Zm supra* vel planiciem / extensionem : extensionem vel pleniorem *R* vel pleniorem et extensionem *Ma* / declaratum *H*
55 *ante* quantitate *add. H* tam (?)

other than its own direction and the direction of it as a straight line, then that extension is breadth, and then a surface is formed. Breadth is not, as many people believe, the line which contains the surface in a direction other than its length. If it were that, as they say, a surface would not be that which "has length and breadth only," but [in fact] breadth would then be length as well. This is so because breadth in their judgement is a line and a line is length. Euclid has already wisely stated this when he has said that a line is only length and a surface is that which has length and breadth. Height, in truth, is the extension of a surface in a direction which is neither that of its length nor that of its breadth; evidently, it is extension upward. Those who thought that breadth is a line also believed that height is a line. The revelation of their error in doing so is the same [as before].

Therefore, it has now been shown what length, breadth, and height are. It is declared in addition that these three quantities, i.e., length, breadth, and height, delimit the magnitude of every body and the extension of every surface. It is further declared that there is no body requiring another [or] fourth quantity to delimit its magnitude. Hence after we have

37 longitudo: السطح (*surface*)
37–38 preter....rectitudinem: فى غير جهة الطول (*in other than the direction of its length*)
38–39 Et[1]...superficies *om. Ar.* ((except as suggested in variant for line 37))
46 *post* latitudinem *add. Ar.* فقط (*only*) / superficiei *om. Ar.*
47–48 scilicet...altum *om. Ar.*
51–52 Iam...quod *om. Ar.*
52–53 scilicet...altitudo *om. Ar.*
54 extensionem: انبساط (*extension*) ((See the introduction for a discussion of this.))
54–60 Et....corporum *om. Ar.*

magnitudo terminetur. Postquam ergo ostensum est illud quod narravimus, tunc oportet ut incipiamus ostendere illud cuius volumus narrationem in hoc nostro libro. Et quoniam nos nolumus significare per illud nisi super scientiam amplitudinis superficierum et magnitudinis corporum et scientia in mensuratione quantitatis illius non comparatur nisi per unum superficiale et per unum corporale et unum superficiale per quod comparatur superficies est superficies cuius longitudo est una et cuius latitudo est una et cuius anguli sunt recti, et unum corporale quo comparatur corpus est corpus cuius longitudo est una et cuius latitudo est una et cuius altitudo est una et elevatio quarundam superficierum eius super alias est super rectos angulos, tunc propter illud oportet ut narremus causam quare posite sunt iste due quantitates quibus comparentur amplitudo superficierum et magnitudo corporum. Causa vero in hoc est, quoniam oportet ut quantitas qua mensurantur superficies et corpora sit talis ut cum duplantur continuentur ad invicem taliter ne dimittat in vacuitatibus aliquid de superficie et corpore mensuratis super quod non veniat. Et est necessarium cum hoc ut sit illud super quod venit mensuratio de superficie aut corpore facile dum non prohibetur eius mensuratio. Et neque est aliquid magis ultimum in facilitate discretionis illius quam ut sit iudicium unius quo comparatur superficies aut corpus in singularitate

57 ut: quod *MaR*
57–58 incipiamus...narrationem: ostendamus illud de quo volumus narare (narrare *R*) *MaR*
57, 58 voluimus *H*
60–62 et... superficies[1]: et per unum corporale et unum superficiale et alia (in alio *Ma*) operatio et scientia in mensuratione quantitatis illius non comparatur nisi per unum superficiales (!) per quod mensuratur comparatur *R*
60 *de* scientia *scr. P Zm mg.* in alio, operatio
61 per[2] *om. S* / et unum: unum autem *S*
62 *de* comparatur *scr. P mg. et Zm supra* vel mensuratur / comparatur: mensurantur et comparantur *S*
63 *post* una[2] *add. H* et cuius altitudo est una / anguli: angeli *R*
65 cuius[1] *om. MaR*
66 quorundam *MaR* / eius *om. S* / super alias: super alia *MaR* superficialis *H*
67–69 tunc...corporum *om. S*
67 narrem *H*
69 quoniam oportet *om. H*
70 qua: que *R* / *post* corpora *add. S* ad invicem / cum duplantur: conduplantur *MaR*
71 continetur *H* / taliter *om. H* / in: de *S* / *de* vacuitatibus *scr. PS mg. et Zm supra* vel (*om. S*) in toto illo / vacuitatibus: vel in toto illo vacuitatibus *MaR* / aliqui *R* alliqui *Ma*
72 mensuratur *H*
73 illud: talis (?) *H*
73 superficiei *R*
74 *de* dum... mensuratio *scr. P mg. et add. H MaRS in textu* (*ante* facile): in (*om. S.Zm?*, vel *H*) alio (*om. S Zm?*) taliter (*om. S*) quod non veniat super ipsum mensuratio eius (*om. Zm*) (*et add. S in text.*: secundam aliam literam)
75 aliquid: aliqui *Ma R* sicut *H*

established what we have described, then it is necessary for us to begin to establish that whose exposition we desire in our book. Now since (1) we do not wish to signify by "that" anything except a knowledge of the amplitude of surfaces and the volume of bodies, and [since] (2) a knowledge of the mensuration of such quantities is compared only by means of a unit surface and by means of a unit body—and a unit surface for the comparison of surfaces is a surface whose length is one, whose breadth is one, and whose angles are right angles, while a unit body for the comparison of bodies is a body whose length is one, whose breadth is one, whose height is one, and wherein the elevation of the certain surfaces upon each other is at right angles—for these reasons, then, it is necessary for us to discuss the cause as to why these two [unit] quantities have been posited for the comparison of the amplitude of surfaces and the volume of bodies. The reason is because it is necessary that, when the quantity by which the surface and body are measured is continuously repeated and one unit placed beside another, it be of such a nature that there must not be in the suface and body uncovered spaces over which the measure does not come. It is necessary in addition that it be easy [to distinguish] that part of the surface or body being measured, so that its measurement is not hindered. And nothing is better for easily distinguishing this than a unit measure for the comparison of a surface or a body which maintains the same character whether used singularly or in duplication, so that by a single

60 scientia: العمل (*procedure*) ((Cf. with the introduction, where it is noted that Gerard gives *operatio* as an alternate reading; presumably the first tradition Gerard used had العلم, which would indeed be rendered by *scientia*.))

60–61 comparatur: يتبين بالقياس (*is investigated by comparison*).

67–69 tunc....est *om. Ar.*

71 in vacuitatibus: في خلله (*in its gaps*) ((This is the first tradition followed by Gerard of Cremona and not that of his marginal notation; see the Latin variants. Presumably the alternate tradition read في كلّه—*in its whole*.))

71–72 de superficie et corpore *om. Ar.*

73–74 sit...mensuratio: تحتاج مع ذلك الى ان يكون تميز ما اتى عليه التقدير مما لم يأت عليه سهلا

(*And it is necessary in addition that the distinction of that which its measure covers from that which it does not cover is easy*) ((cf. the alternate reading)) sua in sua duplatione iudicium unum, ut sit labor in discernendo / illud super quod cadit mensuratio ab eo quod non mensuratur unus. Et hoc quidem non est repertum in aliqua figurarum nisi in quadrato. Et illud est quoniam quando duplatur alteratur eius quantitas sed remanet eius quadratura. Et iterum continuatio unius cum altero quando duplantur est continuatio non dimittens in vacuitatibus suis aliquid de eo quod mensuratur per ipsum super quod non veniat.

Iam ergo manifestum est propter quam causam usi sunt uno quadrato de superficiali et corporali loco quantitatis qua comparetur omnis quantitas superficierum et corporum. Causa autem in utendo orthogonio sine aliis non est nisi quoniam mensurans rem oportet ut sit quantitas qua mensuratur veniens super eam et continens eam velociter. Et non est aliqua figurarum quadratarum velocius continens illud quod cum ea mensuratur quam orthogonia, quoniam est maior earum.

Iam ergo manifestum est propter quam causam ponitur quadratum orthogonium ex superficiebus et corporibus esse quantitas qua comparantur superficies et corpora. Et ita verificatur sermo in eo cuius narrationem voluimus in hoc nostro libro. Incipiamus ergo nunc narrare illud quod volumus.

[I.] OMNIS FIGURE LATERATE CONTINENTIS CIRCULUM MULTIPLICATIO MEDIETATIS DIAMETRI CIRCULI IN MEDIETATEM OMNIUM LATERUM FIGURE CONTINENTIS CIRCULUM EST EMBADUM FIGURE LATERATE.

Verbi gratia, sit circulus *ZDH* contentus a figura *ABG* [Fig. 34].

77 sua[1] *om. R* / duplicatione *S*
79–80 Et illud est *om. S*
80 duplicatur *S* / *post* duplatur *add. P* ne *sed om. Zm HMaRST* / alterantur *MaR*/ quantitates *MaR*
81 quadraturam *MaR* / unius: eius *R*
81–82 duplatur *H* duplentur *Zm*
83 veniat: omnia *R* venia *Ma*
84–89 Iam....Et: utuntur autem orthogonio quia *S*
84 quam causam: quartam *MaR*
85 qua: que *MaR* / comparentur *H*
87 quoniam mensurans *tr. H* / mensuras *MaR*
88 qua: que *MaR* / venies *MaR*
89 Et *om. H* / continet *MaR*
90 mensuraretur *MaR* / quam: que *R* / hortogonia *R*
91–95 Iam....volumus *om. S*
91 propter: ob (?) *H* / quam: qua *MaR* / ponitur *ZmHMaR* potest ? *P*
92 hortogonio *R* / qua: que *R*
94 nostro *om. H* / nunc: nam *H*
95 narrare *PZmR* manifestum *H* narare *Ma*
1 omnes *H*
2 multiplicatio *ZmHMaRTS mg. P*
2–3 medietatem: medietate *MaR hic et ubique* ((I have not noted this elsewhere))
5–21 Verbi.... voluimus: Probatur resolvendo figuram continentem circulum in triangulos, ut si fuerit triangulus in-

effort one can distinguish the part covered by the measure from the part not covered. And this is not found in any figure but the square. This is because when it is duplicated, although its quantity is altered, its squareness remains. And furthermore the continuous juncture of one unit with another in the duplication process is such as not to leave any vacant spaces uncovered by the measure.

It has thus become clear why the unit square has been employed as the unit quantity for the comparison of surfaces and bodies. The reason for using a right-angular unit rather than one with other angles is only because, in measuring something, it is necessary that the quantity used for measuring cover and encompass it quickly. And no other figure of a square-like nature [with equal sides] encompasses that which it measures more quickly than one with right angles [i.e., a square or a cube], since it is the largest of such figures [with equal sides].

Hence it has now become evident why, of surfaces and bodies, the right-angled square-like quantity [i.e., a square or a cube] is posited as the quantity by which surfaces and bodies are compared. And so the terminology for that which we wish to recount in our book is verified. Hence let us now begin our desired narrative.

[I.] IN THE CASE OF EVERY [REGULAR] POLYGON CONTAINING A CIRCLE THE MULTIPLICATION OF THE RADIUS OF THE CIRCLE BY HALF THE PERIMETER OF THE POLYGON CONTAINING THE CIRCLE IS THE AREA OF THE POLYGON.

For example, let circle *ZDH* be contained by figure *ABG* [see Fig. 34].

