



# ARCHIMEDES

BY

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*translated by C. Diksboorn*

With a new bibliographic essay by Wilbur R. Knorr

*Summis ingeniis dux et magister fuit*

(Heiberg, *Archimedis opera omnia* III,  
Prolegomena XCV)

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This follows at once from Prop. 13.

To conclude Book I, the centre of gravity of a trapezium is determined.

**Proposition 15.**

*In any trapezium having two parallel sides<sup>1)</sup> the centre of gravity lies on the straight line joining the middle points of the parallel sides, in such a way that the segment of it having the middle point of the smaller of the parallel sides for extremity is to the remaining segment as the sum of double the greater plus the smaller is to the sum of double the smaller plus the greater of the parallel sides.*

In Fig. 127, in the trapezium  $AB\Gamma\Delta$  ( $A\Delta \parallel B\Gamma$ ) let the non-parallel sides produced meet in  $H$ ;  $Z$  and  $E$  are successively the

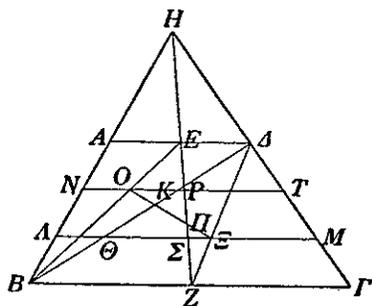


Fig. 127.

middle points of  $B\Gamma$  and  $A\Delta$ . The centres of gravity of the triangles  $HBF$  and  $HAA$  by Prop. 13 lie on  $HZ$ , therefore (Prop. 8 and postulate VII) the required centre of gravity of  $AB\Gamma\Delta$  lies on the line segment  $EZ$ .

Now divide  $BA$  into three equal parts by means of the points  $K$  and  $\Theta$ , and through these points draw  $NT$  and  $AM$  parallel to  $B\Gamma$ . Now the point of intersection  $E$  of  $AM$  and  $\Delta Z$  is the centre of gravity of  $\triangle AB\Gamma$ , and likewise the point of intersection  $O$  of  $N\Gamma$  and  $BE$  is the centre of gravity of  $\triangle BAA$ . The centre of gravity of the trapezium therefore is the point of intersection of  $EZ$  and  $O\Theta$ . Now we have by Prop. 6 or 7:

$$(\Delta B\Gamma, BAA) = (O\Pi, \Xi\Pi) = (P\Pi, \Sigma\Pi),$$

<sup>1)</sup> Trapezium without any further indication refers to a quadrilateral.

and consequently also

$$(B\Gamma, A\Delta) = (P\Pi, \Sigma\Pi),$$

whence

$$\begin{aligned} (B\Gamma, P\Pi) &= (A\Delta, \Sigma\Pi) = (2B\Gamma + A\Delta, 2P\Pi + \Sigma\Pi) \\ &= (B\Gamma + 2A\Delta, P\Pi + 2\Sigma\Pi). \end{aligned}$$

From this it follows that

$$(2B\Gamma + A\Delta, H\Theta) = (B\Gamma + 2A\Delta, Z\Pi),$$

which is equivalent to that which was to be proved.

CHAPTER X

THE METHOD OF MECHANICAL THEOREMS

1. We shall now, deviating from the order in which Archimedes' works appear in Heiberg's edition of the text, first discuss the treatise *The Method of Mechanical Theorems*, for Eratosthenes, to be briefly designated as *The Method*. In fact, when we know this work, it is easier to understand the *Quadrature of the Parabola*, the contents of which in turn are assumed as known in Book II of *On the Equilibrium of Planes*.

The discovery and decipherment of the manuscript of the *Method* has already been discussed in Chapter II. The object of the work becomes clear from the introductory letter to Eratosthenes, which we first give in translation:

*Archimedes to Eratosthenes greeting!*

*On an earlier occasion I sent you some of the theorems found by me, the propositions of which I had written down, urging you to find the proofs which I did not yet communicate at the time. The propositions of the theorems I sent were the following:*

*firstly: if in a right prism having a square<sup>1)</sup> for its base a cylinder be inscribed which has its bases in the squares facing each other and its sides in the other faces of the prism<sup>2)</sup>, and a plane be drawn through*

<sup>1)</sup> The word used is *παράλληλογράμμον*, but it is clear from the context that a square is meant.

