

# CHINA

Below is a scan from the edition of *Jiuzhang Suanshu (The Nine Chapters of the Mathematical Art)*, edited by Shen Kangshen, John Crossley and Anthony Lun, Oxford, 1999. This is the classic of Chinese mathematics. Like Euclid, this is a compendium of the mathematical concepts and techniques which had been developed slowly from perhaps the Zhou (or Chou) dynasty (begins c.1000 BCE) through the Western Han dynasty (ending 9 CE). Unlike Euclid, it is a simple list of actual, perhaps even useful numerical problems and algorithms for their solution, without any indication of proofs. Since then, the *Nine Chapters* had a long history of ups and downs, sometimes being required in civil service exams and sometimes being burned and nearly lost. Each time it was republished though, new commentaries were added, starting with those of the great mathematician Liu Hui in 263 CE and continuing through those in this English translation by Shen, Crossley and Lun. This edition takes a bit of patience to read:

- a. Boldface = text from *The Nine Chapters*
- b. Sans-serif text = Liu's commentary
- c. Italics = Li's commentary
- d. Ordinary text referenced by superscripted numbers = Shen, Crossley and Lun's comments

The excerpt is from the end of Chapter 4, "*Short Width*", where cube roots are introduced and applied (incorrectly) to the problem: what is the diameter of a sphere with given volume. Liu notes their rule is wrong and begins a correct analysis by introducing the 'double umbrella', (= the intersection of two cylinders as in the preface to Archimedes' *Method*). He uses the method of comparing areas of slices to argue that the volume of the double umbrella is  $4/\pi$  times the volume of the sphere. Zu Geng (also called Zu Xuan), in the late 5<sup>th</sup> century CE, found the formula for the volume of the double umbrella, hence the correct formula for the volume of the sphere. This is described in Li's commentary. Amazingly, his method is to add to the volume of the double umbrella the volume of an auxiliary pyramid: as described below, the idea is very close to that of Prop.2 of the *Method* and even closer to Heiberg and Zeuthen's reconstruction of the missing Proposition 15 of the palimpsest. But *he turns the auxiliary pyramid or cone upside down*, resulting in a much simpler proof not needing the balance beam! In a sense, he does Archimedes one better.

# The Nine Chapters on the Mathematical Art

COMPANION AND COMMENTARY

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(middle row) . . . Borrow another rod in the *xiahang* (bottom row).” Thus the Rule follows the coefficients in the binomial expansion:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n.$$

In the middle of the eleventh century Jia Xian (賈憲) studied the table of binomial coefficients up to  $n = 6$ . Volume 16344 of *The Great Encyclopedia of the Yongle Reign* contains his result (Fig. 4.21). In the West, the German P. Apianus (see [Sm], vol. 2, p. 509) gave a table up to  $n = 8$  on the cover of his book *Practical Arithmetic* (Fig. 4.20) in 1527. In France, Blaise Pascal (1623–1662) continued studying the table and gave an inductive definition of the coefficients for all  $n$ , now known as Pascal’s Triangle. However, Jia’s result was obtained several hundred years earlier than those in the West, so it would seem more appropriate to call the table “Jia Xian’s Triangle”.

[Problem 23]

Now given a volume of 45 004 *chi*.<sup>a</sup> Tell: what is the diameter of the sphere?

Answer: 20 *chi*.<sup>α</sup>

(a) Liu: This also means cubic *chi*.

(α) *Li et al.*: According to the precise rate, if the diameter is 20 *chi*, the volume should be 4190 [cubic] *chi*  $\frac{10}{21}$  [cubic] *cun*.<sup>(1)</sup>

(1) Li uses his accurate  $\pi = \frac{22}{7}$  to get  $V_{\text{sphere}} = \frac{2}{3} \cdot \frac{\pi}{4} D^3 = \frac{\pi}{6} D^3 \approx \frac{1}{6} \cdot \frac{22}{7} D^3$ .

[Problem 24]

Given again a volume of 1 644 866 437 500 [cubic] *chi*. Tell: what is the diameter of the sphere?

Answer: 14 300 *chi*.<sup>α</sup>

(α) *Li et al.*: According to the precise rate, the diameter should be 14 643  $\frac{3}{4}$  *chi*.<sup>(1)</sup>

(1) Li calculated the diameter of the sphere to 7 significant figures as:

$$D = \sqrt[3]{\frac{6}{\pi} V} = \sqrt[3]{\frac{21}{11} \cdot 1\,644\,866\,437\,500} \approx 14\,643.75 \text{ } \textit{chi}. \quad (4.6)$$

Diameter of a Sphere [*Kai Liyuan* (開立圓)] Rule<sup>(1)</sup>

Lay down the given number of [cubic] *chi*. Multiply it by 16, divide it by 9. Extract the cube root of the result, giving the diameter of the sphere.<sup>a,α,(2)</sup>

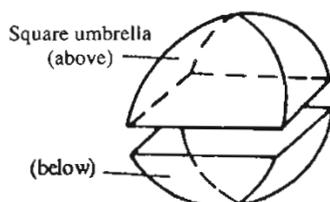


FIG. 4.22. Liu Hui's "joined umbrellas".

(a) Liu: *Liyuan* (立圓) means sphere. The Rule takes the rate of circumference 3 to diameter 1. Suppose the area of a circle is  $\frac{3}{4}$  that of its circumscribed square, then the volume of a cylinder would also be  $\frac{3}{4}$  of its circumscribed cube. Furthermore, suppose that the volume of the cylinder means square rate 12 and that of its inscribed sphere means circle rate 9, so that the volume of the sphere is  $\frac{3}{4}$  that of the circumscribed cylinder. The square of 4 is 16, and the square of 3 is 9, thus the volume of the sphere is  $\frac{9}{16}$  that of the circumscribed cube.<sup>(3)</sup> Therefore, we multiply the given volume of the sphere by 16, and divide it by 9, to obtain the volume of the circumscribed cube. The diameter of the sphere is equal to the side of the circumscribed cube, therefore by extracting its cube root we have the diameter required. But the supposition is incorrect. How can this be verified? Take eight cubic blocks with 1-*cun* sides to form a cube with a 2-*cun* side. Cut the cube horizontally by two identical cylindrical surfaces perpendicular to each other, 2 *cun* both in diameter and in height, then their common part looks like the surface of two four-ribbed umbrellas put together. This solid is called a *mouhefanggai* (牟合方蓋) (joined umbrellas).<sup>(4)</sup> It is composed of eight solid blocks in the form of *yangma* (陽馬), which have two lateral cylindrical surfaces.<sup>(5)</sup> Now see that the joined umbrellas have the square rate, and the inscribed sphere has the circle rate. Is there no defect in thus attributing to the cylinder the square rate?<sup>(6)</sup> With a rate of circumference 3 to diameter 1 the area of a circle would be a little too small. With the cylinder as sphere rate, the volume of the sphere would be somewhat too large. With adjustment, the rate of 9 to 16 would be an approximate value, but is still a little too large.<sup>(7)</sup> Now we consider the space outside the joined umbrellas and inside the cube. It narrows down gradually, but is difficult to quantify. The solid is formed by a mixture of squares and circles. The sections vary in thickness so irregularly that the solid cannot be compared to any regular block.<sup>(8)</sup> I am afraid that it would be unreasonable to make conjectures neglecting the [difference in] shape [between the sphere and the joined umbrellas]. Let us leave the problem to whomever can tell the truth. One cubic *cun* of gold weighs 16 *liang*, and a gold ball one *cun* in diameter weighs 9 *liang*.<sup>(9)</sup> This may be the natural source of the rate, which has never been verified. The *Book of Arts and Crafts*,<sup>(10)</sup> a Chapter in the *Administrative System of the Zhou Dynasty* (周

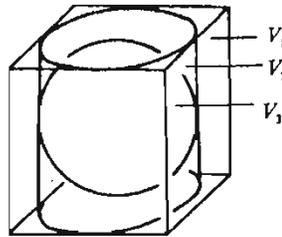
官) [11th century BC–256 BC], says: “Building a standard etalon, Mr Li<sup>(11)</sup> (稟) purified copper with tin, and weighed them on the balance, then measured its volume by water. Then he forged standard weights into the mould.” That is to say, brought to the highest purification, can they be identified as the rates?

