ARCHIMEDES

BY

E. J. DIJKSTERHUIS

translated by C. Dikshoorn

With a new bibliographic essay by Wilbur R. Knorr

Summis ingeniis dux et magister fuit

(Heiberg, *Archimedis opera omnia* III,
Prolegomena XCV)

PRINCETON UNIVERSITY PRESS

Princeton, New Jersey

introductory letter, on the solid determined by the intersection of two cylinders inscribed in a cube is missing. It appears not to be difficult to reconstruct it hypothetically, but we do not wish to go into this here[1]).

CHAPTER XI

QUADRATURE OF THE PARABOLA

1. The theorem on the volume of the cylinder hoof already enabled us to see how Archimedes arrived, from a surmise gained by mechanical means with the aid of the method of indivisibles, at a mathematical proof which satisfied all his requirements of exactness.

An even more beautiful example of such a logical confirmation of an intuitively gained insight is furnished by the theorem on the area of any segment of an orthotome, which formed the subject of the first proposition of the *Method*. In fact, it was to the mathematical proof of this theorem that Archimedes devoted a separate treatise, the *Quadrature of the Parabola*, in which he derives the already known result at great length in two different ways, namely, first with the aid of mechanical considerations and then purely geometrically. As we already observed above, this twofold character of the treatise may be deemed to furnish an argument in favour of the view that when Archimedes denies the demonstrative force of the mechanical method which he explains to Eratosthenes, he does not do so on account of its mechanical nature, but exclusively because it makes use of the method of indivisibles.

2. The treatise *Quadrature of the Parabola* opens with five propositions on properties of the orthotome, which we already incorporated in Chapter III. These are followed by eight propositions in which equalitities and inequalities about plane figures suspended on a balance are enunciated.

[1]) Such a reconstruction is to be found, *inter alia*, in T. L. Heath, *The Method of Archimedes* (Cambridge 1912), pp. 48 *et seq.*, and in E. Rufini, *Il "Metodo" di Archimede e le origini dell'analisi infinitesimale nell'Antichità* (*Per la Storia e la Filosofia delle Matematiche* No 4, Roma 1926), pp. 179–186.

In Props 6 and 7 (Fig. 137) there is suspended from one end A of a balance $A\Gamma$ supported in its middle point B a magnitude Z which balances a triangle $\Gamma H\Delta$, the side $H\Delta$ of which lies in the vertical of B (in Prop. 6, H is moreover placed in B). If Θ be the centre of gravity of the triangle, and if the vertical of Θ meet the straight line $B\Gamma$ in Π, then because $B\Pi=\frac{1}{3}B\Gamma$ we have

$$\triangle\, \Gamma H\Delta = 3Z\,.$$

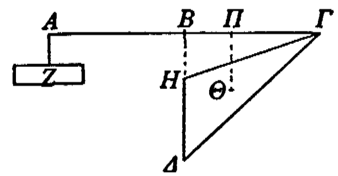

Fig. 137.

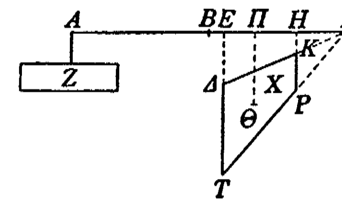

Fig. 138.

Archimedes here deviates from the method followed in *On the Equilibrium of Planes*, where the suspended magnitudes are attached to the balance in their centres of gravity; the centres of gravity are now at a level below that of the balance itself. It is not quite in order that he nevertheless applies the propositions from *On the Equilibrium of Planes*.

The propositions 8–12 are all particular cases of Prop. 13, so that we shall discuss the latter only (Fig. 138). In this case there is suspended from $B\Gamma$ a trapezium X ($TPK\Delta$), the vertical parallel sides of which, *viz.* ΔT and KP, meet $B\Gamma$ successively in E and H, while the other sides ΔK and TP converge towards Γ. The vertical through the centre of gravity Θ meets $B\Gamma$ in Π. Z and X balance. It results from this that

$$(AB, B\Pi) = (X, Z)\,. \qquad (1)$$

We now consider the magnitudes M and N, which would have to hang from A instead of Z in order to balance X, if X were so attached to $B\Gamma$ that the vertical of Θ passed successively through E and through H. M and N are determined by

$$(AB, BE) = (X, M)$$

$$(AB, BH) = (X, N)\,.$$

Since $BE < B\Pi < BH$, we find, by comparing these proportions with (1), the result

$$M < Z < N .$$

This is equivalent to saying that the magnitude at A which balances X grows smaller if X is displaced towards the fulcrum B, and larger if X is displaced in the direction away from the fulcrum B.