<sup>2)</sup> The meaning is that the curved surface of the cylinder touches the vertical faces.

the centre of the circle which is the base of the cylinder and one side of the square in the opposite base, this plane will cut off from the cylinder a portion which is bounded by two planes and the surface of the cylinder, viz. the plane drawn, the plane in which lies the base of the cylinder, and the cylinder surface between the said planes; and the portion cut off from the cylinder is one-sixth of the whole prism<sup>1</sup>).

The proposition of the other theorem is as follows: if in a cube a cylinder be inscribed which has its bases in opposite squares and the surface of which touches the four other faces, and if in the same cube another cylinder be inscribed which has its bases in other squares and the surface of which touches the four other faces, the solid bounded by the surfaces of the cylinders, which is enclosed by the two cylinders, is two-thirds of the whole cube . . .<sup>2</sup>)

I will send you the proofs of these theorems in this book.

Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.

That is the reason why, in the case of the theorems the proofs of which Eudoxus was the first to discover, viz. on the cone and the pyramid, that the cone is one-third of the cylinder and the pyramid one-third of the prism having the same base and equal height, no small share of the credit should be given to Democritus, who was the first to state the fact about the said figure<sup>3</sup>), though without proof.

<sup>1</sup>) The derivation of this result is found in Propositions 12–15.

<sup>2</sup>) The derivation of this result is not to be found in the part of the *Method* that has been preserved.

<sup>3</sup>) It has struck students that Archimedes uses the singular here, whereas he first referred to theorems on the cone and the pyramid. It does not, however, seem likely that any inferences can be made from this.

My own experience is also that I discovered the theorem now published<sup>1</sup>), in the same way as the earlier ones<sup>2</sup>).

I now wish to describe the method in writing, partly, because I have already spoken about it before, that I may not impress some people as having uttered idle talk<sup>3</sup>), partly because I am convinced that it will prove very useful for mathematics; in fact, I presume there will be some among the present as well as future generations who by means of the method here explained will be enabled to find other theorems which have not yet fallen to our share.

We will now first write down what first became clear to us by the mechanical method, viz. that any segment of an orthotome is larger by one-third than the triangle which has the same base and equal height, and thereafter all the things that have become clear in this way. At the end of the book we will give the geometrical proofs of the theorems the propositions of which we sent you on an earlier occasion<sup>4</sup>).

In this exceptionally interesting document Archimedes therefore vouchsafes us a much more intimate glimpse of his mathematical workshop than was ever granted by any other Greek mathematician. In fact, Greek mathematics is characterized—and in this respect, too, it founded a tradition which was to last down to our own time—by a care of the form of the mathematical argument which, superficially viewed, seems almost exaggerated. It demands the inexorably proceeding, irrefutably persuading sequence of logical conclusions constituting the synthetic method of demonstration, but to this it sacrifices the reader's wish to gain also an insight into the method by which the result was first discovered. It is this wish, however, which Archimedes meets in his *Method*: he will reveal how he himself, long before he knew how to prove his theorems, became convinced of their truth.

2. The work opens with a number of lemmas on centres of gravity,

<sup>1</sup>) Here again Archimedes is evidently referring to the two above mentioned propositions.

<sup>2</sup>) This refers to a number of theorems from S.C., C.S., and Q.P., as will become clear in the subsequent discussion of the *Method*.

<sup>3</sup>) In the preface to Q.P. (*Opera* II, 262, lines 11–13) Archimedes says that the theorem on the area of a segment of an orthotome first became clear to him by the mechanical method, and that he then proved it geometrically.

<sup>4</sup>) These are apparently the two theorems mentioned at the beginning of the letter.



and the segment  $\alpha\beta\gamma$  about  $\vartheta$  as centre of gravity, we have (reverse of Pl.Ae. I, 6, 7)

$$(\Delta \alpha\zeta\gamma, \text{segment } \alpha\beta\gamma) = (\kappa\vartheta, \kappa\chi) = (3,1),$$

whence

$$\Delta \alpha\zeta\gamma = 3 \cdot \text{segment } \alpha\beta\gamma.$$

Since

$$\Delta \alpha\zeta\gamma = 2 \cdot \Delta \alpha\epsilon\gamma = 4 \cdot \Delta \alpha\beta\gamma,$$

it is found that

$$\text{segment } \alpha\beta\gamma = \frac{4}{3} \cdot \Delta \alpha\beta\gamma.$$

*This has not therefore been proved by the above, but a certain impression has been created that the conclusion is true. Since we thus see that the conclusion has not been proved, but we suppose it is true, we shall mention the previously published geometrical proof, which we ourselves have found for it, in its appointed place<sup>1</sup>).*

It would seem here that Archimedes intended to collect at the end of the *Method* the exact mathematical proofs of all the theorems found by the method described, even if, as is the case with the theorem in *Quadrature of the Parabola* which has just been discussed, they had already been published previously.

