## About the Cover

Measurement of a circle

This month's cover is the diagram accompanying XII.2 from Euclid's *Elements*, taken from the D'Orville manuscript, copied in 888 A. D. and now at the Bodleian Library of Oxford University. This is one of the two oldest extant complete manuscripts of the *Elements*. The entire manuscript has been photographed and made available at the Clay Institute's website `http://www.claymath.org/library/historical/euclid/`.
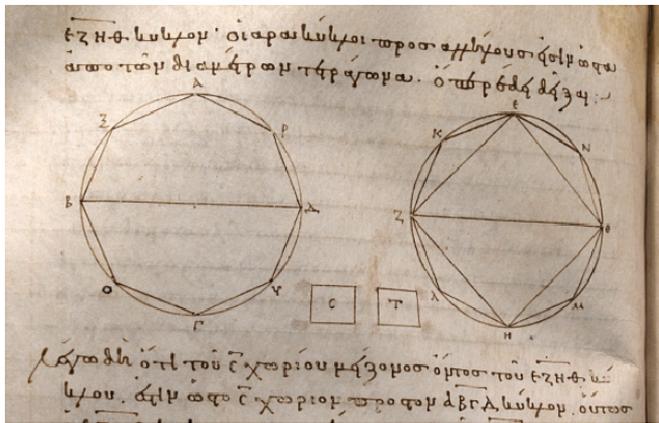
The cover loosely accompanies Len Berggren's review in this issue of *The Archimedes Codex* by Reviel Netz and Will Noel (see also the cover for the August issue of the *Notices*). Much of Archimedes' greatest mathematics involves what was a long time after him called the "method of exhaustion". He himself seems to attribute the technique to Eudoxus, a contemporary of Plato. But our source for Eudoxus' results is Book XII of the *Elements*, which applies it several times, including to the proof of XII.2. In modern notation the Proposition is stated like this: *Given two circles C_i of radii r_i with areas A_i*

$$A_1 : A_2 = r_1^2 : r_2^2 .$$

Euclid didn't have algebraic techniques available to him, but even so results of Book V of the *Elements* then imply that

$$A_1 : r_1^2 = A_2 : r_2^2 .$$

The ratio A/r^2 must therefore be the same for all circles, and it's hard not to think that this must have occurred to Euclid. But he never mentions anything like it in anything that has come down to us.

The Greek notion of ratio is rather complicated, but it is essentially the same as what we would call a real number, and to us the invariant ratio is of course the number π. It was Archimedes who took the matter up in what was perhaps his most popular work, "Measurement of a circle". It has come down to us in a strange form, however—abbreviated, sloppily written, and with no trace of the Sicilian variant of the Doric dialect which is apparent in many of his other writings. This is one of the pieces contained in the Archimedes Palimpsest, and it would have been marvelous if the copy there gave us something new. Unfortunately, as Netz writes to us:

> Substantial fragments of "Measurement of a circle" survive in the Palimpsest. They present serious problems of legibility, as the parchment is very damaged. (They happen to come from pages positioned right at the end of the prayer book). Images are available but are not very informative. The text that can be recovered agrees very well indeed with that known from elsewhere so apparently no new light is thrown on the textual history of the treatise—which is indeed a well known puzzle.

One would like to understand what led Archimedes to look at the circle differently from Euclid, and to approximate π numerically. Was there any earlier discussion of this ratio? Or did Archimedes do something completely new? One technical point new in Archimedes is that he discusses confidently the lengths of curves. The question about the ratio A/r^2 arises without any reference to arc length, but one of Archimedes' insights was to translate it into a question about the length of the circumference of a circle. Modern mathematics defines this to be a limit, but we also know that the limit for arbitrary curves sometimes doesn't exist. Archimedes gets around this difficulty in a clever way, one that seems to me one of his most remarkable achievements. He deals with the problem of lengths of curved paths as well as the area of curved surfaces explicitly in his work "Sphere and cylinder"—he doesn't define the length of a curve, but he axiomatizes its properties for the restricted class of convex arcs. The important part of the characterization is that if one is given two convex arcs between two points P and Q with one arc inside the other, like this—
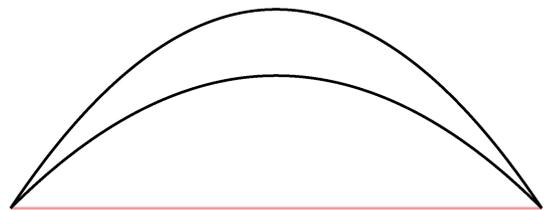
—then the length of the inside arc is less than or equal to that of the outer one. This is somewhat intuitively apparent, but doesn't seem quite obvious enough to qualify it as an axiom. The case where the inner arc is a single line segment and the outer one is a pair of them is Proposition I.20 of the *Elements*, and that in which both consist of a pair of segments is I.21. Other special cases were later discussed by Archimedes' sixth century commentator Eutocius (available in English in Netz' edition of Archimedes' works). The general case is then an application of mathematical induction. One can speculate that Archimedes was reassured by consideration of at least a few cases of this kind.

—Bill Casselman, Graphics Editor