81 eius quadratura: تربيعه (*its squareness*)
81–95 Et....volumus: واظم الاشكال اطربعة احاطه هو القائم الزوايا فهذا هو العلة فى جعل ذلك معيارادون غيره (*And the greatest of figures with a quadrangular perimeter is that which is right angular and this is the reason for using this measure rather than some other one.*) ((Cf. particularly lines 89–90: Et....earum.))
3–4 continentis circulum *om. Ar.*
4 figure laterate *om. Ar.*

[XII.] CUM FUERIT CIRCULUS CUIUS DIAMETER SIT PROTRACTA, ET PROTRAHITUR EX CENTRO IPSIUS LINEA STANS SUPER DIAMETRUM ORTHOGONALITER ET PERVENIENS AD LINEAM CONTINENTEM ET SECATUR UNA DUARUM MEDIETATUM CIRCULI IN DUO MEDIA, TUNC CUM DIVIDITUR UNA HARUM DUARUM QUARTARUM IN DIVISIONES EQUALES QUOTCUNQUE SINT, DEINDE PROTRAHITUR CORDA SECTIONIS CUIUS UNA EXTREMITAS EST PUNCTUM SUPER QUOD SECANT SE LINEA ERECTA SUPER DIAMETRUM ET LINEA CONTINENS ET PRODUCITUR LINEA DIAMETRI IN PARTEM IN QUAM CONCURRUNT DONEC CONCURRUNT ET PROTRAHUNTUR IN CIRCULO CORDE EQUIDISTANTES LINEE DIAMETRI EX OMNIBUS PUNCTIS DIVISIONUM PER QUAS DIVISA EST QUARTA CIRCULI, TUNC LINEA RECTA QUE EST INTER PUNCTUM SUPER QUOD EST CONCURSUS DUARUM LINEARUM PROTRACTARUM ET INTER CENTRUM CIRCULI EST EQUALIS MEDIETATI DIAMETRI ET CORDIS QUE PROTRACTE SUNT IN CIRCULO EQUIDISTANTIBUS DIAMETRO CONIUNCTIS.

Verbi gratia, sit circulus *ABG*, cuius diameter sit linea *AG* et cuius centrum sit punctum *D* [Fig. 46]. Et protrahatur ex eo linea *DB* erecta super lineam *AG* orthogonaliter et dividat arcum *ABG*
c. 2 in duo media. Et / dividam quartam circuli super quam sunt *A*, *B* in divisiones equales quot voluero et ponam eas divisiones *AZ*, *ZL*, *LB*. Et protraham cordam *BL* et faciam ipsam penetrare. Et elongabo iterum lineam *AG*, que est diameter, secundum rectitudinem donec

1 [XII]: 15 *mg. MaR*
1–76 Cum....voluimus *om. S*
2 protrahatur *H*
3 supra *H*
3–4 proveniens *H*
6 dividatur *H* / harum duarum *tr. H* earum duarum *Zm*
9–10 *de* super...continens (*scr. P mg. et Zm supra* (*et add. HMa* ante* super): in (*om. HMa*) alio (*om. HMa*), sectionis medietatis diametri (diametris *Ma*) circuli (*add. Ma mg.* in alio) erecte (erecti *H*) cum linea continente. ((**Ma* puts the phrase in a box.))
12 in: a *H*
15 *de* recta *scr. P mg. Ma mg. et Zm supra* (*et add. Ma post* recta): in alio, divisa
17 protractarum *om. H*
18 corde *H*
25 quot *PZm* quod *H* quo *Ma* / eas easdem *H*

[XII.] WHEN THERE IS A CIRCLE WHOSE DIAMETER IS DRAWN AND THERE IS DRAWN FROM ITS CENTER A LINE PERPENDICULAR TO THE DIAMETER AND TERMINATING AT THE CIRCUMFERENCE SO THAT ONE OF THE TWO HALVES OF THE CIRCLE IS BISECTED, AND THEN WHEN ONE OF THE TWO QUADRANTS IS DIVIDED INTO ANY NUMBER OF EQUAL PARTS AND THE CHORD OF THE SEGMENT, ONE OF WHOSE EXTREMITIES IS THE POINT OF INTERSECTION OF THE LINE ERECTED ON THE DIAMETER AND THE CIRCUMFERENCE, IS PRODUCED WHILE THE DIAMETER IS PRODUCED IN THE DIRECTION OF THEIR INTERSECTION UNTIL THE TWO LINES INTERSECT, AND THERE ARE DRAWN IN THE CIRCLE FROM THE POINTS AT WHICH THE QUADRANT ARC OF THE CIRCLE IS DIVIDED CHORDS PARALLEL TO THE DIAMETER, [IF ALL OF THIS IS DONE,] THEN THE STRAIGHT LINE BETWEEN THE POINT WHERE THE TWO EXTENDED LINES MEET AND THE CENTER OF THE CIRCLE IS EQUAL TO THE SUM OF THE RADIUS PLUS THE CHORDS DRAWN IN THE CIRCLE PARALLEL TO THE DIAMETER.

For example, let there be a circle *ABG* whose diameter is line *AG* and whose center is point *D* [see Fig. 46]. And from the center let line *DB* be drawn perpendicular to *AG*, thus bisecting arc *ABG*. And I shall divide the quadrant *AB* into as many equal parts as I wish, and I shall assume these parts to be *AZ*, *ZL*, *LB*. And I shall draw chord *BL* and make it continue. And I shall also extend line *AG*, the diameter,

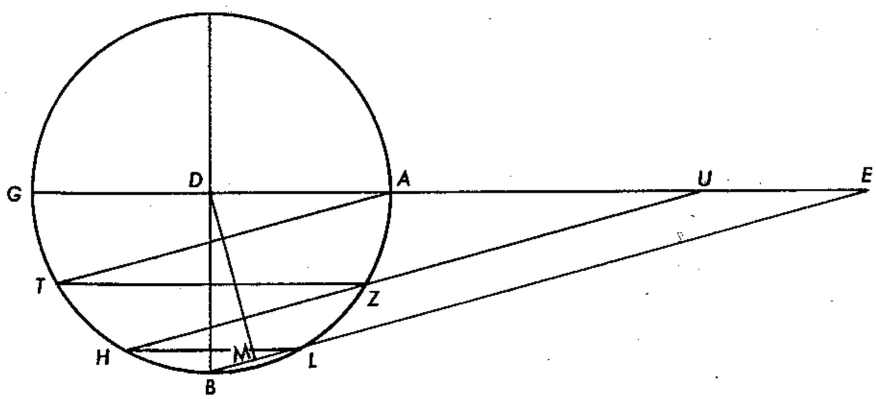


Fig. 46

1–20 Cum...coniunctis *om. Ar.* 23–24 et... media *om. Ar.*
23 lineam AG: القطر (*the diameter*)

concurrant super punctum *E*. Et protraham ex duobus punctis *Z*, *L* duas cordas *ZT*, *LH* equidistantes diametro *AG*. Dico ergo quod linea *DE* est equalis medietati diametri et duabus cordis *ZT*, *LH* coniunctis, cuius hec est demonstratio.

Protraham lineam *TA* et protraham lineam *HZ* et faciam ipsam penetrare secundum rectitudinem donec occurrat linee *EG* super *U*. Et similiter faciam, si quarta circuli super quam sunt *A*, *B* fuerit divisa in divisiones plures istis divisionibus. Linee ergo *TZ*, *HL* sunt equidistantes, quoniam taliter sunt protracte. Et linee *TA*, *HU*, *BE* sunt equidistantes propterea quod due divisiones *TH*, *HB* sunt equales duabus divisionibus *AZ*, *ZL*. Ergo quadratum *TAUZ* est equidistantium laterum. Ergo linea *TZ* est equalis *AU*. Et iterum quadratum *HUEL* est equidistantium laterum. Ergo linea *HL* est equalis *UE*. Ergo tota linea *ED* est equalis duabus lineis *TZ*, *HL* et linee erecte que est medietas diametri coniunctis.

Si ergo nos protraxerimus in hac figura lineam ex centro et secuerit unam cordarum divisionum quarte circuli in duo media, sicut lineam *DM*, tunc secatur linea *LB* super duo media super punctum *M* in duo media. Tunc iam scietur ex eo quod narravimus in hac figura quod multiplicatio medietatis corde *BL* in duas cordas equidistantes diametro et in medietatem diametri coniunctas est minor multiplicatione medietatis diametri in se et maior multiplicatione linee *DM* in se, propterea quod triangulus *DMB* est similis triangulo *EDB* et est similis triangulo *EMD*. Ergo proportio linee *MB* ad *BD* est sicut proportio

28 Z, L: L Z *H*
32 *ante* TA *del. P* TH / HZ: HT *H*
33 occurrat: concurrant *H* / U: N *H hic et ubique*
34 si quarta: super quartam *H*
36 protracta *H* / linee *PZm* linea *HMa* / TA, HU: TA HN TA HN *H*
37 *de* due divisiones *scr. P mg. et Zm supra* (*ed add. Ma ante* due): in (*om. Ma*) alio (*om. RMa*) duo arcus
38 *de* duabus divisionibus *scr. P mg. et Zm supra* (*et add. Ma* ante* duabus): duobus arcubus. ((**Ma* puts the phrase in a box.))
41 ED est *om. H*
44 *de* cordarum... quarte *scr. P mg. Zm.mg.* (*et add. HMa post* secuerit): in alio cordam (corda *Ma*) ex sectionibus (sactionibus! *Ma*) cordarum quarte quarte: quarti *H* / sicut: super *H*
47, 48 in *om. H*

30 *post* diametri *add. Ar.* – ج ا – (*GA*)
31 cuius.... demonstratio *om. Ar.*
32 et... lineam *om. Ar.* / ipsam: – ح ز – (*HZ*)
33 secundum rectitudinem *om. Ar.*
34 quarta.... B *om. Ar.*
35 istis divisionibus *om. Ar.* / TZ, HL: – ج ه – ط ز – ح ل – (*GE*, *TZ*, *HL*)
36 quoniam.... protracte *om. Ar.*

rectilinearly until they [i.e., BL and AG] meet at point E. And I shall draw from the two points Z and L the two chords ZT and LH parallel to diameter AG. I say, therefore, that line DE = radius + $ZT + LH$.

Proof: I shall draw line TA and I shall draw line HZ, continuing the latter rectilinearly until it meets line EG at U. I shall proceed in a similar way if the quadrant AB is divided into more parts than these. Hence lines TZ and HL are parallel, since they are so drawn. And lines TA, HU, and BE are parallel, since $TH = AZ$ and $HB = ZL$. Therefore, the quadrilateral $TAUZ$ is a parallelogram. Therefore, line $TZ = AU$. And also quadrilateral $HUEL$ is a parallelogram. Therefore, line $HL = UE$. Therefore, the whole line $ED = TZ + HL$ + radius.

Hence in this figure we draw a line, e.g., line DM, from the center thus bisecting one of the chords of the quadrant, LB being the line bisected at point M. Then it will already be known, from what we have recounted concerning this figure, that the multiplication of (1) 1/2 chord BL by (2) the sum of the two chords parallel to the diameter plus the radius is less than the square of the radius and is greater than DM^2, because of the fact that the three triangles DMB, EDB, and EMD are similar. Therefore, $MB/BD = DB/BE$. Hence, $DB^2 = MB \cdot BE$, DB being the radius.