3. The method which Archimedes wishes to explain emerges so clearly from the proposition dealt with that we can proceed to discuss it already here.

We may note first of all that it is characterized by the application of two different principles: in the first place it makes use of considerations taken from mechanics in that it conceives geometrical figures to be attached to a lever in such a way that the latter remains in equilibrium, and then draws up conditions for such equilibrium; and it is further based on the view that the area of a plane figure is to be looked upon as the sum of the lengths of all the line segments drawn therein in a given direction and of which the figure is imagined to be made up; this view will be extended to space in the following propositions in the sense that a solid, too, is conceived to be made up of all the intersections determined therein by a plane of fixed inclination that is displaced, and that subsequently also the volume of the solid is looked upon as the sum of the areas of those intersections. We shall designate these two me-

<sup>1</sup>) *Opera* II, 438, lines 16–21.

thodic principles by the references: “barycentric method” and “method of indivisibles”.

We further saw that Archimedes is not prepared to recognize the results obtained with this twofold method as actually proved conclusions. It might now be asked where in his view resides the lack of exactness, in the barycentric character of the arguments, in the application of indivisibles, or in both.

The answer to this question may be given without much doubt: the mathematical deficiency is exclusively a consequence of the use of indivisibles; there is not the least objection from the mathematical point of view against properly founded barycentric considerations, such as we already found applied in Prop. 1.

That this is actually the view of Archimedes is particularly evident from the fact that in his treatise *Quadrature of the Parabola*, which constitutes an official publication satisfying all requirements of exactness, he proves the insight gained in Prop. 1 on the area of any segment of an orthotome once more by means of statical considerations, but this time without indivisibles<sup>1</sup>). Moreover, the way in which in *On the Equilibrium of Planes* he bases the theory of the lever on postulates strongly creates the impression that he does not see any essential difference between his *Elements of Statics* and the systematization of planimetry as Euclid had given it<sup>2</sup>).

Although Archimedes could therefore make use unconcernedly of the barycentric method for dealing with mathematical problems, which had probably been introduced by himself, he was bound to experience a great deal of doubt and uncertainty with regard to the application of the method of indivisibles, for here he touched upon a question which in the centuries preceding his own had given rise

<sup>1</sup>) It is true that he also furnishes the proof once more by purely geometrical means; there is, however, not the slightest cause to assume that he did not consider the two proofs equivalent.

<sup>2</sup>) The view that the application of mechanical methods was the very thing which gave occasion to consider the proofs from the *Method* inexact is taken by H. de Vries, *Historische Studiën* (Groningen 1926), p. 139. It is true that the remark in the introductory letter that the consideration according to the method there mentioned has no demonstrative force, together with the fact that the method is termed mechanical, is an argument in favour of this view. On account of the above argumentation, however, we cannot share it.

to violent controversy more than any other questions in Greek mathematics<sup>1</sup>). It was the profound question of atomism or continuity, on which, though originating from physics, opinions were divided also in mathematics, and which finds its clearest expression in the aporia that worried Democritus: if the circular sections that can be made in a cone parallel to the base are congruent, how can the cone differ from a cylinder; and if they grow smaller towards the vertex, is not then the curved surface, which should be smooth, scalariform?

The great influence which this question exerted on the history of Greek mathematics cannot indeed be reconstructed in detail, but its essential features can be recognized readily enough. The spectacular intervention of Zeno of Elea in the evolution of mathematical thought seems to have been largely caused by the embarrassment into which the human intellect had been thrown by the mathematical continuum; it was perhaps the most powerful source of the famous crisis of principles, which disturbed the gradual growth of mathematics about 400 B.C.; the reconstruction, with which the name of Eudoxus is associated, was brought about not in the last place by the conquest of this intellectual problem.

This marked the end of the unconcern with which the infinite had always been referred to as if the word connoted nothing but something very large, but finite; the method of the indirect limiting process tied down the application of infinite processes to the rigorous forms which so far we have found Archimedes observing in all his works; and if one did not know that in mathematics a discovery is one thing and a proof quite another, and that the method by which the reader is convinced of the truth of a theorem in many cases is quite different from the way in which it was first found, one might believe that the method of indivisibles had disappeared definitely from Greek mathematics after Eudoxus.