Square the diameter of a sphere, divide it by 3 and extract the square root, the result is equal to the side of the cube inscribing the sphere.<sup>(12)</sup> Suppose the side is 5 *chi*, which is regarded as the *gou*. The squared *gou* yields an area of 25 [square] *chi*. Doubling it, we get 50 [square] *chi*, which is considered as the square on the hypotenuse. This is the diagonal of a face of the cube. Take the hypotenuse as the *gu*, 5 *chi* as the *gou* again, and the sum of their squares is 75 [square] *chi*, which is called the area on the long hypotenuse.<sup>(13)</sup> Its square root is the long hypotenuse, the diagonal of the inscribed cube, i.e. the diameter of the sphere. So the square of a side of a cube is one-third that of the diameter of its circumscribing sphere, and the cube of the long hypotenuse is then the volume of the cube circumscribing the sphere.<sup>(14)</sup>

If the square on the long hypotenuse is unextractable, the volume of the circumscribed cube is taken as the side of a square, the area of which is the cube of 75, i.e. the side of a square with area 421 875. 5 [square] *chi* multiply 5 *chi* again giving 125. Squaring it, we get 15 625. Consider the volume of the inscribed cube as the side of a square with area 15 625. Divide both of them by 625. Thus the volumes of the circumscribed and inscribed cubes of a sphere are the sides of squares with areas of 625 and 25 respectively.<sup>(15)</sup> Zhang Heng (張衡) has calculated that if the volume of the circumscribed sphere is the side of a square with an area of 675, say, 26 deficit 1; then the volume of the inscribed sphere is 5,<sup>(16)</sup> i.e. the side of a square with an area of 25. Now I have introduced a cube with an inscribed sphere, that in turn circumscribes a cube. The ratio between the two cubes is just the same as that between the two spheres mentioned by Zhang. It is thus clear that Zhang also derived that from the supposition relevant to the ratio between the two cubes. Zhang said: “If the side [of a sphere] with an area of 64 is the volume of a cube, then the side [of a square] with an area of 25 will be the volume of its inscribed sphere”, i.e. the volume of the sphere is  $\frac{5}{8}$  that of the circumscribing cube.<sup>(17)</sup> Zhang further pointed out that, as the side [of a square] with an area of 8 is to the area of a square, so is the side [of a square] with an area of 5 to its inscribed circle.<sup>(18)</sup> It is thus obvious that in the process of derivation, he also took a cylinder for the sphere rate and its inscribed sphere for the circle rate.<sup>(19)</sup> The error is very serious. In an attempt to make his statement consistent and harmonize his philosophy of *yin* and *yang*, and the doctrine of odd and even, he neglected the precision of the data. The statement runs counter to the facts. It is quite fallacious. Take a cube with a volume of 26, multiply it by 9 and divide it by 16, and we get  $14\frac{5}{8}$ , which is the volume of its inscribed sphere. We reduce it to an improper fraction, and the numerator is 117. The volume of its inscribed cube will be 5. Multiply it by the denominator, get 40. That is to say, the volume of the cube is  $\frac{40}{117}$  that of its circumscribing sphere.<sup>(20)</sup> Here the rate for the sphere is still a little too large. Suppose the side of a square is 2 *chi* and the sum of the four sides, i.e. the perimeter, is 8 *chi*. The diameter of its

inscribed circle is also 2 *chi*, the same as the side of the square. Multiply its radius by half its circumference, get the area of the circle. Multiply half the side by half the perimeter of a square, get the area of the square. So the rate for the square may be derived from the perimeter, and the circle rate from the circumference. According to Zhang, the side of a square with an area of 8 is the rate for the square perimeter and the side of a square with an area of 5 is the rate for its inscribed circumference. Suppose the square perimeter is the side of a square with an area of 64, then its inscribed circumference is the side of a square with an area of 40. Suppose further a circle 1 *chi* in diameter, then the perimeter of its circumscribing square is 4 *chi*. Squaring it, we get 16 [square] *chi* in area. This means the rate for the circumference of a circle is the side of a square with an area of 10,<sup>(21)</sup> and that of the corresponding diameter is the side of a square with an area of 1. Zhang also considered the rate of circumference 3 to diameter 1 is incorrect, and he suggested a new method, but the increased circumference is unnecessarily larger than the exact value.

( $\alpha$ ) *Li et al.*: *Zu Geng said that both Liu Hui and Zhang Heng took the cylinder for the square rate and its inscribed sphere for the circle rate.*<sup>(22)</sup> *And therefore Zu suggested a new hypothesis himself. His Kai Liyuan Rule says: "Double the given volume and extract its cube root, and we have the diameter of the sphere."*<sup>(23)</sup> *But why? Take a cubic block, and with its left rear lower edge as axis and the side as radius, draw a cylindrical surface. Now remove the right-upper part and put the parts together. Again draw a horizontal cylindrical surface and remove the front upper part. Now the cubic block is divided into four parts: inside the surface is called the inner block; whereas the other three outside the surface are called outer blocks.*<sup>(24)(i)</sup> *Again put the blocks together, and cut them horizontally. Now we explain in terms of right-angled triangles. Let the height be the gou, the side of the section of the inner block be the gu and the side of the original cube be the hypotenuse. According to the Gougu Rule, subtract the square on the gou from the square on the hypotenuse, and we obtain the difference as the square on the gu.*<sup>(ii)</sup> *Therefore, subtract the square on the height from the square on the side of the cube to obtain the difference as the area of the section of the inner block. The area of the horizontal face of the cube is the total area of the sections of the four blocks.*<sup>(iii)</sup> *So the square on the height is the total area of the sections of the [three outer] blocks.*<sup>(iv)</sup> *And this is true whatever the height. Indeed, as the saying goes: "All roads lead to one goal." Let us analyse this in further detail by analogy from the general case. Take a yangma of equal sides and height, and setting it inverted, cut a horizontal section and take away its upper part. Then the square of the height of the section is equal to the area of the section.*<sup>(v)</sup> *Combine the three outer blocks together, and the corresponding section areas of the two solids are equal everywhere, so their volumes cannot be unequal.*<sup>(vi)</sup> *From this point of view, if the sections of the three outer blocks are combined as one, they form a yangma.*<sup>(vii)</sup> *Now if the volume of a cube is divided into three equal parts, then obviously the yangma occupies one part, and the inner block two. Combine eight small cubes into a large one, and eight inner blocks into joined umbrellas. The inner block occupies two-thirds of a small cube.*<sup>(viii)</sup> *Undoubtedly*



$$V_1 : V_2 : V_3 = 16 : 12 : 9$$

FIG. 4.23. The ratio of the cube to the cylinder in the *Nine Chapters*.

the joined umbrellas also occupy two-thirds of one large cube. Multiply  $\frac{2}{3}$  by the circle rate 3, and divide by the square rate 4. Simplifying, get the rate for the sphere. Thus a sphere occupies one half of the circumscribing cube.<sup>(ix)</sup> Since this is a precise result, everything is clear. Zhang Heng followed the ancients blindly, making himself a laughing stock for later generations. Liu Hui stuck to the facts, and failed to revise it.<sup>(25)</sup> I am afraid either the problem was too difficult to solve or they had not thought about it deeply. According to the precise rate, the volume of the sphere is the cube of its diameter, multiplied by 11 and divided by 21.<sup>(26)</sup> Here the cube of the diameter is to be found [from the known volume], i.e. multiplied by 21 and then divided by 11. In general, extract the cube root of the cube of a number to restore it to the original number. Thus extracting the cube root, get the diameter of the sphere.<sup>(27)–(32)</sup>

(1) *Kai Liyuan* literally means to extract the root of a sphere, i.e. to find the diameter of a sphere with known volume.