39–40 Et…laterum *om. Ar.*
41 tota linea *om. Ar.* / duabus lineis *om. Ar*
41–22 et….diametri: – ا د – (*AD*)
42 *post* coniunctis *add. Ar.* وذالك ما اردناه
(*And that is what we wished.*)
43–47 in…quod: – د م – عمودا على وتر – ب ل –
(*DM perpendicular to chord BL.*)
47 corde *om. Ar.*
47–48 in…coniunctas: – د ہ – (*DE*)
50–65 et….se: لكون زاويتى – د م ب – ه ب د – قائمتين وزاوية – ب – مشتركة ونسبة – ب م – الى – م د – كنسبة – ب د – الى – د ه – فب م – اعنى نصف – ب ل – فى – د ه – مساو – لب د – فى – د م – و – ب د – فى – د م – اصغر من مربع – ب د – واعظم من مربع – م د – فاذا نصف – ب ل – فى نصف القطر وفى وترى – ط ز – ح ل – جميعا اصغر من مربع نصف القطر واعظم من مربع – د م –
(*For the two angles DMB, EBD are right angles and angle B is common. And* $BM/MD = BD/DE$, *and so* $BM \cdot DE = BD \cdot DM$, *or* [$\frac{1}{2} BL \cdot DE = BD \cdot DM$], $BD \cdot DM < BD^2$, *and* $BD \cdot DM > MD^2$. *And so* [$\frac{1}{2} BL \cdot$ *radius* $\cdot (TZ + HL)$] $<$ *radius*^{2} *and* $> DM^2$.)

DB ad *BE*. Et propter illud erit multiplicatio linee *DB*, que est medietas diametri, in se equalis multiplicationi linee *MB* in lineam *BE*. Verum linea *BE* est longior duabus cordis *ZT*, *LH* et medietate diametri coniunctis, propterea quod iste coniuncte sunt *DE*, et linea *BE* est longior *DE*. Ergo multiplicatio linee *MB* in duas cordas *ZT*, *LH* et in medietatem diametri coniunctas est minor multiplicatione medietatis diametri in se. Et quoniam triangulus *DMB* est similis triangulo *EMD*, erit proportio *BM* ad *MD* sicut proportio *MD* ad *ME*. Et similiter erit multiplicatio linee *BM* in lineam *ME* equalis multiplicationi linee *MD* in se. Sed linea *ME* est minor duabus cordis *ZT*, 60v c. 1 *LH* et medietate / diametri coniunctis, propterea quod iste omnes sunt equales linee *DE*, et linea *DE* est longior *EM*. Ergo multiplicatio *MB* in duas cordas *ZT*, *LH* et in medietatem diametri coniunctas est maior multiplicatione *DM* in se.

Iam ergo ostensum est quod in omni circulo in quo protrahitur ipsius diametrus deinde dividitur una duarum medietatum ipsius in duo media, postea dividitur una duarum quartarum in divisiones equales quotcunque fuerint et protrahuntur ex punctis divisionum omnium corde in circulo equidistantes diametro, tunc multiplicatio medietatis corde unius sectionum quarte circuli in medietatem diametri et in omnes cordas que protracte sunt in circulo equidistantes diametro coniunctim est minor multiplicatione medietatis diametri in se et maior multiplicatione linee que egreditur ex centro et pervenit ad unam cordarum divisionum quarte circuli et dividit eam in duo media in se. Et illud est quod declarare voluimus.

52 *de* Et *scr. P mg. et Zm supra* in alio, similiter / Et *PZm* et similiter *H* similiter et *Ma* ((but *similiter* is in a box in Ma))
54 Verum *PZm om. H* Verum tamen *Ma*
54–55 LH...sunt *om. H*
56 LH: HL *H*
57–58 medietatis *Zm HR, mg. P mg. Ma*
59 BM: MB *Zm*
61 est minor *ZmMa bis P* est maior minori *H*
67, 68 dividatur *H*
70 corde: cordarum *H* / in: in cum (?) *H*
75 divisionem *H* / dividat *H*
76 declarare: demonstrare *H*

---

69 omnium *om. Ar.*
71 quarte circuli *om. Ar.*
72–73 que...coniunctim *om. Ar.*
75 quarte...media *om. Ar.*
76 quod...voluimus: المطلوب (*that which is sought*)

Now line $BE > (ZT + LH + BD)$, since $(ZT + LH + BD) = DE$ and $BE > DE$. Hence, line $MB \cdot (ZT + LH + BD) < BD^2$. And since $\triangle DMB$ is similar to $\triangle EMD$, $BM / MD = MD / ME$. And similarly $BM \cdot ME = MD^2$. But line $ME < (ZT + LH + BD)$, since

$$(ZT + LH + BD) = DE \text{ and } DE > EM.$$

Therefore,

$$MB \cdot (ZT + LH + BD) > DM^2.$$

Therefore it has now been demonstrated that in every circle where the diameter is drawn and one of the two halves of the circle is bisected and one of the two quadrants [thus formed] is then divided into any number of equal parts and from the [dividing] points of the parts are drawn chords in the circle parallel to the diameter, then the multiplication of one half of the chord of one of the segments of the quadrant by the sum of the radius plus all the chords drawn in the circle parallel to the diameter is less than the square of the radius and greater than the square of the line going out from the center which meets and bisects the chord of one of the parts of the quadrant. And this is what we wished to show.

[XIII.] CUM CECIDERIT IN MEDIETATE SPERE CORPUS QUOD CONTINEAT MEDIETAS SPERE ET FUERIT CORPUS COMPOSITUM EX PORTIONIBUS PIRAMIDUM COLUMPNARUM QUOTCUNQUE FUERINT, ET FUERIT SUPERFICIES SUPERIOR CUIUSQUE PORTIONIS EXISTENS BASIS PORTIONIS QUE EST SUPER EAM, ET FUERINT SUPERFICIES BASIUM PORTIONUM OMNIUM EQUIDISTANTES, ET FUERIT BASIS PORTIONIS INFERIORIS IPSA BASIS MEDIETATIS SPERE, ET FUERIT PORTIO SUPERIOR PIRAMIDIS PIRAMIS CAPITIS, ET PUNCTUM CAPITIS EIUS EST POLUS MEDIETATIS SPERE, ET FUERINT LINEE RECTE QUE EGREDIUNTUR EX OMNIBUS BASIBUS PORTIONUM AD ILLUD QUOD EST ALTIUS IN EIS SECUNDUM RECTITUDINEM EQUALES, ET CUM CECIDERIT IN CORPORE MEDIETAS SPERE QUAM CONTINEAT CORPUS, ET FUERIT SUPERFICIES BASIS HUIUS MEDIETATIS SPERE POSITA IN SUPERFICIE BASIS CORPORIS: TUNC EMBADUM SUPERFICIEI HUIUS CORPORIS ERIT MINUS DUPLO EMBADI SUPERFICIEI BASIS MEDIETATIS SPERE QUE CONTINET CORPUS ET MAIUS DUPLO EMBADI SUPERFICIEI BASIS MEDIETATIS SPERE QUAM CONTINET CORPUS.

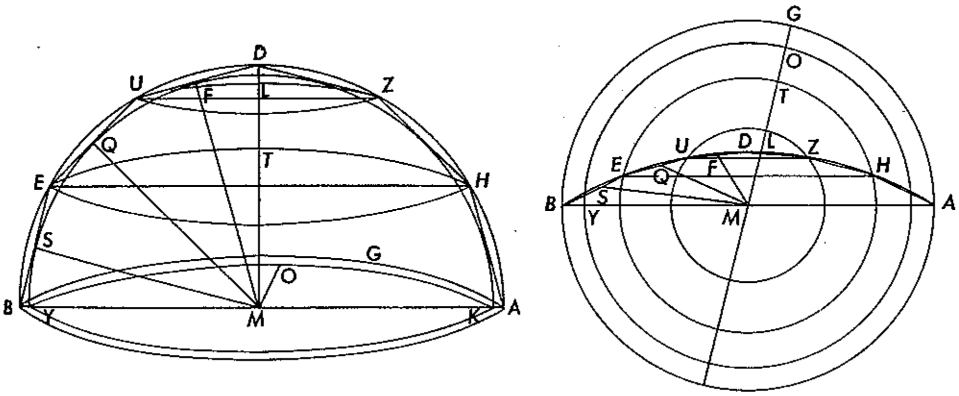
Verbi gratia, sit medietas spere *ABGD* [Fig. 47]. Et circulus *ABG* sit circulus magnus, et eius superficies sit basis medietatis spere *ABGD*.

1 [XIII]: 16 *mg. Ma mg. R*
1–153 Cum....eius *om. S*
5 cuiusque: cuiuscunque *H*
7 equidistantes *PZm* extremitates *H*
13 altius *PZm* alteri *H* alterum *Ma*
18 erit: eius *H*
19 medietatis *HR om. PZmMa*
19–21 que...spere *om. H*
20–21 et...corpus *om. R*
22 ABGD: ABG *H*

6–7 et...equidistantes *tr. Ar. post* spere *in linea 11*
14 et cum: ثُمّ (*then*)
17 corporis: النصف الاول (*of the first half*)
18–19, 20 embadi superficiei *om. Ar.*
19–20 que...corpus: الاول (*of the first*)
21 quam...corpus: الثانية (*of the second*)
22 medietas *om. Ar.*
23 et...ABGD *om. Ar.*

[XIII.] WHEN THERE IS A BODY WHICH FALLS WITHIN A HEMISPHERE—AND WHICH [CONSEQUENTLY] THE HEMISPHERE CONTAINS—AND THE BODY IS COMPOSED OF ANY NUMBER OF SEGMENTS OF CONES* SUCH THAT THE UPPER [PLANE] SURFACE OF ANY SEGMENT IS THE BASE OF THE SEGMENT [IMMEDIATELY] ABOVE IT AND THE BASE SURFACES OF ALL THE SEGMENTS ARE PARALLEL, AND SUCH THAT THE BASE OF THE BOTTOM SEGMENT IS THE BASE OF THE HEMISPHERE, WHILE THE TOP SEGMENT IS ITSELF A CONE WITH ITS VERTEX A POLE OF THE HEMISPHERE, AND SUCH THAT THE SLANT HEIGHTS OF THE SEGMENTS ARE EQUAL, AND WHEN THERE IS INSCRIBED WITHIN THE BODY A HEMISPHERE WHICH THE BODY CONTAINS AND WHOSE BASE IS PLACED WITHIN THE SURFACE OF THE BASE OF THE BODY—[WHEN ALL OF THIS IS TRUE,] THEN THE SURFACE AREA OF THE BODY IS LESS THAN DOUBLE THE AREA OF THE BASE OF THE HEMISPHERE CONTAINING THE BODY AND MORE THAN DOUBLE THE AREA OF THE BASE OF THE HEMISPHERE WHICH THE BODY CONTAINS.

For example, let there be a hemisphere *ABGD* [see Fig. 47], and *ABG* a great circle of it, whose surface is the base of the hemisphere *ABGD*.



[Reconstructed] Fig. 47 [As given in Arabic texts and in MSS *H* and *P*]

Note: *AMB* is drawn off center in MS *P*, and lines *MF*, *MQ*, and *MS* are incorrectly drawn in MS *H*.

* As in Proposition XI, I have used here the expression "segment of a cone" to translate "portio piramidis columpne." Note that the authors use this to stand both for a frustum of a cone and for a small cone which stands on the uppermost frustum.