The *Method* has revealed to us—and it is this which constitutes the eminent importance of its discovery—that the indivisibles had only been banished from the published treatises, but that in the workshop of the producing mathematician they held undiminished sway, as they were to do so frequently in later periods, e.g. in Cavalieri, Galilei, Huygens, Leibniz. Unconcerned about the ra-

<sup>1</sup>) For a more detailed discussion of this matter reference is made to *Elements of Euclid* I, p. 41 et seq.

tional untenability of the view, and as a remarkable instance of the fertility which may be inherent to irrational modes of thought even in the most rational of all sciences, Archimedes, as long as he is seeking to find new results, considers a segment of a parabola as the sum of all its ordinates, or a solid as the sum of all its intersections in parallel planes; and when he is speaking about a figure, we find him already glibly using the expressions “all the lines” or “all the circles” which are to become current coin in the 17th century and which denote the set of all parallel indivisibles which “fill up” the figure, as it is technically called.

This is the important new insight which the publication of the *Method* has furnished us. Moreover, as will be shown more in detail with examples, it enables us to follow in various cases the development of a proposition from its non-rigorous, intuitively convincing discovery to its impeccable, abstractly persuasive exposition in a published work.

Archimedes always continues to distinguish sharply between these two phases of the process: Democritus has found the theorem that any pyramid is one-third of the prism having the same base and height (perhaps also by means of indivisibles), but only Eudoxus has proved it. That any segment of a parabola is greater by one-third than the triangle having the same base and vertex is to be expected in view of Prop. I of the *Method*; but it is only in the long chain of propositions contained in the treatise *Quadrature of the Parabola* that this assertion is to be raised to the rank of a proved assertion.

4. Before proceeding to the discussion of the other propositions of the *Method*, we still have to give an explanation of the mechanical part proper of the method applied. It is found that each time one of the straight lines of the figure is considered as an immaterial balance (*ζυγός*) supported in its centre of gravity, that on this balance in one or more points plane or solid figures are so attached that the point of attachment is at the same time the centre of gravity of the figure, and that of other figures “all the lines” or “all the intersections” are transferred to one of the ends of the lever (in Prop. 1 all the ordinates  $o\xi$  of the segment of the orthotome being transferred to  $\vartheta$ ). The idea is then that, after this transfer, from all these lines or intersections the figure from which they are taken is built up again, but in such a way that the end in which

each of them is placed with its centre of gravity is also the centre of gravity of the reconstructed figure again. The fact that the established equilibrium of the lever is not disturbed by this is guaranteed by postulate VI of *On the Equilibrium of Planes*: the reconstruction leaves the weight and the place of the figure unchanged, and the equilibrium once established is thus maintained. The fundamental importance which this postulate is thus found to possess for the *Method* convinces us once again of the correctness of the identical interpretation which we gave of it, in accordance with Toeplitz, in *On the Equilibrium of Planes*.

5. We will now continue our discussion of the propositions.

**Proposition 2.**

*That the volume of any sphere is four times that of the cone which has its base equal to the greatest circle of the sphere and its height equal to the radius of the sphere, and that the volume of the cylinder which has its base equal to the greatest circle of the sphere and its height equal to the diameter of the sphere is one and a half times that of the sphere, is recognized according to this method as follows.*

In Fig. 129 let  $\alpha\beta\gamma\delta$  be a greatest circle of the sphere,  $\alpha\gamma$  and  $\beta\delta$  two diameters of it at right angles to each other. Consider the cone with vertex  $\alpha$ , whose base is the greatest circle in the plane through  $\beta\delta$  at right angles to  $\alpha\gamma$ . The extended surface of this cone intersects the plane through  $\gamma$  at right angles to  $\alpha\gamma$  in a circle on  $\varepsilon\zeta$  as diameter. This circle is the base of a cylinder  $\varepsilon\zeta\eta\lambda$  with height  $\alpha\gamma$ .

Make  $\alpha\vartheta = \alpha\gamma$ , and consider  $\gamma\vartheta$  as a balance with fulcrum  $\alpha$ . A variable plane  $\mu\nu$  at right angles to  $\alpha\gamma$  intersects cone, sphere, and cylinder in circles whose diameters are successively  $\pi\rho^1$ ,  $\xi\sigma$ , and  $\mu\nu$ , and which we designate by  $\mathbf{K}(\pi\rho)$ ,  $\mathbf{K}(\xi\sigma)$ ,  $\mathbf{K}(\mu\nu)$ .