In Prop. 12, Δ fell in E and K in H, in Prop. 11 E fell in B, in Prop. 10 moreover Δ fell in E; in Props 8 and 9 P, K, H coincided in Γ, so that the trapezium passed into a triangle; the magnitude N (which would now be equal to X) is not considered in these propositions.

This part is followed by the proof of the main theorem, which occupies the propositions 14–17. For convenience we summarize the contents of the propositions 14 and 16 in the following argument:

3. A segment of an orthotome, a chord $B\Gamma$ of which is at right angles to the diameter (Fig. 139), is attached along the chord on the balance $A\Gamma$, which is supported in its middle point B.

The tangent at Γ meets the line, drawn through B parallel to the diameter, in Δ. $B\Delta$ is divided into equal segments BE, EH, etc., the number of which (n) is yet to be defined. The points of division are joined to Γ, and through the points in which these lines meet the curve are drawn straight lines parallel to the diameter, which meet $B\Gamma$ in the points $M, N \ldots$, and $\Gamma\Delta$ in the points $\alpha, \beta, \gamma \ldots$. The segments BM, $MN \ldots$ are now also equal (*vide* the

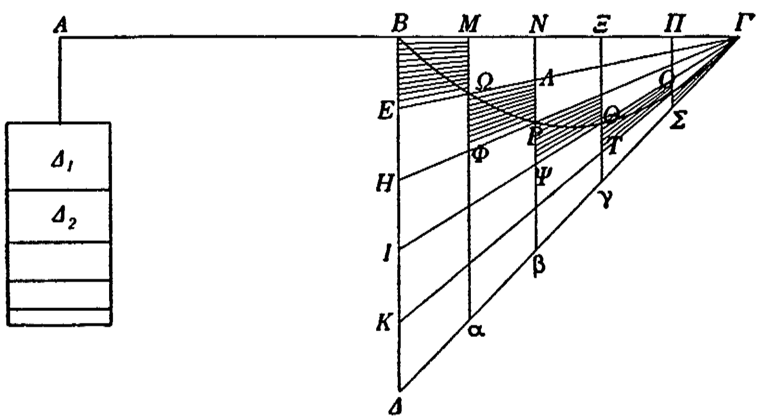

Fig. 139.

Note). In the figure several trapezia are thus formed, among which we distinguish three different types:

1. the trapezia $B\alpha$, $M\beta$, ..., which we call partial trapezia of $\triangle B\Gamma\Delta$; to be denoted by $d_1, d_2, \ldots d_n$.

2. the trapezia EM, ΦN, ..., which we call circumscribed trapezia to the segment; to be denoted by $c_1, c_2, \ldots c_n$; together they form the figure C_n.

3. the trapezia ΩN, $P\Xi$, ..., which we call inscribed trapezia to the segment; to be denoted by $i_2, i_3, \ldots i_n$; together they form the figure I_n. It is obvious that c_n and i_n are triangles with vertex Γ.

The derivation is given with the aid of the difference form of the compression method (III; 8.21); for this it is necessary that the difference $C_n - I_n$ can, by the choice of n, be made less than any assigned magnitude. Now $C_n - I_n$ is made up of the trapezia EM, $\Phi\Lambda$, ... With regard to these it is true that

$$\Phi\Lambda = \Omega N, \text{ etc.}$$

In fact, from $BE = EH$ it follows that $M\Omega = \Omega\Phi$ and $N\Lambda = \Lambda P$, etc.

The difference $C_n - I_n$ therefore is equal to $\triangle \Gamma BE$, so that it is indeed possible, by the choice of n, to make it less than any assigned magnitude.

It has now to be proved that the segment $B\Theta\Gamma$ is equal to $\frac{1}{3}.\triangle B\Delta\Gamma$. This is done according to the principle of the compression method by proving the inequality

$$I_n < \tfrac{1}{3}.\triangle B\Delta\Gamma < C_n . \tag{1}$$

Once this has been proved, what was required to be proved follows in the usual way, because the segment is then comprehended with $\frac{1}{3}.\triangle B\Delta\Gamma$ between boundaries the difference between which can be made less than any assigned magnitude.