(2) Let the known volume of a sphere be  $V$ , the Rule says that  $D = \sqrt[3]{\frac{16V}{9}}$  which would make  $\pi = 3.375$ .

(3) Let the volume of a cube, its inscribed cylinder and inscribed sphere be  $V_1$ ,  $V_2$  and  $V_3$  respectively. The *Nine Chapters* then says  $V_1 : V_2 : V_3 = 16 : 12 : 9$  (Fig. 4.23).

(4) *Mouhefanggai* literally means “equally combined square umbrellas”. It is a symmetrical solid as if two square four-ribbed umbrellas were put together (Fig. 4.22). We shall call *mouhefanggai* “joined umbrellas”. Figure 4.24 shows its generation from a cube and cylinders.<sup>24</sup> Archimedes also studied the solid as the accumulation of thin slices<sup>25</sup> as in the usual method for finding the volume of a solid in his treatise *The Method*.

In the introductory greeting from Archimedes to Erastosthenes in the *Method* (see [HA]) we find a theorem: “2. If in a cube a cylinder be inscribed which has its bases in the opposite squares and touches with its surfaces the remaining four

<sup>24</sup>In 1979, an American teacher, Stannard, wrote a paper about the solid which appeared in *The Mathematical Teacher*; he called it a “birdcage”, see [261]. He subsequently wrote another article for the case where the axes are not orthogonal [262].

<sup>25</sup>Ancient mathematicians of the Western or Eastern world all considered these slices as being without thickness.

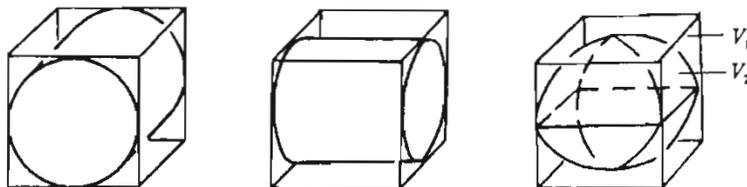


FIG. 4.24. Generating the “joined umbrellas” by intersecting two cylinders.

faces, and if there also be inscribed in the same cube another cylinder which has its bases in other opposite squares and touches with its surface the remaining four faces, then the figure bounded by the surfaces of the two cylinders, which is within both cylinders, is two-thirds of the whole cube.” In fact, the solid that Archimedes describes here, i.e. the common part of two perpendicular cylinders, is the same as the one Liu Hui called “joined umbrellas.” Proposition 15 of *The Method* gives the volume formula for this common part. It is a pity that the actual pages about the derivation have been lost. However, Zeuthen tried to restore the proof by imitating Proposition 2 and the diagram, see [247] and cf. note (27), below. His reconstruction is quite plausible. For explanation we use the diagram in Fig. 4.25. In the cube  $XYWV$ , there are two inscribed cylinders with central axes  $BD$  and  $TU$ , passing through  $O$  and perpendicular to  $BD$  and  $AC$ . Outside the cube there are a rectangular parallelepiped with square base, whose side  $LG$  is twice the side of the cube and whose height is as long as the side of the cubes, and a pyramid with the same base as the parallelepiped, with  $A$  as its vertex. Let any horizontal plane (e.g.  $MN$ ) cut the pyramid, the common part of the two inscribed cylinders and the parallelepiped. There are square sections  $QR^2$ ,  $KP^2$  and  $MN^2$  respectively. Extend  $CA$  to  $H$  so that  $CA = AH$ . Archimedes’ Method was to work out the volume by balancing it, slice by slice, against a known volume. Taking  $CH$  as the lever and  $A$  as the fulcrum, Archimedes derived  $QR^2 \cdot AH + KP^2 \cdot AH = MN^2 \cdot AS$ . Summing up the moments about  $A$  of all these horizontal sections, he got (Pyramid + the common part)  $\cdot AH = \text{parallelepiped} \cdot \frac{1}{2}AC$ , from which he obtained the common part, i.e. joined umbrellas  $= \frac{2}{3} \text{cube} = \frac{2}{3}D^3$ .

Although Archimedes’ *Method* had been lost for about 2000 years, the idea of the distinguished seventeenth century Italian mathematician Cavalieri for finding the volume of solids and the area of plane figures essentially coincides with that of Archimedes. He published a book entitled *Six Geometric Exercises* [237], in which he considers a line as made up of points like a string of beads, a plane as made up of lines like a cloth of threads, or a solid as made of plane areas like a book is made of pages. His theorem is now known as Cavalieri’s Principle, cf. note (24), below.

Before calculus was transmitted from the West into China, Li Shanlan independently discovered similar facts in treating volume problems. He wrote

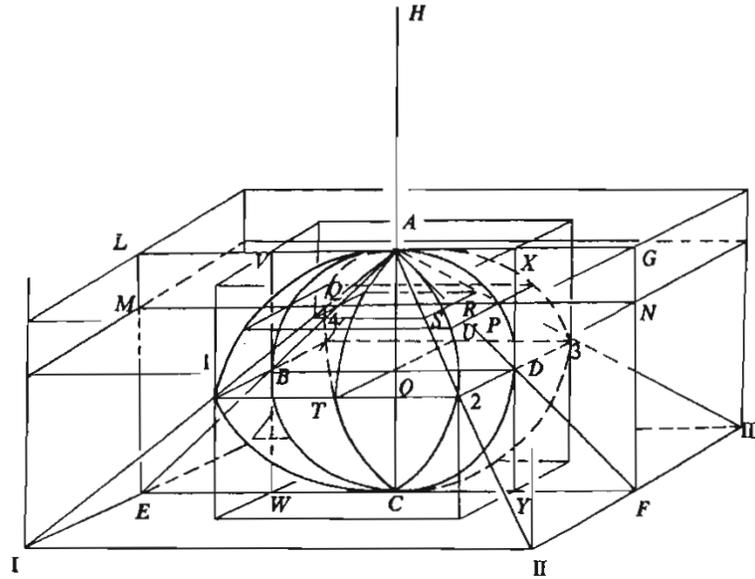


FIG. 4.25. The “joined umbrellas” as constructed by Archimedes in *The Method*.

this idea in one of his treatises, *Explanation of Secrets in Squares and Circles* (Fangyuan Chanyou, 方圓闡幽), as follows: “A book 1 *chi* thick is made up of sheets of paper, whereas taffeta 1 *zhang* long is made up of threads of silk.” It is interesting that he held the same ideas as Cavalieri, but Cavalieri did not know that his Principle had been summarized by the Chinese mathematician Zu Geng about 1000 years earlier. Using the same Principle but in a simpler way, Zu Geng had already derived the volume of the joined umbrellas.

In 1984, during the Third International Congress of Chinese Sciences in Beijing, J.C. Martzloff, the French historian of mathematics, told one of the authors [Shen] that Archimedes had studied the joined umbrellas. At that time Shen was unaware of Archimedes’ work. But now in reading Archimedes’ original treatise *The Method*, it is very gratifying to see that the mathematicians of East and West had independently solved the same problem in interestingly different ways; they had reached the same goal by different routes. The final result obtained was the same, but the approaches in East and West were very different. Archimedes’ *Method* is the only place we know where, as Heath says (see [HA]), “we have a sort of lifting of the veil, a glimpse of the interior of Archimedes’ workshop as it were.” Otherwise Euclidean mathematics is always very formal. In the work of Liu Hui and Zu Geng, however, we see the gradual working towards a solution and the final result is as elegant and clear as any mathematician could desire.