Et polus huius circuli magni sit punctum *D*. Et signabo in medietate spere in primis corpus compositum ex portionibus quot voluero piramidum columpnarum secundum modum quem narravimus. Et ponam corpus in hac descriptione compositum ex tribus portionibus, que sint portiones *ABGH*, *EHTZ*, *ULZD*. Et basis corporis et basis medietatis spere *ABGD* est una. Et est superficies circuli *ABG*. Et caput c. 2 / corporis est punctum *D*. Et est polus medietatis spere *ABGD*. Dico ergo quod embadum superficiei corporis *ABGD* compositi ex embadis superficierum trium portionum piramidum, quarum una est superficies que elevatur ex circulo *ABG* secundum rectitudinem ad circulum *HTE* et superficies alia est illa que elevatur ex circulo *HTE* secundum rectitudinem ad circulum *ULZ* et superficies tertia est que elevatur ex circulo *ULZ* secundum rectitudinem ad punctum *D*, est minus duplo embadi superficiei circuli *ABG*, cuius hec est demonstratio.

Protraham in spera *ABGD* medietatem circuli qui est ex circulis magnis qui cadunt in spera transeuntis super polum qui est punctum *D*, qui sit arcus *ADB*. Et protraham lineam *AB*, que sit diameter spere, et dividam eam in duo media super punctum *M*. Et notum est quod punctum *M* est centrum spere. Et protraham duas lineas *HE*, *UZ*. Et notum est etiam quod utreque sunt equidistantes, et equidistantes linee *AB*, propterea quod linee *AB*, *HE*, *ZU* sunt differentie communes super quas superficies circuli *ADB* secat superficies tres equidistantes, scilicet superficies circulorum *ABG*, *ETH*, *ULZ*. Et manifestum est quod linee *AB*, *HE*, *UZ* sunt corde circulorum *ABG*, *ETH*, *ULZ*, qui sunt bases portionum ex quibus componitur corpus *ABGD*, propterea quod polum horum circulorum omnium est punctum *D* super quod transit medietas circuli *ADB*. Et protraham in

24 *de* circuli *scr. P mg. et Zm supra* in alio, spere / *ante* circuli *add. Ma* spere
26 quam *H*
27–28 que...portiones *ZmMa mg. P om. H*
28 ABGH...ULZD *scr. et del H* / EHTZ *corr. ex* EHLZ *in MSS*
30 *post* corporis *add. P et del.* eius
31 embadum *PZm* embado *H* embade *Ma*
34 *de* HTE[1] *scr. P mg.* in alio / HTE[1, 2]: ELH *Zm et Zm scr. supra* in alio, HTE / HTE[1]: pro alio habere (?) *H*
34–35 HTE[2]...ULZ *om. H*
34 HTE[2] *de scr. P mg.* in alio
35, 36 *de* ULZ *scr. P mg.* in alio / ULZ: UTZ *Zm et scr. Zm supra* in alio ULZ
35 U-: N- *H hic et ubique*
39 qui: quod *H*
42 notum: necessarium *H*
43 quod *bis H*
44 et equidistantes *om. H*
46 ADB: ABD *Zm*
48 corde: diametri *Zm*
49 corpus: totum corpus *H*

And the pole [of the axis] of this great circle is point *D*. And I shall first describe in the hemisphere a body composed of as many segments as I wish in the way that we have recounted. I shall posit as the body so described the body composed of three segments: the segments *ABGH*, *EHTZ*, and *ULZD*. And the base of this body is also the base of hemisphere *ABGD*, this base being the surface of circle *ABG*. And the vertex of the body is point *D*, which is also the pole of the hemisphere *ABGD*. I say, therefore, that the surface area of body *ABGD* which is composed of three segments of cones—i.e., (1) the surface extended rectilinearly [i.e., the lateral surface] between circle *ABG* and circle *HTE*, and (2) the surface extended rectilinearly between circle *HTE* and circle *ULZ*, and (3) the surface extended rectilinearly between circle *ULZ* and point *D*—is less than double the surface area of circle *ABG*.

Proof: I shall draw in [hemi]sphere *ABGD* a semicircle of one of the great circles of the sphere [and this is the semicircle] passing through polar point *D*. This semicircle is the arc *ADB*. And I shall draw line *AB* as a diameter of the sphere and I shall bisect it at point *M*. And it is known that *M* is the center of the sphere. And I shall draw the two lines *HE* and *UZ*. It is known also that they are both parallel [to each other] and parallel to line *AB*, since lines *AB*, *HE*, and *ZU* are the common sections of circle *ADB* and the three parallel surfaces of circles *ABG*, *ETH*, and *ULZ*. And it is evident that lines *AB*, *HE*, *UZ* are chords of circles *ABG*, *ETH*, and *ULZ* which are [themselves] the bases of the segments of which body *ABGD* is composed, since the pole [of the axis] of all these circles is point *D*, through which the semicircle *ADB* passes. And I shall draw

24–36 Et[2]....D: وليكون فيه مجسم على ما وصفنا مركب من ثلث قطع اولاها يرتفع من دائرة – ا ب ج – الى دائرة – ه ط ح – والثانية ترتفع منها الى دائرة – و ل ز – والثالثة ترتفع منها الى نقطة – د – نقول فالسطوح المستديرة المحيط بهذا الجسم جميعا (*And according to what we have described let there be* [*inscribed*] *in it a body compounded of three segments. The first extends from circle ABG to circle ETH and the second extends from it* (*ETH*) *to circle ULZ, and the third extends from it* (*ULZ*) *to point D. And so we say that the sum of the areas of the conic segments comprising the body*)

37–38 cuius...demonstratio *om. Ar.*

39 *ante* spera *add. Ar.* نصف (*half*)

41 arcus *om. Ar.*

42 punctum *om. Ar.*

42–43 Et...spere *om. Ar.*

44 notum...quod *om. Ar.*

45 linee...ZU *om. Ar.*

47 equidistantes....superficies *om. Ar.*

47–48 ABG....corde: وهما قطر (*And they are diameters*)

48 ABG *om. Ar.*

49–51 qui....ADB *om. Ar.*

51–52 in...portionibus *om. Ar.*

omnibus portionibus ex basibus earum ad ipsarum altiora lineas rectas, que sint linee *BE*, *EU*, *UD*. Et notum est quod ipse sunt equales, propterea quod ita posite sunt. Ergo medietatis circuli *ADB* iam protracta est diametrus et est *AB*, et divisa est medietas circuli in duo media super *D*. Et divisus est arcus *DB* in divisiones equales, que sunt arcus *BE*, *EU*, *UD*, et protracte sunt ex duobus punctis *E*, *U* due corde equidistantes diametro, que sunt *UZ*, *EH*. Ergo multiplicatio medietatis unius cordarum *BE*, *EU*, *UD*, quecunque fuerit, in duas lineas *UZ*, *HE* et in medietatem linee *AB* coniunctim est minor multiplicatione medietatis linee *AB* in se, propter illud cuius premisimus demonstrationem. Et iterum corpus *ABGD* est compositum ex portionibus piramidum columpnarum, et bases portionum omnium sunt equidistantes, et portio superior habet caput quod est piramis, et linee recte que protrahuntur in omnibus portionibus ex basibus earum ad earum superiora secundum rectitudinem sunt equales. Ergo propter illud erit multiplicatio linee unius earum que protrahuntur ex basibus portionum ad superiora earum secundum rectitudinem in medietatem linee continentis basim portionis inferioris et in omnes lineas continentes bases portionum que sunt super portionem infe-
61r c. 1 riorem est embadum superficiei corporis, sicut o/stendimus in illis que sunt premissa. Ergo multiplicatio linee *BE* in duos circulos *ULZ*, *ETH* et in medietatem circuli *ABG* coniunctim est embadum superficiei corporis *ABGD*. Verum multiplicatio linee *BE* in duos circulos

53 sint: sunt *Zm*
60 et: etiam *H*
62 est *PZm* eorum *H* ex *Ma*
64 equidistantes: extremitates *H*
67 erit: est *H*
68 in *om. H*
69 basim *PZm mg. H* basis *Ma*
71–72 illis...premissa: premissis *H*
74 *add. Zm mg. + quod refert ad sequentes lineas in fol. 84r:*
Set multiplicatio linee EB in multiplicationen linee ZU in proportionem circumferentie circuli ULZ ad lineam ZU et in multiplicationem linee EH in proportionem circumferentie circuli ETH ad lineam EH et in multiplicationem medietatis diametri AB in proportionem medietatis circumferentie circuli ABG ad lineam MB que est medietas diametri est superficies corporis ABGD. Set multiplicatio EB in multiplicationem linee ZU in proportionem circumferentie circuli UZL ad lineam ZU est sicut multiplicatio EB in lineam ZU et eius quod provenit in proportionem circumferentie circuli UZL ad diametrum UZ. Ergo multiplicatio linee EB in lineas ZU, EH, MB et eius quod provenit in proportionem circumferentie circuli UZL ad lineam ZU est superficies corporis ABGD. Ergo et multiplicio medietatis linee EB in predicta omnia est medietas superficiei corporis ABGD. Et multiplicatio linee BS in lineas UZ, EH, MB est minus multiplicatione MB in se ipsam. Ergo multiplicatio linee BS in lineas ZU, EH, MB et eius quod provenit in proportionem circumferentie circuli UZL ad lineam ZU est minus

in all of the [surfaces of the] segments from their bases to those [bases immediately] above them straight lines, [i.e., slant heights] and these are the lines *BE*, *EU*, *UD*. And it is known that these lines are equal, since they were posited to be so. Hence a diameter of semicircle *ADB* has already been drawn and it is *AB*. And the semicircle [i.e., arc *ADB*] has been bisected at *D*. And arc *DB* has been divided into equal parts, namely, the arcs *BE*, *EU*, and *UD*. And from the two points *E* and *U* the two chords *UZ* and *EH* have been drawn parallel to the diameter. Therefore, the multiplication of one half of any one of the chords *BE*, *EU*, and *UD* by the sum ($UZ + HE + 1/2\ AB$) is less than $(1/2\ AB)^2$, as we have demonstrated before [in Proposition XII]. Furthermore, body *ABGD* is composed of segments of cones in such a way that the bases of all the segments are parallel, the upper segment is a cone, and the straight lines drawn in all [the surfaces of] the segments from their bases to their upper [plane surfaces] rectilinearly [i.e., the straight lines constituting the slant heights] are equal. Therefore, as we demonstrated before [in Proposition XI], the multiplication of (1) one of these lines [i.e., slant heights] drawn from the bases of the segments to the upper [plane surfaces] rectilinearly by (2) the sum of one half the circumference of the base of the lowest segment plus all the circumferences of the bases of the segments above the lowest one is [equal to] the surface area of the body. Therefore,

$$BE \cdot (\text{circum } ULZ + \text{circum } ETH + 1/2 \text{ circum } ABG) = \text{surf area body } ABGD.$$

53 que...linee *om. Ar.* / notum...quod *om. Ar.*
54–58 Ergo...EH *om. Ar.*
59 cordarum...UD: منها (*of them*)
60 duas lineas *om. Ar.*
61–96 propter....spere: وايضا سطح واحد منها في نصف محيط دائرة – ا ب ج – وفي محيطي دائرة – ح ه ط – ز و ل – جميعا مثل السطح المحيط بالمجسّم لما مرّ وسطح واحد منها في نصف

– ا ب – وفي – ه ح – و ز – جميعا. ثم الحاصل فيما اذا ضرب فيه القطر حصل المحيط مساويا لسطح واحد منها في نصف محيط دائرة – ا ب ج – وفي محيطي دائرتي – ح ه ط – ز و ل – جميعا اعني السطح المحيط بالمجسّم وهو اقل من ضعف الحاصل من ضرب مربع نصف – ا ب – في ما اذا ضرب فيه القطر حصل المحيط ومربع نصف