Now because

$$\rho\sigma = \alpha\sigma,$$

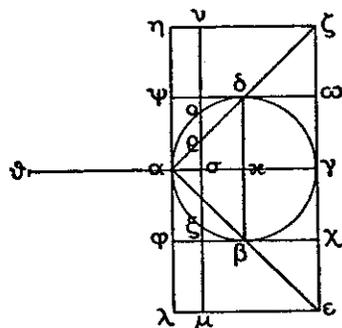


Fig. 129.

<sup>1)</sup> At the point of intersection of the straight lines  $\alpha\varepsilon$  and  $\mu\nu$  the letter  $\pi$  has been dropped.

we have

$$\mathbf{T}(\rho\sigma) + \mathbf{T}(\sigma\sigma) = \mathbf{T}(\alpha\sigma) = \mathbf{O}(\alpha\sigma, \alpha\gamma),$$

whence because

$$\nu\sigma = \alpha\gamma,$$

we have

$$\begin{aligned} [\mathbf{T}(\nu\sigma), \mathbf{T}(\rho\sigma) + \mathbf{T}(\sigma\sigma)] &= [\mathbf{T}(\alpha\gamma), \mathbf{O}(\alpha\sigma, \alpha\gamma)] = \\ &(\alpha\gamma, \alpha\sigma) = (\alpha\vartheta, \alpha\sigma) \end{aligned} \tag{1}$$

or

$$[\mathbf{K}(\mu\nu), \mathbf{K}(\pi\rho) + \mathbf{K}(\xi\sigma)] = (\alpha\vartheta, \alpha\sigma).$$

The circle  $\mathbf{K}(\mu\nu)$  *suo loco*<sup>1)</sup> can therefore balance the circles  $\mathbf{K}(\pi\rho)$  and  $\mathbf{K}(\xi\sigma)$ , both placed in  $\vartheta$  (*i.e.* so placed that their centres of gravity fall in  $\vartheta$ ). Consequently there is also equilibrium between the cylinder  $\varepsilon\zeta\eta\lambda$  *suo loco* and the combination in  $\vartheta$  of the sphere  $\alpha\beta\gamma\delta$  and the cone  $\alpha\varepsilon\zeta$  (for the three solids are filled up by the above-mentioned circles if the plane  $\mu\nu$  moves from  $\eta\lambda$  to  $\zeta\varepsilon$ ).

Since  $\kappa$  is the centre of gravity of the cylinder, the following relation holds:

$$(\text{Sphere} + \text{Cone}, \text{Cylinder}) = (\alpha\kappa, \alpha\vartheta),$$

whence

$$\text{Cylinder} = 2 (\text{Sphere} + \text{Cone}).$$

The cylinder is three times the cone, so that it follows from the last-mentioned relation that

$$\text{Cone } \alpha\varepsilon\zeta = 2 \cdot \text{Sphere } \alpha\beta\gamma\delta,$$

whence

$$\text{Sphere } \alpha\beta\gamma\delta = 4 \cdot \text{Cone } \alpha\beta\delta.$$

The theorem on the ratio between the sphere and the cylinder, too, is now immediately obvious.

*When this had been recognized, the suspicion arose that the surface of any sphere is four times that of a greatest circle on the sphere. In fact, it was assumed that as any circle is equal to a triangle which has*

<sup>1)</sup> By this Latin translation of the Greek expression *αὐτοῦ μένων* we indicate that the figure under consideration remains where it is.

the circumference of the circle for its base and whose height is equal to the radius of the circle, so any sphere is equal to a cone which has the surface of the sphere for its base and whose height is equal to the radius of the sphere.

A truly striking sidelight is here thrown on the way in which Archimedes found his two famous theorems 33 and 34 of *On the Sphere and Cylinder*, for it is revealed that the theorem on the volume was the first to be found, and that subsequently the suspected analogy with the relation between the surface and the circumference of the circle led to the theorem on the surface.

**Proposition 3.**

By this method it is also recognized that the cylinder which has a base equal to the greatest circle of a spheroid and a height equal to the axis of the spheroid is one and a half times the spheroid; when this has been recognized, it is obvious that if a spheroid be cut by a plane through the centre at right angles to the axis, half the spheroid will be the double of the cone which has the same base as the segment and the same axis.

As regards the mechanical argument, the proof is identical with that of Prop. 2. The proportion (1), however, requires a longer derivation, because in Fig. 129  $\alpha\beta\gamma\delta$  now has to be conceived as an oxytome, which generates the spheroid (ellipsoid of revolution) by revolution about one of its axes,  $\alpha\gamma$ .