(5) *Yangma*, cf. Problem 5.15, see p. 269. Here a *yangma* with the two lateral cylindrical surfaces means one of the eight segments of the joined umbrellas (Fig. 4.26).

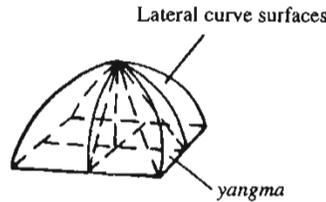


FIG. 4.26. “Joined umbrellas” and the *yangma*.

(6) Liu used the logical law of the excluded middle in one place only. If it is correct that the ratio of the volume of the joined umbrellas to that of the inscribed sphere is the same as the square rate to the circle rate, then it is not wrong to consider the cylinder to its inscribed sphere has the same ratio. However, his general approach is very constructive (see [144]).

(7) This means that  $\frac{3}{4}D^2 < \frac{\pi}{4}D^2$ , hence  $\frac{3}{4}D^3 < \frac{\pi}{4}D^3 = V_{\text{cylinder}}$ . On the other hand,  $V_{\text{cylinder}} > V_{\text{umbrellas}} > V_{\text{sphere}}$ , hence  $\frac{9}{16}D^3 \approx V_{\text{sphere}}$ . However, Liu said nothing in his derivation about  $\frac{9}{16}D^3 \approx V_{\text{sphere}}$ .

(8) “Regular blocks” mean special solids whose volumes are known, such as the cube, *qiandu* (整塔, prism), *yangma* (pyramid) and *bienao* (cf. Chapter 5, p. 251).

(9) In the Han Dynasty, 1 *cun* was about 2.31 cm. 1 cubic *cun* of pure gold should be  $19.32 \times 2.31^3 \approx 238$  g. Liu said it weighed 16 *liang*, i.e. about 250 g. Cf. Section 3 in the Introduction to this book, p. 9.

(10) *Kaogong Ji* (考工記) is Chapter VI of the *Administrative System of the Zhou Dynasty* (*Zhouguan*) (周官).

(11) Mr Li (栗) was an artisan in the Zhou Dynasty.

(12) Let the side of the cube be  $a$  and the diameter of the circumscribing sphere be  $D$ . Then  $a = \sqrt{D^2/3}$  (Fig. 4.27). A version of the same rule can be found as Proposition 15, Book 13 in Euclid’s *Elements*: “To construct a cube and comprehend it in a sphere like the pyramid; and to prove that the square on the diameter of the sphere is the triple of the square on the side of the cube.”

(13) This is checking the proposition given in note (12).  $BC^2 = \text{hypotenuse}^2 = \text{gou}^2 + \text{gu}^2 = AB^2 + AC^2 = 2(\text{gu})^2 D^2 = EB^2 = (\text{long hypotenuse})^2 = BC^2 + CE^2 = 3a^2 = 75$ .

(14) The volume of a cube circumscribing a sphere of diameter  $D$  is  $D^3$ .

(15) Since  $\sqrt{75}$  is not a perfect square, let  $D = \sqrt{75}$  and  $a = 5$ , then

$$\text{volume of the circumscribing sphere} : \text{volume of the inscribed sphere} = D^3 : a^3 = \sqrt{D^6} = \sqrt{a^6}.$$

That is,

$$(\sqrt{75})^3 : 5^3 = \sqrt{75^3} : \sqrt{125^2} = \sqrt{421\,875} : \sqrt{15\,625}$$

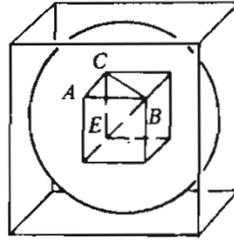


FIG. 4.27. Constructing a cube in a sphere.

$$= \sqrt{\frac{421\,875}{625}} : \sqrt{\frac{15\,625}{625}} = \sqrt{675} : \sqrt{25} = \sqrt{676-1} : 5 \approx 26 : 5.$$

(16) Zhang's approximation means  $\sqrt{675} : \sqrt{25} = \sqrt{26^2 - 1} : \sqrt{25} \approx 26 : 5$ .

(17) Zhang's second point is that the ratio of the volume of the cube to its inscribed sphere is  $\sqrt{64} : \sqrt{25} = 8 : 5$ .

(18) Zhang's third point: The ratio of the area of the square to its inscribed circle is  $\sqrt{8} : \sqrt{5}$ .

(19) Here Liu Hui applied the homogenizing and uniformizing operations on irrational numbers. There are two sets of rates:  $V_{\text{cube}} : V_{\text{cylinder}} = A_{\text{square}} : A_{\text{circle}} = \sqrt{8} : \sqrt{5}$ ,  $V_{\text{cube}} : V_{\text{sphere}} = 8 : 5$ . Thus  $V_{\text{cube}} : V_{\text{inscribed cylinder}} : V_{\text{inscribed sphere}} = 8 : \sqrt{5} \times \sqrt{8} : 5$ , where Liu uniformized the  $V_{\text{cube}}$ , to 8, and homogenized the  $V_{\text{inscribed cylinder}}$  to  $\sqrt{5} \times \sqrt{8}$ . Thus Zhang regarded

$$\begin{aligned} V_{\text{inscribed cylinder}} : V_{\text{inscribed sphere}} &= \sqrt{5} \times \sqrt{8} : 5 = \sqrt{8} : \sqrt{5} \\ &= A_{\text{square}} : A_{\text{circle}} = \text{square rate} : \text{circle rate}. \end{aligned}$$

(20) Suppose the volume of a cube is 26, then  $V_{\text{inscribed sphere}} = 26 \times \frac{9}{16} = \frac{117}{8}$ . According to Zhang's or Liu's ideas, the ratio of the volume of the cubes is  $26 : 5$ . Thus  $V_{\text{cube}} : V_{\text{circumscribed sphere}} = 5 : \frac{117}{8} = 40 : 117$ .

(21) Perimeter of square : circumference of its inscribed circle =  $\frac{A_{\text{square}}}{\text{half side}} : \frac{A_{\text{circle}}}{\text{radius}} = \sqrt{8} : \sqrt{5} = \sqrt{64} : \sqrt{40} = 4 : \sqrt{10}$ , i.e.  $\pi \approx \sqrt{10}$ .

(22) Li's criticism of Liu is not justified, since Liu himself always maintained: it was incorrect to take the cylinder as the square rate and its inscribed sphere as the circle rate.

(23) This means  $D = \sqrt[3]{2V_{\text{sphere}}}$ , where  $\pi \approx 3$ .

(24) (i) Draw a cylindrical surface with  $OX, OY$  as axes,  $\frac{D}{2}$  as radius, cutting the cube into 4 blocks: the inner is  $U_1$ , the outer are  $U_2, U_3$  and  $U_4$  (see Fig. 4.28).

(ii) Horizontal section  $F_1$ , with height  $z$  and side  $AG$ , where  $AG^2 = (\frac{D}{2})^2 - z^2$  (see Fig. 4.29).

(iii)  $(\frac{D}{2})^2 = F_1 + F_2 + F_3 + F_4$ .