ULZ, *ETH* et in medietatem circuli *ABG* est equalis ei quod fit ex multiplicatione linee *BE* in duas lineas *UZ*, *EH* et in medietatem linee *AB* coniunctim et multiplicationi eius quod agregatur inde in quantitatem in quam cum multiplicatur diameter est illud quod agregatur inde ipsa linea circumdans, propterea quod linee *UZ*, *EH*, *AB* sunt diametri circulorum *ULZ*, *ETH*, *ABG*. Ergo multiplicatio linee *BE* in duas lineas *UZ*, *EH* et in medietatem linee *AB* coniunctim et multiplicatio eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans est embadum superficiei corporis *ABGD*. Sed multiplicatio medietatis linee *BE* in duas lineas *UZ*,*EH* et in medietatem linee *AB* coniunctim et multiplicatio eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans est equalis medietati superficiei corporis *ABGD*. Et ipsa est minor multiplicatione medietatis linee *AB* in se et multiplicatione eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans. Sed multiplicatio medietatis linee *AB* in se et multiplicatio eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans est embadum superficiei circuli *ABG*, propterea quod linea *AB* est eius diameter. Ergo superficies circuli *ABG* qui est basis corporis et medietatis spere que continet

multiplicatione linee MB in se et eius quod provenit in proportionem circumferentie circuli UZL ad lineam ZU. Set multiplicatio quadrati linee MB in proportionem circumferentie circuli UZL ad lineam ZU que est eadem proportioni circumferentie circuli ABG vel medietatis eius ad diametrum AB vel medietatem eius est superficies circuli ABG quia si posuero superficiem que provenit ex multiplicatione medietatis diametri AB in medietatem circumferentie circuli ABG que est equalis superficiei circuli ABG et quadravero diametri medietatem erit proportio illius superficiei equalis circulo ABG ad quadratum medietatis diametri AB sicut proportio medietatis circumferentie ad medietatem diametri. Cum ergo diviserimus superficiem illam equalem circulo per quadratum medietatis diametri proveniet proportio circuli ad quadratum que est proportio medietatis circumferentie ad medietatem diametri. Si ergo multiplicaverimus quadratum illud in illam proportionem proveniet circuli supreficies. Set multiplicatio quadrati in proportionem est maior multiplicatione medietatis linee EB in lineas ZU, HE, MB et eius quod provenit in proportionem quod est equale medietati superficiei corporis ABGD. Ergo duplum circuli est maius superficie corporis.

75 ei: etiam (?) *H*
77 inde *om. H*
78 inde *PZm* in *HMa*
81 et[1] *om. H*
84 BE: EB *H*
85 medietatem *ZmHMa* medietate *P*
90–91 *post* circumdans *add. H, Mamg. lineas 97–118* ("Quod....forma")
92 cum *om. H*

Now

$$BE \cdot (\text{circum } ULZ + \text{circum } ETH + 1/2 \text{ circum } ABG) = BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi^*,$$

since UZ, EH, and AB are the diameters of circles ULZ, ETH, and ABG. Therefore, $BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi =$ surf area body $ABGD$. But $1/2\ BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi = 1/2$ area body $ABGD$, and so $1/2\ BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi < (AB/2)^2 \cdot \pi$. But $(AB/2)^2 \cdot \pi =$ area circle ABG, since line AB is its diameter. Therefore, circle ABG,

* I have used here the modern symbol π to stand for the phrase "the quantity which when multiplied by the diameter produces the circumference."

– ا ب – فيما اذا ضرب فيه القطر مساو لسطح
الدائرة لأن ضرب نصف – ا ب – فيما اذا ضرب
فيه القطر حصل المحيط هو نصف المحيط وضربه
مرّة اخرى فى نصف – ا ب – هو سطح الدائرة
فالسطح المحيط بالمجسّم اقل من ضعف * سطح
دائرة – ا ب ج –

((* ضعف in Paris MS, نصف in printed text.)) (*And also the product of one of them and ½ the circumference of circle ABG and the circumferences of circles HET, ZUL together is equal to the area of the surface of the body, according to what has passed* [*in Proposition IX*]; *and the product of one of them and ½ AB and (EH + UZ), and that whole product multiplied by* π, *is equal to the product of one of them and ½ the circumference of circle ABG and the sum of the circumferences of circles HET and ZUL, i.e., to the area of the surface of the body. And this is less than* $2\pi\,(AB/2)^2$ and $\pi(AB/2)^2$ *is equal to the area of the circle, because* $(AB/2) \cdot \pi$ *is equal to ½ the circumference of the circle and* $(AB/2) \cdot \pi \cdot (AB/2)$. *One half AB is the area of the circle. And so the area of the surface of the body is less than double the area of circle ABG.* ((As in the translation of Gerard's text, I have throughout rendered by the symbol π the phrase "that quantity which when multiplied by the diameter produces the circumference.))

corpus est plus medietate embadi corporis cadentis in medietate spere.

[Quod multiplicatio medietatis diametri in se et eius quod provenit in quantitatem in quam cum multiplicatur diameter provenit linea circumdans circuli sit equale superficiei circuli, ita ostenditur. Quoniam ponam *ET* equalem medietati circumferentie et *EZ* equalem medietati diametri (Fig. 48), et unam multiplicabo in alteram, erit ergo superficies *ZT* equalis superficiei circuli, et super *ZE* constituam quadratum, quod sit *ZL*, et ponam quod quantitas in quam cum multiplicatur diameter provenit circumferentia sit quantitas *RU*. Et quod diameter cum multiplicatur in *RU* proveniet circumferentia, ergo circumferentia cum dividitur per diametrum provenit *RU*. Ergo *RU* est proportio circumferentie ad diametrum. Sed proportio totius ad totum est sicut medietatis ad medietatem et *ET* equatur medietati circumferentie et *EL* medietati diametri. Ergo proportio *TE* ad *EL* est *RU*. Sed proportio *TE* ad *EL* est sicut proportio *TZ* ad *ZL*. Ergo proportio *TZ* ad *ZL* est *RU*. Ergo *TZ* cum dividetur per *RU* proveniet *ZL*. Ergo et *ZL* cum multiplicabitur in *RU* proveniet *ZT*. Sed *ZL* est quadratum medietatis diametri et *RU* est quantitas in quam cum multiplicatur diameter provenit circumferentia. Et *ZT* equatur superficei circuli. Ergo multiplicatio medietatis diametri in se et eius quod provenit in quantitatem in quam cum multiplicatur diameter provenit circumferentia equatur superficiei circuli. Et hoc est quod voluimus, cuius hec est forma.]

96 medietate: medietati *H*
97–118 [Quod....forma] *mg. P*; *om. Zm?*; *cf. var. lin. 90–91*
97 Quod: sed quod *H*
98–99 linea...circuli[2] *corr. ex* linea circumdans est superficies circuli ABG in *mg. H et* superficies circuli *in mg. P et* vel circumferentia circuli sit equale superficiei circuli *supra mg. P et* vel superficies circuli *in Ma*
102 ZT: ET *H*
103 quod[1] *HMa om. P* / ZL: Z *H*
104, 106 proveniet *H*
107 ad[1] *PMa mg. H*
108 ET equatur: erit equatum *H*
109 medietati: medietate *H* / EL[2]: DL *H*
110 RU: NZ *H* / ad[1]: et *H*
111 RU[1]: NR *H*
112 Ergo *om. H*
113–114 et....diameter *om. H*
114 ZT: ZD *H* / equatum *H*
115–16 proveniet *H*
117 equatur: equalis *H* / *post* quod *add. H* demonstrare
118 cuius...forma *om. H* / *post* forma *add. Ma* vel circumferentia circuli sit equale superficiei circuli (*cf. var. lin. 98–99*).

97–118 [Quod....forma] *om. Ar.*

the base of the body and the hemisphere which contains the body, is equal to more than one half the area of the body falling within the hemisphere.

*[That the multiplication of the square of the radius by π is equal to the area of the circle is demonstrated as follows (see Fig. 48). Since I assume ET to be equal to one half the circumference and EZ equal to

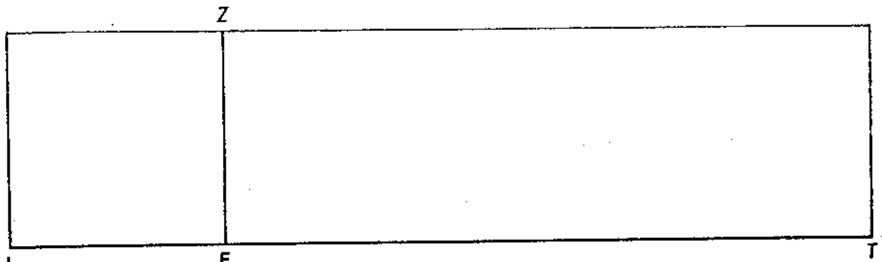


Fig. 48

Note: I have not reproduced line RU (which equals π).

the radius and I shall multiply one into the other, therefore surface ZT will be equal to the surface of the circle. And I shall construct a square on ZE, which square is ZL. I shall posit RU as the quantity which when multiplied by the diameter produces the circumference (i.e., as π). And since the multiplication of the diameter by RU will produce the circumference, therefore, when the circumference is divided by the diameter, RU is produced. Hence, $RU =$ curcumference/diameter. But the ratio of the whole to the whole is as that of the half to the half, and ET is equal to half the circumference while EL is half the diameter. Hence, $TE/EL = RU$. But $(TE/EL) =$ (area TZ/area ZL). Therefore, (area TZ/area TL) $= RU$. Hence, (area TZ/RU) $=$ area ZL. Hence, (area $ZL \cdot RU$) $=$ area ZT. But ZL is the square of the radius, $RU = \pi$, and ZT is the area of the circle. Hence, the multiplication of the square of the radius by π is equal to the area of the circle. And this is what we wished. This is its form (Fig. 48).]

* For the doubtful authenticity of lines 97–118, included here in brackets, see the Introduction, division 2, footnote 8, of this chapter.