By the symptom of the oxytome (III; 3.0) we have

$$[\mathbf{T}(\alpha\sigma), \mathbf{O}(\alpha\sigma, \gamma\sigma)] = [\mathbf{T}(\delta\kappa), \mathbf{T}(\alpha\kappa)] = [\mathbf{T}(\varrho\sigma), \mathbf{T}(\alpha\sigma)],$$

whence

$$\begin{aligned} [\mathbf{T}(\alpha\sigma), \mathbf{T}(\varrho\sigma)] &= (\gamma\sigma, \alpha\sigma) = (\nu\zeta, \alpha\sigma) = (\nu\varrho, \varrho\sigma) \\ &= [\mathbf{O}(\nu\varrho, \varrho\sigma), \mathbf{T}(\varrho\sigma)], \end{aligned}$$

whence

$$\mathbf{T}(\alpha\sigma) = \mathbf{O}(\varrho\sigma, \nu\varrho).$$

From this it follows that

$$\mathbf{T}(\varrho\sigma) + \mathbf{T}(\alpha\sigma) = \mathbf{O}(\varrho\sigma, \varrho\sigma + \nu\varrho) = \mathbf{O}(\varrho\sigma, \nu\sigma).$$

Now we have

$$[\mathbf{T}(\nu\sigma), \mathbf{T}(\varrho\sigma) + \mathbf{T}(\alpha\sigma)] = (\nu\sigma, \varrho\sigma) = (\zeta\gamma, \varrho\sigma) = (\alpha\gamma, \alpha\sigma) = (\alpha\vartheta, \alpha\sigma).$$

Further the proof proceeds on the same lines as in Prop. 2.

The exact proof of the theorem on half the spheroid has been furnished in C.S. 27.

**Proposition 4.**

That any segment cut off from an orthoconoid by a plane at right angles to the axis is one and a half times the cone which has the same base as the segment and the same axis, is recognized by this method as follows.

In Fig. 130 let the orthotome  $\alpha\beta\gamma$  generate, by revolution about the diameter  $\alpha\delta$ , an orthoconoid, from which a segment is cut off by a plane through  $\delta$  at right angles to  $\alpha\delta$ . A variable plane  $\mu\nu$  at right angles to  $\alpha\delta$  intersects the orthoconoid in a circle on  $o\xi$  as diameter, and the cylinder which has its base and height in common with the segment in a circle on  $\mu\nu$  as diameter. Again make  $\alpha\vartheta = \alpha\delta$ , and consider  $\delta\vartheta$  as a balance with fulcrum  $\alpha$ .

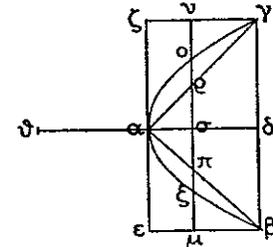


Fig. 130.

Again make  $\alpha\vartheta = \alpha\delta$ , and consider  $\delta\vartheta$  as a balance with fulcrum  $\alpha$ .

We now have by the symptom of the orthotome (III; 2.0):

$$[\mathbf{T}(\nu\sigma), \mathbf{T}(\alpha\sigma)] = [\mathbf{T}(\gamma\delta), \mathbf{T}(\alpha\sigma)] = (\alpha\delta, \alpha\sigma) = (\alpha\vartheta, \alpha\sigma).$$

From this it appears that  $\mathbf{K}(\mu\nu)$  *suo loco* balances  $\mathbf{K}(\alpha\xi)$  in  $\vartheta$ , so that also the cylinder *suo loco* balances the segment in  $\vartheta$ . Since the centre of gravity of the cylinder is the middle point  $\kappa$  of  $\alpha\delta$ , it follows from this that

$$\text{Cylinder } \beta\gamma\zeta\epsilon = 2 \cdot \text{Segment } \alpha\beta\gamma,$$

whence

$$\text{Segment } \alpha\beta\gamma = \frac{2}{3} \cdot \text{Cone } \alpha\beta\gamma.$$

The exact proof is to be found in C.S. 21.

6. In the propositions hitherto discussed it was possible to find, from the consideration of the equilibrium of the balance, the volume of the solid which, resolved into its intersections, had been transferred to one of the ends, because with regard to the solid which was attached to the balance *suo loco* both the volume and the position of the centre of gravity were known. If, however, the