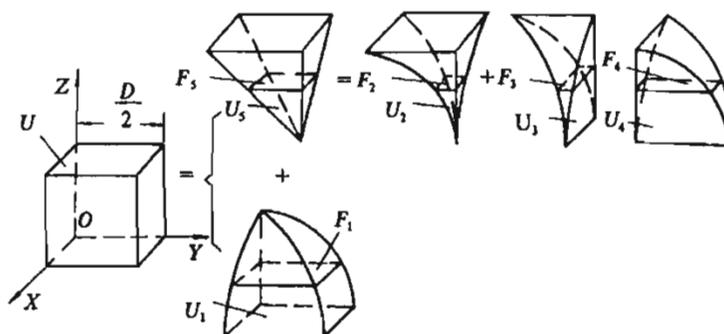


FIG. 4.28. Zu Geng's construction of the "joined umbrellas".

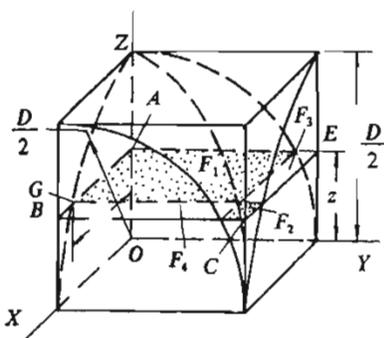


FIG. 4.29. Zu Geng's construction for the rate of the volume of the sphere to that of the "joined umbrellas".

(iv)  $z^2 = F_2 + F_3 + F_4$ .

(v)  $F_5 = z^2$ . (See Fig. 4.28.)

(vi) We call this proposition the Liu-Zu Principle, which is equivalent to Cavalieri's Principle in the West. Cavalieri was a professor at Bologna and Galileo's disciple. His book on indivisibles [236] was the first textbook on what we now call the method of integration. This work, regards an area as a sum of indivisibles, the line segments of which it is composed, and volumes as sums of plane areas. The book shows how to measure plane areas and solid volumes by comparing the indivisibles of one with those of another. By taking these indivisibles parallel to one another, Cavalieri invented the Principle which is still called by his name.

Cavalieri's Principle: "If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and if between the same parallel planes any

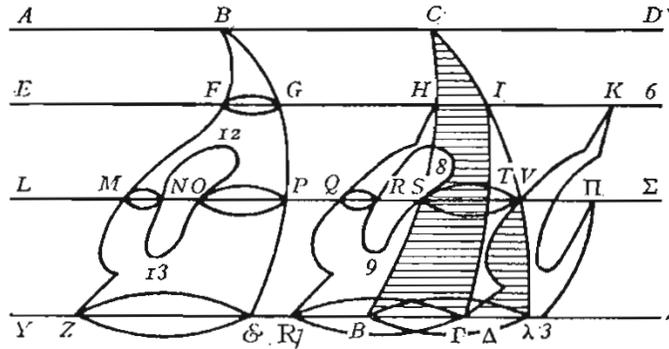


FIG. 4.30. Cavalieri's Principle. (From [236].)

solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the plane so drawn are equal, the solid figures are likewise equal to one another." (See Fig. 4.30 and [263], pp. 209–219.)

In modern notation Cavalieri's Principle is thus: Let the two lines  $x = a, x = b$  together with the continuous curves  $y = f_1(x)$  and  $y = f_2(x)$  enclose an area  $A_1$ ; and let the same lines together with the continuous curves  $y = \varphi_1(x); y = \varphi_2(x)$  enclose  $A_2$ . If for all  $x, a \leq x \leq b, f_2(x) - f_1(x) = \varphi_2(x) - \varphi_1(x)$ , then  $A_1 = \int_a^b (f_2(x) - f_1(x))dx = A_2 = \int_a^b (\varphi_2(x) - \varphi_1(x))dx$ .

Cavalieri's Principle is still useful in teaching the mensuration of solids since it allows the easy computation of definite integrals without the formal apparatus of the calculus, while the results are correct because they can be justified by limit arguments.

(vii)  $U_2 + U_3 + U_4 = U_5$ .

(viii)  $V_{\text{umbrellas}} = 8U_1 = \frac{2}{3}V_{\text{cube}}$ .

(ix)  $V_{\text{sphere}} = \frac{\pi}{4} \cdot \frac{2}{3}V_{\text{cube}} \approx \frac{1}{2}V_{\text{cube}}$ , where  $\pi \approx 3$ .

(25) Li's criticism is not appropriate, cf. note (22), above.

(26)  $V_{\text{sphere}} = \frac{11}{21}V_{\text{cube}}$ , where  $\pi \approx \frac{22}{7}$ .

(27) In his work *On the Sphere and the Cylinder* (see [HA], pp. 1–44), Archimedes, using the method of indirect proof and with 33 propositions as lemmas, gave his conclusion that  $V_{\text{sphere}} = 4V_{\text{cone}}$ , where the cone has the great circle of the sphere as its base and the radius as its height. On the other hand, in Proposition 2 of *The Method*, he discussed the sphere-volume formula further. There he regarded a solid as the accumulation of area elements, and deduced the same result with only one proposition, using a lever-balance equation.

Let  $ABCD$  be a great circle of a sphere, and  $AC, BD$  diameters at right angles to each another (Fig. 4.31). Let a circle be drawn about  $BD$  as diameter and in a plane perpendicular to  $AC$ , and on this circle as base let a cone be described with  $A$  as vertex. Let the surface of this cone be produced and then

# INDIA

Below are two scanned documents on the Indian calculation of the area and volume of the sphere:

1. A secondary source, Kim Plofker, *Mathematics in India*, Princeton Univ Press, 2009, Chapter 6, pp. 196-201, describing the work of Bhaskara II (1114-1185) on the sphere.

2. A literal translation by Takao Hayashi of verses 58-61 in the “Gola” or sphere section of Bhaskara’s treatise *Siddhānta Śīromani*, (Crest-jewel of the Siddhāntas) plus Bhaskara’s his own (prose) commentary.

## Some background:

a. Indian math was developed first in service of Vedic ceremonies, later in service of astronomy which, in turn, was driven by the market for astrology. All “documents” were originally in very concise cryptic Sanskrit verse, meant to be memorized and passed down with oral instruction by each guru. Only later are any justifications described.

b. Most likely under Greek influence in the wake of Alexander, planetary models with epicycles appear. Using these for predictions demanded tables of sines. Sines were made into integers by multiplying by a radius  $R$  equal to 3438 (which makes the length of one arc minute of the circumference very close to 1) and the resulting ‘Rsines’ were then rounded and computed at intervals of  $3\frac{3}{4}$  degrees (=30/8 degrees), using half angle formulas. The Rsine table itself would require a lot of memorization and indeed it was their *first differences* which were versified (using three different systems, the most colorful being using words -- such are ‘arms’ for 2).

c. At least by the mid 5<sup>th</sup> century CE, in the earliest surviving siddhanta, it is noted that the *second difference* of the sine table is proportional to the sine table itself, which allows the whole table to be constructed from scratch. The discovery of the discrete second order difference equation for sines, I believe, is the key which led to all the Indian work in calculus.

d. Bhaskara follows exactly Archimedes’ approach in ‘*On the Sphere and Cylinder*’: he finds the area of the sphere by integrating over latitude and is led to its approximation using the sum of sines over a regular sequence of angles from 0 to 90. Note that Bhaskara II also realized also that the *first differences* of sines are cosines (see Plofker below; this was not apparent earlier because it needs half-angles to come out right). Likewise differences of cosines are sines, so the required sum of sines could be found directly from this fact – and this appears explicitly some 2 centuries later in the work of Mādhava. But instead of using this identity, Bhaskara *adds the sines numerically* using his table in order to find the area! He clearly prefers numerical justifications to theoretical: a bias present in most of Indian mathematics (e.g. when they found Gregory’s series for pi, they also found ingenious ways to approximate the remainder in order to usefully sum it).