Amplius describam in corpore *ABGD* medietatem spere quam continet corpus et sit superficies basis medietatis spere in superficie basis corporis, que est superficies circuli *ABG*, et est superficies circuli *OKY*. Et dividam lineas *BE*, *EU*, *UD* in duo media super puncta *S*, *Q*, *F* et protraham lineas *MS*, *MQ*, *MF*. Et notum est quod ipse sunt equales, propterea quod punctum *M* est centrum circuli *ABG* et corde *BE*, *EU*, *UD* sunt equales. Et faciam in superficie huius circuli lineam *MO* non in superficie circuli *ADB*. Ergo puncta *S*, *Q*, *F*, *O* quattuor non sunt in superficie una. Et ad ea quidem omnia protracte sunt linee ex puncto *M*, que sunt linee *MS*, *MQ*, *MF*, *MO*, et sunt linee equales. Ergo punctum *M* est centrum spere quam continet corpus *ABGD* et linea *MS* est medietas diametri eius. Et circulus *KOY* est basis medietatis spere. Ergo multiplicatio linee *MS* in se, deinde eius quod agregatur in quantitatem in quam cum multiplicatur diametrus est linea circumdans, est embadum circuli *KOY*. Sed multiplicatio medietatis linee *BE* in duas lineas *UZ*, *EH* et in medietatem linee *AB* coniunctim est maior multiplicatione linee *MS* in se, propter illud

119 *de* describam *scr. P mg.* vel signabo *et Zm mg.* in alio, signabo / describam *P* signabo describam *HMa* / ABGD: ABG (?) *H*
119–20 continet *HMa* contineat *P Zm*
120 in superficie *PZm* in superficies *Ma* inferioris *H*
123 MF, MQ, MS *H*
125 EU: EM *H* / Et: quod *H*
126 non: que non sit *Zm* / punctum *H*
129 centrum: centrum circuli *Zm*
130 diametri: digitus *H*
131–35 deinde.... coniunctim *ZmHMa mg. P*
133 est *om. Zm.*
133–49 *de* Sed.... ABGD *scr. Zm in infer. mg. fol. 84r*: Set multiplicatio medietatis linee EB in lineas ZU, EH, MB coniunctas est maior multiplicatione linee MS in se, que est equalis MY que est medietas diametri circuli, scilicet KOY. Ponam proportionem medietatis circumferentie circuli KOY ad medietatem diametri ipsius KMY que est eadem que totius circumferentie ad totam diametrum. Ergo multiplicatio medietatis EB in lineas ZU, HE, MB coniunctas et eius quod provenit in proportionem circumferentie ad diametrum vel medietatis circumferentie ad medietatem diametri et hoc est medietas superficiei corporis ABGD, quia proportio circumferentie ad diametrum suum est una est maius multiplicatione linee MY in se et eius quod provenit in proportionem circumferentie vel medietatis eius ad diametrum vel medietatem eius. Set multiplicatio linee MY in se et eius quod provenit in proportionem est superficies circuli KOY, ut probatum est. Ergo superficies circuli KOY est minus medietate superficiei corporis ABGD. Ergo duplum circuli est minus superficie totius corporis et hoc est quod demonstrare voluimus.
134 UZ: QZ *H*
135 maior: minor *Zm* / linee... se: medietas linee BE in duas lineas UZ, EH et in medietatem linee AB coniunctim *Zm*
135 propter illud *scr. et del.* (?) *H*

Now, further, I shall describe in body *ABGD* [Fig. 47] a hemisphere which the body contains and let the base of the hemisphere be inside the surface of the base of the body, i.e., inside the surface of circle *ABG*, and it (the base of the hemisphere) is the surface of circle *OKY*. And I shall bisect lines *BE*, *EU*, and *UD* at points *S*, *Q*, and *F*, and I shall draw lines *MS*, *MQ*, and *MF*. And it is known that these lines are equal, since point *M* is the center of circle *ABG* and the chords *BE*, *EU*, and *UD* are equal. And I shall produce in circle *OKY* line *OM*, which will not be in the surface of circle *ADB*. Therefore, the four points *S*, *Q*, *F*, and *O* are not in a single [plane] surface. And to these points the equal lines *MS*, *MQ*, *MF*, *MO* have been drawn from point *M*. Therefore, point *M* is the center of the sphere which body *ABGD* contains and line *MS* is its radius. And the circle *KOY* is the base of the hemisphere. Therefore, $MS^2 \cdot \pi =$ area circle *KOY*. But $BE \cdot (UZ + EH + 1/2\ AB) > MS^2$, as we have

121–22 et...OKY: يكون اصغر منها
(*will be smaller than it*)
123 Q: ع ((Note: here and everywhere))
124–25 propterea....equales: لانهـا اعمدة
من المركز على اوتار متساوية و نرسم على مركز
— م — وبعد — م س — فى سطح دائرة — ا ب ح
— دائرة — ك ص ى —
(*For they are perpendiculars* [*drawn*] *from the center to equal chords and we describe circle KOY on center M with radius MS and within circle ABG.*) ((Note: Gerard represents ص by O, no doubt because he had already used S for س.))
126–53 Ergo....eius: ولأن خطوط — م س —
م ع — م ف — م ص — الاربعة المتساوية التى
ليست فى سطح واحد خرجت من نقطه — م —
الى محيط الكرة الدخلة يكون — م — مركزا لها و —
م س — نصف قطر لها ودائرة — ك س ى —
قاعدة لها ومربع — م س — اضعر من سطح نصف
— ب ه — فى نصف — ا ب — وفى — ه ح —
و ز — جميعا فربع — م س — فى المقدار اللذى اذا
ضرب فيه القطر حصل المحيط اعنى سطح دائرة
— ك ص ى — اصغر من سطح نصف — ب ه —
فى نصف — ا ب — وفى — ه ح — و ز — جميعا
ثم الحاصل فى المقدار الـذى اذا ضرب فيـه
القطر حصل المحيط اعنى نصف سطح المجسّم
المحيـط بنصف الكرة الداخلـة فجميـع سطـح
المجسّم اعظم بن ضعف سطح دائرة — ك ص ى —
وذلك ما اردناه

(*And because lines MS, MQ*, MF, and MO** are four equal lines which are not in one surface and are drawn from point M to the surface of the inside sphere, M is its center. And MS is half of its diameter; and circle KSY* ((!KOY?)) *is its base. And* $MS^2 < [\frac{1}{2} BE \cdot \frac{1}{2} AB \cdot (EH + UZ)]$. *And so* $MS^2 \cdot \pi =$ *area circle KOY, and* $(MS^2 \cdot \pi) < [\frac{1}{2} BE \cdot \frac{1}{2} AB \cdot (EH + UZ) \cdot \pi]$, *i.e.,* $(MS^2 \cdot \pi) < \frac{1}{2}$ *area of the body contained by the interior sphere. And so the area of the whole body* $< 2 \cdot$ *area circle KOY. Q.E.D.*) ((*Rendering ع by *Q*, as Gerard does)) ((**Rendering ص by *O*.))

c. 2 cuius demonstrationem premisi/mus. Ergo multiplicatio linee *MS* in se et multiplicatio eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans est equalis superficiei circuli *KOY*. Ergo superficies circuli *KOY* est minor multiplicatione medietatis linee *BE* in duas lineas *UZ*, *EH* et in medietatem linee *AB* et multiplicatione eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans. Sed multiplicatio medietatis linee *BE* in duas lineas *UZ*, *EH* et in medietatem linee *AB* et multiplicatio eius quod agregatur in quantitatem in quam cum multiplicatur diameter est illud quod agregatur ipsa linea circumdans est equalis medietati embadi superficiei corporis *ABGD*. Ergo embadum superficiei corporis *ABGD* est maius duplo embadi superficiei circuli *KOY*, que est basis medietatis spere quam continet corpus *ABGD*

Iam ergo ostensum est quod embadum superficiei corporis *ABGD* est minus duplo embadi basis medietatis spere que continet corpus et maius duplo embadi basis medietatis spere quam continet corpus *ABGD*. Et illud est quod declarare voluimus. Et hec est forma eius.

[XIV.] EMBADUM SUPERFICIEI OMNIS MEDIETATIS SPERE EST DUPLUM EMBADI SUPERFICIEI MAIORIS CIRCULI QUI CADIT IN EA.

Verbi gratia, sit medietas spere *BGAD*, et maior circulus qui cadit in ea sit circulus *ABG*, et punctum *D* sit polus huis circuli [Fig. 49]. Dico ergo quod embadum superficiei medietatis spere *ABGD* est duplum embadi superficiei circuli *ABG*, quod sic probatur.

Si non fuerit duplum embadi circuli *ABG* equale superficiei medietatis spere *ABGD*, tunc sit duplum eius aut minus superficie medietatis spere *ABGD* aut maius ea [, si fuerit possibile]. Sit ergo in primis duplum embadi circuli *ABG* minus embado superficiei medietatis spere *ABGD*, si fuerit illud possibile. Et sit duplum embadi circuli

136–37 cuius...se *om. H*
138 ipsa: illa *H*
139 KOY[1]: KOIB *H* / Ergo *tr. H post* circuli[2]
149 basis: embadum spere basis *H*
150 quod: quod cum *H*
152 et: est *H*
153 declarare: demonstrare *H* / Et...eius *om. H*
1 [XIV]: 17 *mg. MaR*
4–36 Verbi....est[1]: Ex quo infert *S*
4 BGAD: ABGD *Ar*
10 [si... possibile] *sic. PZmHMa*; *sed delendum est?*
12–13 si....ABG *om. H*

demonstrated earlier. Therefore, [since] $MS^2 \cdot \pi =$ area circle KOY, then area circle $KOY < 1/2\ BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi$. But

$$1/2\ BE \cdot (UZ + EH + 1/2\ AB) \cdot \pi = 1/2 \text{ surf area body } ABGD.$$

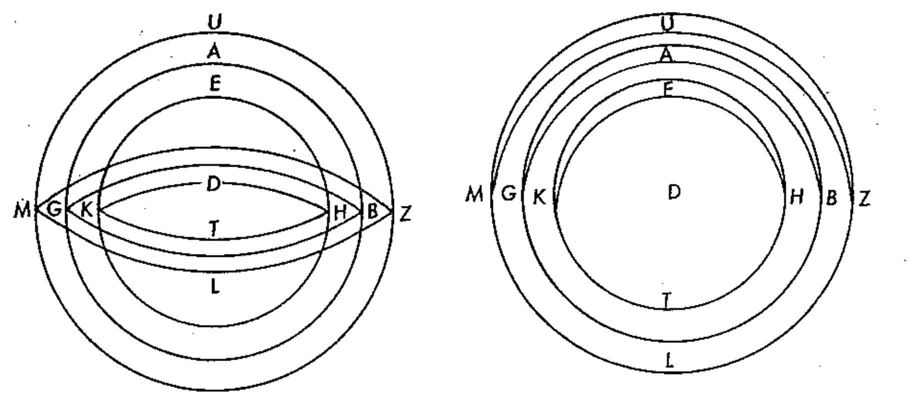
Therefore, the surface area of body $ABGD$ is greater than double the area of circle KOY, circle KOY being the base of the hemisphere which body $ABGD$ contains.

Therefore, it has now been demonstrated that the surface area of body $ABGD$ is less than double the area of the base of the hemisphere which contains the body and greater than double the area of the base of the sphere which body $ABGD$ contains. And this is what we wished to show. And this is its form [Fig. 47].

[XIV.] THE SURFACE AREA OF EVERY HEMISPHERE IS DOUBLE THE AREA OF THE GREATEST CIRCLE WHICH FALLS IN IT.

For example, let there be the hemisphere $BGAD$ and circle ABG the greatest circle falling in it, and let point D be the pole of this circle [see Fig. 49]. I say, therefore, that the surface area of hemisphere $ABGD$ is equal to double the area of circle ABG.

Proof: If double the area of circle ABG is not equal to the area of hemisphere $ABGD$, then it is less than the area of hemisphere $ABGD$ or greater than it. First, let double the area of circle ABG be less than the area of hemisphere $ABGD$, if that is possible. And let double the



[Reconstructed] Fig. 49 [As given in MS P and in Arabic texts]

4–5 qui...ea: التي هو قاعدته (*which is its base*)
6–7 Dico...probatur *om. Ar.*
9–12 ABGD....possibile: فليكن اولا اصغر منه (*then let it at first be less than it*)
12–13 duplum...ABG *om. Ar.*

ABG equale superficiei medietatis spere minoris medietate spere *ABGD*, que sit medietas spere *EHTK*, Cum ergo fiet in medietate spere *ABGD* corpus compositum ex portionibus piramidum columnarum, cuius basis sit superficies circuli *ABG* et cuius caput sit punctum *D*, et ponetur ut corpus non tangat / 61v c. 1 medietatem spere *EHTK*, tunc oportebit ex eis que premisimus ut embadum superficiei corporis *ABGD* sit minus duplo embadi superficei circuli *ABG*. Sed embadum superficiei corporis *ABGD* est maius embado superficiei medietatis spere *EHTK*, quoniam continet ipsam. Ergo embadum superficiei medietatis spere *EHTK* est multo minus duplo embadi superficiei circuli *ABG*. Et iam fuit ei equalis. Hoc vero contrarium est et impossibile.