In order to prove the inequality (1), relations of equilibrium between the various trapezia distinguished above are derived (in Prop. 14). By the theorem **Q.P.** 5 (III; 2.7) we have

$$(\alpha\Omega, \Omega M) = (\Gamma M, BM),$$

whence

$$(\alpha M, \Omega M) = (B\Gamma, BM) = (AB, BM).$$

Similarly we have

$$(\beta N, PN) = (AB, BN).$$

Now

$$(d_1, c_1) = (\Delta B+\alpha M, EB+\Omega M).$$

Since, however,

$$(\Delta B, EB) = (\alpha M, \Omega M) = (\Delta B+\alpha M, EB+\Omega M)$$

$$(AB, AH) = (X, N),$$

we also have

$$(d_1, c_1) = (\Delta B, EB) = (\alpha M, \Omega M).$$

Similarly it is seen that

$$(d_2, c_2) = (\alpha M, \Phi M) = (\beta N, PN), \text{ etc.}$$

However, we also have

$$(d_2, i_2) = (\alpha M, \Omega M) = (\beta N, \Lambda N), \text{ etc.}$$

From this it follows that

$$(d_1, c_1) = (d_2, i_2) = (\Delta B, EB).$$

Since it follows from

$$EB = EH, \text{ etc.}$$

that

$$BM = MN, \text{ etc.},$$

we have

$$(\Delta B, EB) = (\Gamma B, MB),$$

whence

$$(d_1, c_1) = (\Gamma B, MB) = (AB, BM).$$

Similarly

$$(d_2, c_2) = (d_3, i_3) = (AB, BN).$$

From this it appears that

c_1 at A balances d_1 at M

c_2 at A balances d_2 at N, etc.

and also

i_2 at A balances d_2 at M

i_3 at A balances d_3 at N, etc.

We now conceive to be suspended at A the magnitudes $\Delta_1, \Delta_2 \ldots$ which successively balance $d_1, d_2, \ldots$ *suis locis*. By the propositions 8, 10, and 12 we now have

$$c_k > \Delta_k \quad (k = 1 \ldots n)$$

$$i_k < \Delta_k \quad (k = 2 \ldots n).$$

From this it follows that

$$i_2 + \ldots + i_n < \Delta_1 + \ldots + \Delta_n < c_1 + \ldots + c_n$$

or

$$I_n < \Delta_1 + \ldots \Delta_n < C_n.$$

By Prop. 6, however, we have

$$3(\Delta_1 + \ldots + \Delta_n) = \triangle B\Delta\Gamma,$$

from which follows the desired inequality

$$I_n < \tfrac{1}{3}.\triangle B\Delta\Gamma < C_n. \quad (1)$$

The argument further proceeds automatically (in Prop. 16).

If the segment $B\Theta\Gamma$ is not equal to $\frac{1}{3}.\triangle B\Delta\Gamma$, it is either greater or smaller.

Case I. Suppose

$$\text{segment } B\Theta\Gamma > \tfrac{1}{3}.\triangle B\Delta\Gamma.$$

Now determine n such that

$$C_n - I_n < \triangle BE\Gamma < \text{segment } B\Theta\Gamma - \tfrac{1}{3}.\triangle B\Delta\Gamma,$$

then *a fortiori*

$$\text{segment } B\Theta\Gamma - I_n < \text{segment } B\Theta\Gamma - \tfrac{1}{3}.\triangle B\Delta\Gamma,$$

whence

$$I_n > \tfrac{1}{3}.\triangle B\Delta\Gamma,$$

which is contrary to (1).

Case II. Suppose

$$\text{segment } B\Theta\Gamma < \tfrac{1}{3}.\triangle B\Delta\Gamma.$$

Now determine n such that

$$C_n - I_n < \triangle BE\Gamma < \tfrac{1}{3}.\triangle B\Delta\Gamma - \text{segment } B\Theta\Gamma,$$

then *a fortiori*

$$C_n - \text{segment } B\Theta\Gamma < \tfrac{1}{3}.\triangle B\Delta\Gamma - \text{segment } B\Theta\Gamma,$$

whence

$$C_n < \tfrac{1}{3}.\triangle B\Delta\Gamma,$$

which is contrary to (1).