In this case, what we would write as the quadratic equation in  $x$ , where  $x$  is the desired number of monkeys in the troop, that is,  $\left(\frac{1}{5}x - 3\right) + 1 = x$ , has been solved by Bhāskara as the equation  $yā va 1 yā 55 rū 0 [=] yā va 0 yā 0 rū 250$ , where  $yā va$  stands for  $yāvattāvat vāryā$ , “the square of the unknown.” The two resulting values for  $x$ , from the two square roots in the quadratic, are 50 and 5. Only the first of these values, Bhāskara says, is “applicable” to this particular problem, because the second would make the “fifth part of the troop, minus three” equal to a negative number, which makes no sense in this context.

**Section 10. Equations in more than one unknown.** The remaining three sections build on the basic rules of Brahmagupta in the *Brāhma-sphuṭa-siddhānta*.<sup>37</sup> Linear equations in more than one unknown are solved by finding one unknown in terms of the others. If there are more unknowns than equations, so that the problem is indeterminate, the pulverizer is used.

**Section 11. Elimination of the middle with more than one unknown.** A single quadratic equation with more than one unknown is to be transformed into a square-nature problem if possible, and solved by indeterminate methods.

**Section 12. Products of unknowns.** Equations containing products of two or more unknowns are solved with arbitrarily chosen numbers, as directed by Brahmagupta.

**Section 13. Conclusion.** The last few verses contain information about Bhāskara’s background and his work, surveyed at the beginning of this section.

### 6.2.3 The *Siddhānta-śīromāṇi*

This work (literally “Crest-jewel of *siddhāntas*”) was composed when Bhāskara was 36, that is, in 1150. It lived up to the boast in its title by gaining a high place among astronomical treatises, although as an orthodox Brāhmaṇa pakṣa work it could not supplant the canonical texts of the other pakṣas. The *Siddhānta-śīromāṇi* is sometimes described as containing the *Līlāvātī* and *Bīja-gaṇita* in addition to its two sections on astronomy proper, and it is evident that Bhāskara considered the subject matter of all these compositions to be very closely linked. But since the arithmetic and algebra texts have individual titles and have usually been copied as individual manuscripts, we will follow the tradition of considering the *Siddhānta-śīromāṇi* a separate work devoted to astronomy.

The treatise is divided into two sections: the first, on planetary calculations, presents standard computational algorithms like discussed in section 4.3 for calculating mean motions, true motions, the Three Questions, lunar and solar eclipses, and so forth. The second, on *gola*, contains chapters on the following subjects: praise of the sphere, the form of the sphere, the

sphere of the earth, explanations of various subjects from the *gaṇita* section, instruments, a poetic description of the seasons, and questions to test the student’s knowledge. As it is not possible to do justice here to even a substantial portion of this comprehensive work, we will content ourselves with pointing out a few examples of Bhāskara’s ingenious manipulations of small quantities and his explanations in his own commentary *Vāsanā-bhāṣya*, or “Commentary of rationales.”

The first example, from the first section’s chapter on true motions, involves calculating the speed of a planet’s motion through the sky. We have seen in section 4.3.2 how to calculate a planet’s mean speed  $R/D$ , where  $R$  is the number of the planet’s integer revolutions in a given time period and  $D$  is the number of days in that period. But usually a planet’s apparent motion will be slower or faster than that mean motion. Roughly, the speed will be least when the planet’s anomaly  $\kappa$  (see section 4.3.3) is zero (i.e., when the planet is at apogee), greatest when the anomaly is  $180^\circ$ , and close to the mean speed when the anomaly is about  $90^\circ$ . If we want to know how fast a planet appears to be moving at some given time, how should we compute that? This problem of *tātkālika* or “at-that-time” motion was tackled in various ways by Indian astronomers; one typical strategy involved calculating the difference between the true and mean speeds as approximately proportional to the Sine-difference corresponding to the value of the anomaly. In following this approach, Bhāskara made use of a concept he called an “instantaneous Sine-difference,” computed from the Cosine by a Rule of Three Quantities:

The difference between today’s and tomorrow’s true [positions of a] planet ... is the true [daily] speed... [At some point] within that time [or, on average in that time] the planet is required to move with that speed. Yet this is the approximate speed. Now the accurate [speed] for that time [or, instantaneous (*tātkālika*) speed] is described...

If a Sine-difference equal to five-two-two [i.e., 225] is obtained with a Cosine equal to the Radius, then what [is obtained] with a desired [Cosine]? Here, five-two-two is the multiplier and the Radius is the divisor of the Cosine. The result is the accurate Sine-difference at that time.<sup>38</sup>

In other words, the “at-that-time” Sine-difference  $\Delta \text{Sin}$  for a given arc  $\alpha$  is considered simply proportional to the Cosine of  $\alpha$ :

$$\Delta \text{Sin} = \text{Cos } \alpha \cdot \frac{225}{R}$$

It has been noted<sup>39</sup> that this and related statements reveal similarities between Bhāskara’s ideas of motion and concepts in differential calculus. (In fact, perhaps these ratios of small quantities are what he was referring to in

<sup>38</sup> *Vāsanā-bhāṣya* on *Siddhānta-śīromāṇi* Ga.2.36–38, [SasB1989], pp. 52–53. The method is explained in detail in [Ike2004].

<sup>39</sup> For example, in [Rao2004], pp. 162–163.

<sup>37</sup> They are described more fully in [DatSii1962], vol. 2, pp. 57–59, 181–193, and 199–201, respectively.

his commentary on *Līlāvātī* 47 when he spoke of calculations with factors of  $0/0$  being “useful in astronomy.”) This analogy should not be stretched too far: for one thing, Bhāskara is dealing with particular increments of particular trigonometric quantities, not with general functions or rates of change in the abstract. But it does bring out the conceptual boldness of the idea of an instantaneous speed, and of its derivation by means of ratios of small increments.

The *gola* section of the *Siddhānta-śīromāṇi* begins with an exhortation on the importance of understanding astronomy’s geometric models that is somewhat reminiscent of Lalla’s remarks quoted in section 4.5, but also emphasizes the need for their mathematical demonstration:

A mathematician [knowing only] the calculation of the planets [stated] here [in the chapters on] mean motions and so forth, without the demonstration of that, will not attain greatness in the assemblies of the eminent, [and] will himself not be free from doubt. In the sphere, that clear [demonstration] is perceived directly like a fruit in the hand. Therefore I am undertaking the subject of the sphere as a means to understanding demonstrations.

Like flavorful food without ghee and a kingdom deprived of [its] king, like an assembly without a good speaker, so is a mathematician ignorant of the sphere.<sup>40</sup>

An example of what Bhāskara means by a demonstration can be seen in his chapter on the terrestrial globe, where he criticizes the value for the size of the earth given by Lalla in the *Śiṣya-dhā-vṛddhida-tantra*, arguing that it is erroneous because of Lalla’s erroneous rule for the surface area of a sphere. This rule, which Lalla allegedly stated in a now lost work on *gaṇita*, says that the surface area is equal to the area of a great circle times its circumference (or  $2\pi^2r^3$ ). Bhāskara remarks, “Because of the error in the computation stated by Lalla, the surface area of the [spherical] earth is wrong too.”<sup>41</sup> He justifies his criticism by explaining his own rules from *Līlāvātī* 199–201 for the sphere’s surface area and volume, which are equivalent to  $4\pi r^2$  and  $\frac{4}{3}\pi r^3$ , respectively, for a sphere of radius  $r$ . He starts out by imagining equidistant great circles like longitude circles on the sphere’s surface, and approximating the area of one spherical lune (a portion of the surface cut off between two adjacent great semicircles, like the skin of a segment of an orange):

The circumference of a sphere is to be considered [as having] measure equal to the amount of Sines, times four [i.e.,  $24 \times 4 = 96$ ].