Et iterum sit duplum embadi superficiei circuli *ABG* maius embado superficiei medietatis spere *ABGD*, si fuerit possibile illud. Et sit equale superficiei medietatis spere maioris medietate spere *ABGD*, que sit medietas spere *UZLM*. Cum ergo fiet in medietate spere *UZLM* corpus compositum ex portionibus piramidum columpnarum, cuius basis sit superficies circuli *UZLM* et cuius caput sit punctum *D*, et non sit corpus tangens medietatem spere *ABGD*, tunc oportebit ex eo quod premisimus ut sit embadum superficiei corporis *UZLM* maius duplo embadi circuli *ABG*. Verum embadum superficiei medietatis spere *UZLM* est maius embado superficiei corporis *UZLM*. Ergo embadum medietatis spere *UZLM* est maius duplo embadi superficiei circuli *ABG*. Sed iam fuit ei equale. Hoc vero est contrarium et impossibile.

Iam ergo ostensum est quod embadum superficiei omnis spere est quadruplum embadi superficiei maioris circuli cadentis in ea. Et illud est quod declarare voluimus. Et hec est forma eius.

13 equale *PZmMa* equalis *H*
18 *ante* tunc *del. H* cum ?
19 ABGD: ABG *H* / ABG: ABGD *P*
21 spere: spei *H*
23 ei: dg *H* / est: fuit *H*
24 Et *om. H*
25 ABGD: AABGD *P*
26 equale *ZmPMa* equalis *H* / medietate: medietatis *H*
27 U-: N- *H hic et ubique in hac propositione*
29 LZLM *H*
31 ut: quod *H*
33 maius: magis *H*
33–34 embado....maius *om. H hic sed cf. var. lin. 35*
34 superficiei *om. H*
35 *post* ABG *add. H* omisimus embadum superficiei medietatis spere NZLM est magis embado superficiei circuli AB (!) / Sed: et *H* / equale *corr. ex* equalis *in PHMa*
37–38 Et....eius *om. S*
38 Et...eius *om. H*

15 compositum...columnarum: كما وصفنا (*just as we have described*)

18–23 tunc....impossibile: كان سطحه اصغر من ضعف سطح دائرة – ا ب ج – واعظم من سطح

area of circle *ABG* be equal to the area of a hemisphere smaller than hemisphere *ABGD*, namely, hemisphere *EHTK*. When, therefore, there is described in hemisphere *ABGD* a body composed of segments of cones, the base of which body is the surface of circle *ABG* and its vertex is point *D*, and it is posited that the body does not touch hemisphere *EHTK*, then from what we have proved before [in Proposition XIII] it will follow that the surface area of body *ABGD* is less than double the area of circle *ABG*. But the surface area of body *ABGD* is greater than the surface area of hemisphere *EHTK*, since the one contains the other. Therefore, the surface area of hemisphere *EHTK* is much less than double the area of circle *ABG*. But it was posited as equal to it. This indeed is a contradiction and is impossible.

Now again let double the area of circle *ABG* be greater than the surface area of hemisphere *ABGD*, if that is possible. Let it be equal to the area of a hemisphere greater than hemisphere *ABGD*, namely, hemisphere *UZLM*. When, therefore, there is inscribed in hemisphere *UZLM* a body composed of segments of cones, the base of which body is circle *UZLM* and its vertex is point *D* and the body does not touch hemisphere *ABGD*, then it will follow from what we have proved before that the surface area of body *UZLM* is greater than double the area of circle *ABG*. But the surface area of hemisphere *UZLM* is greater than the surface area of body *UZLM*. Therefore, the surface area of hemisphere *UZLM* is greater than double the area of circle *ABG*. But it was posited as equal to it. This indeed is a contradiction and is impossible.

Therefore, it has now been demonstrated that the surface area of any sphere is quadruple the area of the greatest circle falling in it. And this is what we wished to show. And this is its form [Fig. 49].

نصف كرة – ه ح ط ك – وضعف سطح دائرة – ا ب ج – المساوى لسطح نصف كرة – ه ح ط ك – اعظم كثيرا منه هذا خلف
(*Its surface was less than double the surface of circle ABG and greater than the surface of hemisphere EHTK. And double the area of circle ABG, which is equal to the area of the hemisphere EHTK, is much greater than it.* [*But*] *this is a contradiction.*)

25 si....illud *om. Ar.*

26–27 maioris...spere[1] *om. Ar.*

28–29 compositum...D: كما وصفنا (*just as we have described*)

30–31 ex eo quod premisimus: لما مر (*according to what went before.*) ((Note: this phrase is transferred to a position after ABG in line 32.))

32 Verum: و (*And*)

33 UZLM[2] *om. Ar. sed add.* لكونه محيطا به (*because the one contains the other*)

34 maius: اعظم كثيرا (*much greater*) / duplo *om. Ar.*

35 et impossibile *om. Ar. et hic add.* فاذا الحكم ثابت وذلك ما اردناه (*And so the rule is established. Q.E.D.*) ((Note: the Q.E.D. is given here instead of in lines 37–38 where it appears in the Latin text.))

[XV.] MULTIPLICATIO MEDIETATIS DIAMETRI OMNIS SPERE IN TERTIAM EMBADI SUPERFICIEI SUE EST EMBADUM MAGNITUDINIS SPERE.

Verbi gratia, sit spera *ABGD*, et medietas diametri eius sit linea *SB* [Fig. 50]. Dico ergo quod multiplicatio linee *SB* in tertiam embadi superficiei spere *ABGD* est embadum magnitudinis spere *ABGD*, cuius hec est demonstratio.

Si non fuerit ita, tunc sit multiplicatio linee *SB* in tertiam embadi superficiei spere minoris aut maioris spera *ABGD* ipsum embadum magnitudinis spere *ABGD*. Ponam ergo in primis multiplicationem linee *SB* in tertiam embadi superficiei spere maioris spera *ABGD* ipsum embadum magnitudinis spere *ABGD*. Et sit spera *ZULM*, cuius centrum et centrum spere *ABGD* sit unum. Ergo multiplicatio
c. 2 *SB* in tertiam emba/di spere *UZLM* est embadum magnitudinis spere *ABGD*. Ergo cum fiet super speram *ABGD* corpus habens superficies continentes ipsum et non tangat speram *UZLM*, oportebit ex eo quod premisimus ut multiplicatio linee *SB* in tertiam embadi superficiei corporis quod continet speram *ABGD* sit maior embado spere *ABGD*. Sed multiplicatio linee *SB* in tertiam superficiei spere *UZLM* est embadum magnitudinis spere *ABGD*. Ergo tertia embadi spere *UZLM* est minor tertia embadi superficiei corporis habentis superficies, et spera *UZLM* continet corpus. Hoc autem est contrarium.

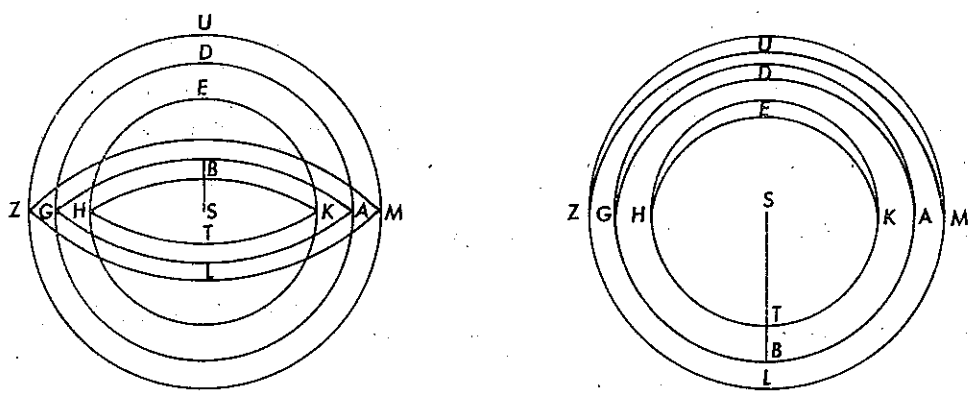
1 [XV]: 18 *mg. Ma mg. R*
2–3 *post* embadum *add. S* et
4–38 Verbi....voluimus *om. S*
7 cuius: secundum *H*
11 *ante* superficiei *scr. et del. H* et embadum
12–13 Et...unum: sit primum *H*
13 *ante* et *del. P* sit E
14 U-: N- *H hic et ubique in hac propositione*
16 tangat *HR* tangant *PZm* tagat *Ma*
17 *de* premisimus *scr. P. mg.* in conclusione 1 figure huius / superficiei: superficiem NZLM est embadum magnitudinis spere ABGD *H* (*Sed cf. var. lin. 19-20*)
19–20 UZLM....ABGD *om. H*

5 SB: سف (*SF*)
5–7 Dico... demonstratio *om. Ar.*
8–15 Si....ABGD:[1] فان لم يكن – س ف – ى ثلث سطح كرة – ا ب ج د – عظمها فليكن اولا اصغر من عظمها وليكن – س ب – فى ثلث سطح كرة اعظم من كرة – ا ب ج د – مساويا لعظم كرة – ا ب ج د – مثلا ككرة – و ز ل م – فليكن مركزاهما واحدا
(*And if the product of SF and* $\frac{1}{3}$ *the area of sphere ABGD is not equal to its volume, then at first let it be less than its volume. And let the product of SB and* $\frac{1}{3}$ *the area of a sphere larger than sphere ABGD be equal to the volume of sphere ABGD—and* [*this larger sphere is*] *for example UZLM. And let the centers of the two spheres be common.*)
15–16 habens....ipsum: كما وصفنا
(*just as we have described*)
18–22 quod....corpus: يساوى المجسّم ويكون اكبر من كرة – ا ب ج د – ويلزم منه ان يكون ثلث سطح المجسّم اعظم من ثلث كرة – و ز ل م – المحيط به

[XV.] THE MULTIPLICATION OF THE RADIUS OF EVERY SPHERE BY ONE THIRD OF ITS SURFACE AREA IS THE VOLUME OF THE SPHERE.

For example, let there be a sphere *ABGD* with radius *SB* [see Fig. 50]. I say, therefore, that the multiplication of line *SB* by one third of the surface area of sphere *ABGD* is the volume of sphere *ABGD*.

Proof: If this is not so, then let the multiplication of line *SB* by one third of the surface area of a sphere either larger than or smaller than sphere *ABGD* be [equal to] the volume of sphere *ABGD*. Hence, I shall posit first that the multiplication of line *SB* by one third of the surface area of a sphere larger than sphere *ABGD* is [equal to] the volume of sphere *ABGD*. Let this sphere be *ZULM* [*UZLM*], concentric with sphere



[Reconstructed] Fig. 50 [As given in MS *P* and in Arabic texts]

ABGD. Hence ($SB \cdot 1/3$ area sphere *UZLM*) = vol sphere *ABGD*. Hence, when there is described about sphere *ABGD* a body having surfaces which bound it but which body does not touch sphere *UZLM*, it will follow from what we have proved before that the multiplication of line *SB* by one third of the surface area of the body which contains sphere *ABGD* is greater than the area of sphere *ABGD*. But the multiplication of line *SB* by one third of the area of sphere *UZLM* is [equal to] the volume of sphere *ABGD*. Therefore, one third of the area of sphere *UZLM* is less than one third of the surface area of the body having surfaces, while the sphere *UZLM* contains the body. This, however, is a contradiction.