Note: It appears to be essential to the proof that from the equality of the segments BE, etc. of $B\Delta$ should follow the equality of the segments BM, MN, etc. of $B\Gamma$. Now Archimedes starts by assuming the latter equality in Prop. 14, in which he derives the inequality (1), and he then applies Prop. 14 in Prop. 16, in which not $B\Gamma$, but $B\Delta$ is divided into a number of equal segments, but he does not prove that by this the condition of Prop. 14 is also satisfied. This slight gap is naturally filled by observing that by **Q.P.** 5 (III; 2.7)

$$(\Gamma M, BM) = (\alpha\Omega, \Omega M) = (\Delta E, EB),$$

so that BM is found to be the nth part of $B\Gamma$, if BE is the nth part of $B\Delta$. Proceeding in this way, we find: $BM = MN$, etc.

In Prop. 15 the inequality (1) is proved for the case that the chord of the segment is not at right angles to the diameter. It is now attached to the balance in such a way that the diameter is vertical, one end of the chord lies at Γ, and the other in the vertical of B. The argument is scarcely altered by it. The only difference is that we have to use Props 7, 9, 11, and 13 instead of 6, 8, 10, and 12.

Finally in Prop. 17 the theorem is enunciated in the form in which Archimedes is accustomed to apply it:

Proposition 17.

This having been proved, it is manifest that the area of any segment which is comprehended by a straight line and an orthotome is greater by one-third than the triangle which has the same base as the segment and equal height.

Here (Fig. 140) "height" is meant to denote the distance from the chord $B\Gamma$ of the point Θ (vertex), where the tangent is parallel to the chord. Apparently we have

$$B\Delta = 2.ZE = 4.Z\Theta,$$

whence

$$\text{segment } B\Theta\Gamma = \tfrac{1}{3} \triangle B\Delta\Gamma = \tfrac{4}{3} \triangle B\Theta\Gamma.$$

Prop. 17 is followed by definitions of the terms base, height, and vertex of the segment, which, however, have already been used in Prop. 17. The height is now defined as the greatest distance of a point of the branch $B\Gamma$ of the curve from the base, the vertex as the point which has this greatest distance from the base. In Prop.

18 it is proved that the vertex is the point where the straight line through the middle point of the chord, parallel to the diameter, meets the curve.

4. With Prop. 18 the second part of the treatise has already started, in which Archimedes proves the theorem on the area of any segment of an orthotome once more by purely geometrical

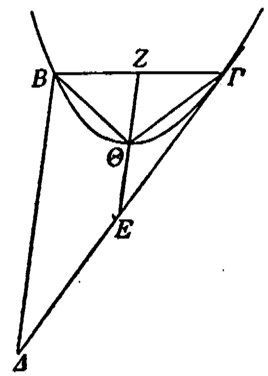

Fig. 140.

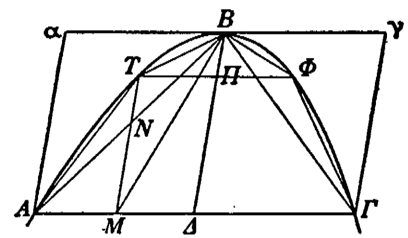

Fig. 141.

means. He here applies that form of indirect limiting process which we have referred to in the general discussion as approximation method (III; 8.30). He therefore inscribes in the segment a figure, the area of which is more than the half of that of the segment, treats the remaining part in the same way, and thus, by the lemma of Euclid, approximates the area to be found as accurately as is required.

In the following we summarize the contents of Props 19–24 (Fig. 141). Let Σ be the segment $AB\Gamma$ of the orthotome with base $A\Gamma$ and vertex B. Construct the parallelogram $A\Gamma\gamma\alpha$ with side $A\Gamma$, the opposite side of which passes through B, then it will appear in Prop. 20 that

$$\triangle AB\Gamma = \tfrac{1}{2} A\Gamma\gamma\alpha > \tfrac{1}{2}\Sigma.$$

Since we can deal with each of the segments AB, $B\Gamma$ in the same way, the lemma of Euclid applies (Prop. 20; Corollary). Now consider the triangle ATB inscribed in the segment AB(cf. III; 2.4). The vertex T lies on the straight line drawn through the middle point N of AB, parallel to the diameter, *i.e.* parallel to $B\Delta$; this straight line therefore will also pass through the middle point M of $A\Delta$. Join B and M. We will now compare the triangles $AB\Gamma$ and ABT.