<sup>40</sup> *Siddhānta-śīromāṇi* Go.1.2–3, [SasB1989], pp. 175–176. See also [Srin2005], pp. 228–229.

<sup>41</sup> *Vāsanā-bhāṣya* on *Siddhānta-śīromāṇi* Go.3.54–57, [SasB1989], p. 187. See also [Chal1981] 2, pp. xx–xxi, 250–251, and [Hay1997b], pp. 198–199.

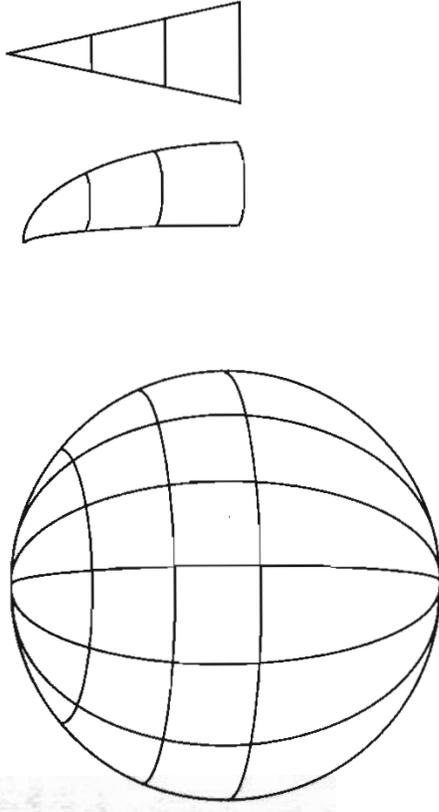


Figure 6.1 Dividing the surface of the sphere to compute its area.

Spherical lunes [literally “mounds” or “circumferences”] are perceived [when the surface is] divided by multiple lines going from the top to the bottom as on the surface of the ball of an *amla* [Indian gooseberry] fruit. When one has set out spherical lunes [equal in] number to the prescribed [divisions of the] circumference . . . the area in one lune is to be determined thus. . . .

The circumference in the sphere is assumed [to be] equal to ninety-six cubits, and that many lunes are made with vertical lines at each cubit. Then when one has made horizontal lines at each cubit-interval on half of one lune, twenty-four lune-portions are imagined, [equal in] number to the Sines. Then the Sines separately divided by the Radius are the measures of the horizontal lines. In that case the lowest line is equal to [one] cubit, while [the lines] above are successively somewhat less, in accordance with [the sequence of] the Sines. The altitude is always just equal to [one] cubit. When one has found the areas of the portions equal to the sum of the base and the top multiplied by the altitude, [they are all] added. That is the area in half a lune; that times two is the area in one lune.<sup>42</sup>

The following explanation and amplification of Bhāskara’s arguments rely on the sphere shown in figure 6.1, with circumference  $4\pi r$ , where  $\pi$  is the number of Sines in a quadrant. (Bhāskara takes  $\pi = 24$ , but our figure for

<sup>42</sup> *Vāsanā-bhāṣya* on *Siddhānta-śīromāṇi* Go.3.58–61, [SasB1989], p. 188. This rationale is preceded in Bhāskara’s discussion by a similar one that imagines a hemisphere divided into zones by small circles parallel to the equator, with a spherical cap on top. The areas of the zones spread out into long trapezoidal strips, plus the circular area of the spherical cap, add up to the total area of the hemisphere (*Vāsanā-bhāṣya* on *Siddhānta-śīromāṇi* Go.3.54–57, [SasB1989], pp. 187–188). See the description of both methods in [Sara1979], pp. 211–213, and the translation and explanation in [Hay1997b], pp. 202–217.

simplicity shows  $n = 3$ .) The sphere's surface is divided vertically into  $4n$  equal lunes of unit width at the equator, and the top half of each lune is divided horizontally into  $n$  segments by small circles parallel to the equator at unit intervals.

The radius of each  $i$ th parallel circle (taking  $i = 0$  at the equator) is proportional to the Cosine of its elevation above the equator. (If this is not immediately obvious, think—as Bhāskara certainly would have—of parallel day-circles on the celestial sphere whose radii are equal to the Cosines of their declinations, as explained in section 4.3.4.) If we let  $A_L$  be the area of one lune, we can consider that the area  $A_L/2$  of the individual half-lune shown on the right side of figure 6.1 is approximately equal to that of the triangular figure corresponding to it. That triangle is a stack of  $(n - 1)$  trapezoids with a triangle on top, all of which are considered to have unit altitude. The equally spaced horizontal line segments dividing them are the chords of the corresponding unit arcs of the parallel circles in the half-lune. Assuming that the lowest of these line segments  $s_0$  has unit length, and that the unit arc on the sphere contains  $u$  degrees, we can express the length  $s_i$  of each  $i$ th horizontal line segment by

$$s_i = 1 \cdot \frac{\text{Cos}(iu)}{R} = 1 \cdot \frac{\text{Sin}(90 - iu)}{R}.$$

Hence, as Bhāskara says, the measures of the line segments  $s_i$  are the Sines separately divided by the Radius. Then the area of the lowest trapezoid in the half-lune will be  $\frac{s_0 + s_1}{2} \cdot 1$ , and so on up to the top triangle, whose area

will be  $\frac{s_{n-1} + 0}{2} \cdot 1$ . The sum of all of them will be the total area  $A_L/2$  of the half-lune, which we may express in terms of the  $n$  Sines as follows:

$$\begin{aligned} \frac{A_L}{2} &= \left( \frac{\text{Sin}_n + \text{Sin}_{n-1}}{2} + \frac{\text{Sin}_{n-1} + \text{Sin}_{n-2}}{2} + \dots + \frac{\text{Sin}_2 + \text{Sin}_1}{2} + \frac{\text{Sin}_1 + 0}{2} \right) \frac{1}{R} \\ &= \left( \frac{\text{Sin}_n}{2} + \sum_{i=1}^{n-1} \text{Sin}_i \right) \frac{1}{R} = \left( \sum_{i=1}^{n-1} \text{Sin}_i - \frac{\text{Sin}_n}{2} \right) \frac{1}{R}. \end{aligned}$$

The area  $A_L$  of the whole lune must be twice that amount. Since  $\text{Sin}_n = R$ , area  $A_L$  will indeed be the sum of all the Sines minus half the Radius and divided by half the Radius, just as Bhāskara says.

To find a simpler expression for that sum of all the Sines, he then switches from a geometrical demonstration to a numerical illustration—that is, just adding up their known values:

For the sake of determining that [lune area], this rule [was stated]:

“The sum of all the Sines is decreased by half the Radius [and divided by half the Radius]” [verse Go.3.60cd]. Here the sum of all the sines beginning with 225 is 54,233. [When] that is decreased by half the Radius, the result is 52,514. [When] that is

divided by half the Radius, the result is the area of one lune [and equal to the diameter, 30;33. Because the diameter of a sphere with circumference ninety-six is just that much, 30;33, and the lunes are equal [in number] to [the divisions of] the circumference, therefore the area of the surface of the sphere is equal to the product of the circumference and the diameter; thus it is demonstrated.<sup>43</sup>

The sum of all the Sines in the standard Sine table, from  $\text{Sin}_1 = 225$  to  $\text{Sin}_{24} = 3438 = R$ , is 54232 (Bhāskara says 54,233), which diminished by  $R/2$  equals 52,513 (Bhāskara says 52,514). Dividing by  $R/2$ , we get a little over  $30 \frac{1}{2}$ , or 30;33 to the nearest sixtieth, which is in fact the diameter of a circle with circumference ninety-six units. From this Bhāskara infers the general result that the area  $A_L$  of the lune is equal to the diameter of the sphere. Since there are as many lunes as there are units in the sphere's circumference, the total area  $A$  of the sphere's surface therefore is just its circumference times its diameter.