(*equals the body and will be greater than sphere ABGD, and it follows from this that* $\frac{1}{3}$ *of the area of the body is greater than* $\frac{1}{3}$ *of* [*the area of*] *sphere UZLM containing it.*)

Et sit multiplicatio linee *SB* in tertiam embadi superficiei spere minoris spera *ABGD* ipsum embadum magnitudinis spere *ABGD*. Et sit spera illa spera *EHTK*, cuius centrum et centrum spere *ABGD* sit unum. Ergo multiplicatio *SB* in tertiam embadi superficiei spere *EHTK* est embadum magnitudinis spere *ABGD*. Cum ergo fiet in spera *ABGD* corpus habens superficies continentes ipsum et non tangat speram *EHTK*, oportebit ex eo quod premisimus ut sit multiplicatio linee *SB* in tertiam embadi superficiei corporis habentis superficies quod continet spera *ABGD* minor embado spere *ABGD*. Sed multiplicatio linee *SB* in tertiam embadi superficiei spere *EHTK* est embadum magnitudinis spere *ABGD*. Ergo tertia embadi superficiei spere *EHTK* est maior tertia embadi superficiei corporis habentis superficies, et corpus continet speram *EHTK*. Hoc vero est contrarium. Iam ergo declaratum est quod multiplicatio medietatis diametri spere in tertiam embadi superficiei eius est embadum magnitudinis eius. Et illud est quod declarare voluimus.

[XVI.] VOLO OSTENDERE QUOMODO PONANTUR INTER DUAS QUANTITATES DUE QUANTITATES ITA UT CONTINUENTUR QUANTITATES QUATTUOR SECUNDUM PROPORTIONEM UNAM.

Scientia enim illius valde utile est ei qui geometrie querit scientiam. Et hac eadem operatione extrahatur latus cubi, quod est quoniam quando illud quod est in cubo de unitatibus et partibus est no/tum et (62r c. 1) ponuntur inter numerum cubi et inter unum duo numeri continui secundum proportionem ⟨unam⟩, tunc ille qui sequitur unum ex duobus numeris mediis est latus cubi.

25 spera² *om. H* | ETHK *H* | ét: est *H*
27 fiet: fiat *H*
29 tangat *HMa* tangant *PZm* | *de* premisimus *scr. P mg.* in conclusione 2 figure huius
30–31 superficies: superficiem *H*
36 diametri *om. H*
37 embadi superficiei *tr. H*
38 quod...voluimus *om. H*
1 [XVI]: 19 *mg. Ma mg. R*
1–62 Volo....ostendere *om. S*
5 Scientia...scientiam *PZmMa mg. H* | Scientia enim: Nota quod scientia *H*
9 ⟨unam⟩ *supplevi*

23–38 Et....eius: ثم ليكن — س ب — فى ثلث سطح كرة اصغر من كرة — ا ب ج د — ككرة — ہ ح ط ك — مساويا لعظم كرة — ا ب ج د — ونعمل فى كرة — ا ب ج د — مجسّما كما وصفنا بحيث لا يماس كرة — ہ ح ط ك — ويجب مما مرّ ان — س ب — فى ثلث مساحة سطح المجسّم اصغر من مساحة كرة — ا ب ج د — فثلث سطح — ہ ح ط ك — اعظم من ثلث سطح المجسّم المحيط به هذا خلف ، فاذا الحكم ثابت

And let the multiplication of line *SB* by one third of the surface area of a sphere less than sphere *ABGD* be [equal to] the volume of sphere *ABGD*. Let that [lesser] sphere be sphere *EHTK*, concentric with sphere *ABGD*. Therefore, the multiplication of *SB* by one third of the surface area of sphere *EHTK* is [equal to] the volume of sphere *ABGD*. When, therefore, there is inscribed in sphere *ABGD* a body having sufaces which bound it but which body does not touch sphere *EHTK*, it will follow from what we have proved before that the multiplication of line *SB* by one third of the surface area of the body having surfaces which the sphere *ABGD* contains is less than the area of sphere *ABGD*. But the multiplication of line *SB* by one third of the surface area of sphere *EHTK* is [equal to] the volume of sphere *ABGD*. Therefore, one third of the surface area of sphere *EHTK* is greater than one third the surface area of the body having surfaces, while the body contains sphere *EHTK*. This indeed is a contradiction. Therefore, it has now been shown that the multiplication of the radius of the sphere by one third of its surface area is [equal to] its volume. And this is what we wished to show.

[XVI.] I WISH TO DEMONSTRATE HOW TWO QUANTITIES ARE PLACED BETWEEN TWO QUANTITIES SO THAT THE FOUR QUANTITIES ARE IN CONTINUED PROPORTION.

For a knowledge of this [proposition] is very useful to anyone who seeks a knowledge of geometry. By this same operation is extracted the side of a cube; for when the cube is known in terms of units and parts, and between the number representing the cube and 1 are placed two [other] numbers in continued proportion [with the cube number and 1], then that number [representing that one] of the two mean numbers which follows 1 is the side of the cube.*

* See Commentary for a representation of this statement in modern notation.

(*Then let the product of SB and* $\frac{1}{3}$ *the area of a sphere EHTK—a sphere smaller than sphere ABGD—be equal to the volume of sphere ABGD. And let us inscribe in sphere ABGD a body such as we have described before so that it does not touch sphere EHTK. And it is necessary from what has gone before* [*in the first part of the theorem*] *that the product of SB and* $\frac{1}{3}$ *the area of* [*the inscribed*] *body is less than the volume of sphere ABGD. And so* $\frac{1}{3}$ *the area of EHTK is greater than* $\frac{1}{3}$ *the area of the body containing it.* [*But*] *this is a contradiction. And so the rule is established.*)

38 declarare *om. Ar.*

1–62 ((Note: al-Ṭūsī seems to have followed the original text more closely here than in most propositions.))

6 hac... extrahatur: به يعرف (*by means of this is known*)

7–8 illud... ponuntur: عرفنا مقدارين يقعان (*we know two quantities which fall*...)

$A_1 = \frac{1}{2} c_1 \cdot (p + s)$ and $A_2 = \frac{1}{2} c_2 \cdot p$.

Now [1]

$$A \text{ seg} = A_1 - A_2 = \tfrac{1}{2} c_1 (p + s) - \tfrac{1}{2} c_2 \cdot p$$
$$= (\tfrac{1}{2} c_1 \cdot p) + (\tfrac{1}{2} c_1 \cdot s) - (\tfrac{1}{2} c_2 \cdot p).$$

But $c_1/c_2 = (p + s)/p$ [by similar triangles].

Or [2] $c_1 \cdot p = (c_2 \cdot p) + (c_2 \cdot s)$. Substituting [2] in [1], we arrive at the required proposition, namely, $A \text{ seg} = \frac{1}{2} s \cdot (c_1 + c_2)$. Cf. Leonardo Pisano, *Practica geometrie*, ed. cit., pp. 181–82; this was drawn directly from the Banū Mūsā text. Cf. Archimedes, *De sphaera et cylindro*, Proposition 16. Cf. also *De curvis superficiebus*, Proposition IV (see Chapter Six below). It is evident that the original proposition of Archimedes has been considerably altered by the author of the *De curvis superficiebus* and by the Banū Mūsā.

52–71 The material in brackets is added in the margin of *P*, *Zm*, and *Ma* and in the text of *H* and *R*. I have put it in brackets because I believe it to be an addition of Gerard's rather than a part of the text. For it is missing in the Arabic text and it cites a specific proposition of Euclid, a practice not followed in the Banū Mūsā.

Proposition XII

1–76 "Cum.... voluimus." Cf. Archimedes, *De sphaera et cylindro*, Propositions 21 and 22, and the *De curvis superficiebus*, Proposition V. Again we note that Leonardo Pisano drew from the Banū Mūsā: see Leonardo Pisano, *Practica geometrie*, ed. cit., p. 183.

43–76 "Si.... voluimus." This passage is an extension of, or corollary to, the enunciation which does not indicate the object of this additional proof.

Proposition XIII

1–153 "Cum.... eius." Cf. Archimedes. *De sphaera et cylindro*, Propositions 25 and 30. Also see Leonardo Pisano's *Practica geometrie*, ed. cit., 183–85.

77–79 "quantitatem... circumdans." This is the rhetorical expression used by the Banū Mūsā to designate what came to be designated as π. Notice that Leonardo Pisano, in taking over this proof, simply substitutes "3 1/7" whenever this expression appears in his proof (*Ibid.* 184–85).

97–118 "Quod.... forma." This is the section proving that $A = \pi r^2$. The more common formulation for the area of the circle in antiquity and the Middle Ages is $A = \frac{1}{2} c \cdot r$, as it appears in the *De mensura circuli* of Archimedes.

Proposition XIV

1–38 "Embadum.... eius." Cf. Archimedes, *De sphaera et cylindro*, Proposition 33, and the *De curvis superficiebus*, Proposition VI, Corollary. See also Leonardo Pisano, *Practica geometrie*, ed. cit., 185–86, and the anonymous *De ysoperimetris*, Appendix III, paragraph 8.

Proposition XV

1–38 "Multiplicatio.... voluimus. Cf. Archimedes, *De sphaera et cylindro*, Proposition 34, and the *De curvis superficiebus*, Proposition VIII. See Leonardo Pisano, *Practica geometrie*, ed. cit., 186–87 and the anonymous *De ysoperimetris*, Appendix III, paragraph 8.

Proposition XVI

6–10 "Et... cubi." The substance of this statement in modern notation is this: If a and 1 are the given quantities, a being equal to the cube, and $a/x = x/y = y/1$, x and y being the mean proportionals, then $y = \sqrt[3]{a}$ for $a \cdot y = x^2$, and $x = y^2$. Thus $a \cdot y = y^4$, or $a = y^3$.

12 "Mileus... geometria." The Banū Mūsā say that they took this proof from a *Liber in geometria* attributed to Mileus (i.e., Menelaus, as in the Arabic text*). While such a Greek text has not survived, we find this work mentioned in the *Fihrist* (cf. H. Suter, "Das Mathematiker-Verzeichniss im Fihrist des Ibn Abî Ja'kûb an-Nadîm," *Abhandlungen zur Geschichte der Mathematik*, 6. Heft [1892], p. 19). This work is also mentioned by al-Bīrūnī (Suter, *Bibliotheca Mathematica*, 3. Folge, vol. *11*, p. 69). Menelaus probably took it from Archytas (see next remarks).

12–62 "Sint.... ostendere." While attributed here to Menelaus, this is the solution which Eutocius in his commentary on the *De sphaera et cylindro* (ed. of Heiberg, *Arch. opera omnia*, vol. *3*, pp. 84.12–88.2) assigns to Archytas on the authority of Eudemus. Cf. the English translation of the passage by Ivor Thomas, *Selections Illustrating the History of Greek Mathematics*, vol. *1* (London, Cambridge, Mass., 1951), pp. 284–89. Thomas analyses the solution in modern notation which I have here adapted to the lettering found in the *Verba filiorum* (see Fig. 51 in the text): Take AB as the x axis, a perpendicular to A in the plane of $ADBG$ as the y axis, and a perpendicular to the plane of $ADBG$ at A as the z axis. AB, AG are the two given quantities, and we let $AB = a$, $AG = b$. The solution depends on finding point

* Obviously the Arabic form was without any diacritical mark for the nūn, so that Gerard read ميلاوس instead of منلاوس.