To this end it is first proved (Prop. 19) that

$$B\Delta = \tfrac{4}{3}TM .$$

This follows from the symptom of the orthotome; for if the straight line through T parallel to $A\Gamma$ meet the straight line $B\Delta$ in Π, we have

$$(B\Delta, B\Pi) = [\mathbf{T}(A\Delta), \mathbf{T}(T\Pi)] = [\mathbf{T}(A\Delta), \mathbf{T}(M\Delta)] ,$$

whence

$$B\Delta = 4.B\Pi ,$$

therefore

$$TM = \Pi\Delta = \tfrac{3}{4}B\Delta .$$

Since $NM = \frac{1}{2}B\Delta$, it is found that $NM = 2.TN$.

From this it follows that

$$\triangle\ ABT = 2.\triangle\ BNT = \triangle\ BNM = \tfrac{1}{2}\triangle\ ABM = \tfrac{1}{8}\triangle\ AB\Gamma. \quad \text{(Prop. 21)}$$

If we now represent the area of $\triangle\ AB\Gamma$ by Z, the sum of the areas of the two triangles inscribed in the segments AB and $B\Gamma$ is: $Z_1 = \frac{1}{4}Z$; proceeding in this way, we find for the sum of the areas of the triangles which are inscribed in the remaining four segments: $Z_2 = \frac{1}{4}Z_1$, etc. Archimedes expresses this by introducing magnitudes H, Θ, I, etc., so that

$$H = \tfrac{1}{4}Z,\ \Theta = \tfrac{1}{4}H,\ I = \tfrac{1}{4}\Theta .$$

By this it has been proved (Prop. 22) that

$$\Sigma > Z+H+\Theta+I ,$$

or in general

$$\Sigma > Z+Z_1+\ldots+Z_{n-1} .$$

In Prop. 23 a lemma on the sum of a geometrical progression with the proportion $\frac{1}{4}$ is then derived. By this lemma (discussed in III; 7.60) we have

$$Z+H+\Theta+I+\tfrac{1}{3}I = \tfrac{4}{3}Z \qquad (1)$$

or in general

$$Z+Z_1+\ldots+Z_{n-1}+\tfrac{1}{3}Z_{n-1} = \tfrac{4}{3}Z . \qquad (2)$$

This is followed, in Prop. 24, by the proof of the main theorem. Let K be equal to $\frac{4}{3}Z$, then it has to be proved that $\Sigma = K$. Suppose this is not true, then either $\Sigma > K$ or $\Sigma < K$.

Case I. Suppose $\Sigma > K$. Now continue inscribing triangles in each of the segments obtained until the sum I_n of all the inscribed triangles satisfies the relation

$$\Sigma - I_n < \Sigma - K \qquad \text{(lemma of Euclid III; 0.5)} ,$$

whence

$$I_n > K .$$

Archimedes assumes this to be the case for $n=4$, and then writes

$$Z+H+\Phi+I > K ,$$

which is contrary to (1).

More in general it is found that

$$Z+Z_1+\ldots+Z_{n-1} > K , \qquad \text{which is contrary to (2).}$$

Case II. Suppose $\Sigma < K$. We cannot now, as with the compression method, argue on entirely analogous lines to Case I, because there is no sum C_n of circumscribed figures. Archimedes now continues inscribing triangles until the sum of the ultimately obtained areas satisfies the relation

$$I < K - \Sigma . \qquad (3)$$

We further have

$$K-(Z+H+\Theta+I) = \tfrac{1}{3}I < I ,$$

whence

$$I > K - I_n > K - \Sigma , \qquad \text{which is contrary to (3).}$$

In general we get

$$Z_{n-1} < K - \Sigma . \qquad (4)$$

Now since

$$Z+Z_1+\ldots Z_{n-1}+\tfrac{1}{3}Z_{n-1} = K ,$$

we also have

$$K-(Z+Z_1+\ldots+Z_{n-1}) = \tfrac{1}{3}Z_{n-1} < Z_{n-1} ,$$

whence

$$Z_{n-1} > K - I_n$$

and *a fortiori*

$$Z_{n-1} > K - \Sigma , \qquad \text{which is contrary to (4).}$$