Now that the formula for the surface area is demonstrated, Bhāskara uses it to explain the formula for the volume:

And in the same way, that area produced from the surface of a sphere, multiplied by the diameter [and] divided by six, is called the accurate solid [volume] within the sphere.... Here is the demonstration: Square pyramidal holes [literally “needle-excavations”] with unit [base]-sides [and] depth equal to the half-diameter, [equal in] number to [the divisions of] the area, are imagined in the surface of the sphere. The meeting-point of the tips of the pyramids is inside the sphere. Thus the sum of the pyramid amounts is the solid amount; thus it is demonstrated.<sup>44</sup>

Using the same imagined unit grid, this time covering the whole of the sphere's surface, Bhāskara now considers it as made up of unit squares which are the bases of square pyramidal holes bored into the sphere, with depth equal to the sphere's radius  $r$ . The sum of the volumes of all the pyramidal holes is the total volume of the sphere. Bhāskara leaves it to the reader to recall that the volume of each pyramid will be one-third the product of its depth and the area of its base. So the sum of the volumes will be one-third the product of the total surface area times the depth, or  $\frac{1}{3} A \cdot r = \frac{1}{6} A \cdot 2r$ .

Elsewhere in the *Siddhānta-śiromaṇi*, in his chapters on eclipses, Bhāskara again criticizes Lalla's mathematics, this time concerning the geometry underlying computations for eclipse diagrams. As we saw in section 4.3.5, these computations involve a quantity called the “deflection” or deviation of the path of the ecliptic away from the east-west direction on the disk of the

<sup>43</sup> *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.58–61, [SasB1989], pp. 188–189.

<sup>44</sup> *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.58–61, [SasB1989], p. 189. See also [Sara1979], p. 213.

Now, in order to teach the beginner, one should show <the derivation of the correct formula> on a sphere. Having made an earthen or wooden globe of the earth, having supposed that it has a circumference equal to the <number of> minutes, 21600, of a disc, and having put a dot at its summit, one should produce, from that dot <as a center>, a circular line <on the surface> by means of <a thread>, which, corresponding to a ninety-sixth part of the globe, has a length of <the arrows, two, a twin>, 225, and a form of an arc. Again, from the same dot, <one should produce> another <circular line> by means of a thread twice as long as that, and another by means of <a thread> thrice as long as that, and so on, up to twenty-four times as long. Twenty-four circles are produced <in this way>.

The radii of these circles will be the half chords (i. e., Sines) beginning with ‘<the arrows, eyes, arms>’, 225, <which have been versified in the Sine table of Lalla.<sup>28)</sup>>

From them by proportion the sizes of <the circumferences of> the circles <are obtained>. Among them, first of all, the size of the last circle is the minutes in a disc, 21600. Its radius is the Sine of Three <Houses> (i. e., of ninety degrees), 3438. The half chords (i. e., Sines), when multiplied by the minutes of a disc (21600) and divided by the Sine of Three (3438), become the sizes of the circles.

There is one geometric figure having the shape of a belt (*valaya*) between every two consecutive circles. They are twenty-four in number. In a case where many <more> Sines are supposed, there would be many <more belt-like figures>.

There, when one has supposed the lower, greater circle to be the base, the above, smaller one to be the face, and <the arrows, two, a twin> (225) to be the perpendicular, the area of each <belt-like figure is calculated> severally by means of <the rule>:

Half the sum of the base and the face is multiplied by the perpendicular. (Tr 42d)

The sum of those areas is the surface area of the hemisphere. That multiplied by two is the surface area of the whole sphere. It shall be equal to the product of the diameter and the circumference. <The end of the commentary on Stanzas> 54—57.

Now, <the derivation of the rule> is explained in a different way.

58. The circumference of a sphere should be supposed to be measured by

the number of the Sines multiplied by «the Vedas» (4). Just as segments (*vapṛakas*) on an *āmalaka*<sup>29)</sup> fruit are observed to be <separated> by means of lines passing through the top and the bottom,

59. just so one should suppose segments on a sphere, <separated> by means of lines made vertically, as many as <the units in> the circumference told above.

60. There, the area of one segment is obtained by means of parts (*khaṇḍas*); that is, the sum of all the Sines is decreased by half the Sine of Three Houses and divided by half the Sine of Three.

61. Thus is <obtained> the area of a segment. Since it must be equal to the diameter of the sphere, the area of the surface of a sphere is remembered to be the product of the circumference and the diameter.

Here, the number of the Sines <tabulated> in any optional book is multiplied by four. On a sphere, the circumference should be regarded as being measured by it. Just as, on the surface of the globe of an *āmalaka* fruit, segments divided by natural lines passing through the top and the bottom are observed, just so, on the surface of any optional sphere, when one has supposed segments <divided> by lines going from the top to the bottom as many as the supposed units in the circumference, the area on one segment should be obtained. It is as follows.

In this *Dhivyḍhida*<sup>30)</sup> twenty-four Sines <have been tabulated>. The circumference is, therefore, supposed to be ninety-six *hastas* (i. e., units). The same number of segments are made with vertical lines <drawn> at every *hasta*. On the <upper or lower> half of one of those segments, when one has made horizontal lines at intervals of one *hasta*, as many parts as the number of the <tabulated> Sines, twenty-four <in the present case>, are supposed. The Sines there, severally divided by the Sine of Three, become the sizes of the horizontal lines. The low<est> line among them is measured by one *hasta*, while the upper ones decrease little by little due to <the diminution of> the Sines. The perpendicular is measured by one *hasta* everywhere. When one has calculated the areas of the parts by means of <the rule>:

Half the sum of the base and the face is multiplied by the perpendicular, (Tr 42d)

they are made into one (i. e., summed up). That is the area on the half of a segment. That multiplied by two becomes the area on one segment. What exists here in order to obtain it is this rule:

The sum of all the Sines is decreased by half the Sine of Three Houses, etc.  
(Stanza 60cd)

Here (in our case), 'the sum' of 'all the Sines', that is, '«the arrows, eyes, arms» (225)', etc.,<sup>31)</sup> is equal to «the gods, a twin, Kṛta, arrows», 54233. This, decreased by half the

Sine of Three, becomes what is measured by 《Manu, the principles, five》, 52514. This, divided by half the Sine of Three, becomes the area on one segment, 30 ; 33, which is equal to the diameter, since this much will be the diameter, 30 ; 33, of a sphere whose circumference is ninety-six. The segments are equal (in number) to (the number of units in) the circumference. It has been proved (or derived), therefore, that 'the product of the circumference and the diameter is the surface area of a sphere'. The same has also been stated in our mathematics of algorithms (*pāṭiganita*):

In a circular figure, a quarter of the diameter multiplied by the circumference is the area, which, multiplied (lit. pounded) by 《the Vedas》 (4), is the surface area (lit. the fruit produced from the surface) of the sphere like a net all around the surface of a hand ball. That, too, when multiplied by the diameter and divided by six, becomes the exact thing called a cube (or solid) (contained) inside the sphere. (L 201)

A sixth part of the surface area of a sphere multiplied by the diameter shall be the volume. Proof of this: Needle-like ditches, as many as (the units in) the surface area, whose arms (sides) are unity, and whose depth is equal to the radius, should be supposed on (the inner side of) the surface of a sphere. The tips of the needles fall together to the center of the sphere. In this way, the sum of the volumes of the needles is the volume (of the sphere). Thus has been proved (the formula for the volume of a sphere).

'The area multiplied by the square root of the area shall be the volume': the teacher Caturveda (Pṛthūdakasvāmin) stated this<sup>32)</sup> perhaps as an opinion of others.<sup>33)</sup> (The end of the commentary on Stanzas) 58-61.