from which it follows that

$$\text{area of parabolic segment} = \tfrac{4}{3}\triangle ABC.$$

The same sort of argument is used for solids, *plane sections* taking the place of *straight lines*.

Archimedes is careful to state once more that this method of argument does not constitute a *proof*. Thus, at the end of the above proposition about the parabolic segment, he adds: "This property is of course not proved by what has just been said; but it has furnished a sort of *indication* (ἔμφασίν τινα) that the conclusion is true."

Let us now turn to the passage of Plutarch (*De Comm. Not. adv. Stoicos* XXXIX. 3) about Democritus above referred to. Plutarch speaks of Democritus as having raised the question in natural philosophy (φυσικῶς): "if a cone were cut by a plane parallel to the base [by which is clearly meant a plane indefinitely near to the base], what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal circles, which is very absurd." The phrase "*made up* of equal...circles" (ἐξ ἴσων συγκείμενος...κύκλων) shows that Democritus already had the idea of a solid being the sum of an infinite number of parallel planes, or indefinitely thin laminae, indefinitely near together: a most important anticipation of the same thought which led to such fruitful results in Archimedes. If then one may hazard a conjecture as to Democritus' argument with regard to a pyramid, it seems probable that he would notice that, if two pyramids of the same height and equal triangular bases are respectively cut by planes parallel to the base and dividing the heights in the same ratio, the corresponding sections of the two pyramids are equal, whence he would infer that the pyramids are equal as being the sum of the same infinite number of equal plane sections or indefinitely thin laminae. (This would be a particular anticipation of Cavalieri's proposition that the areal or solid contents of two figures are equal if two sections of them taken at the same height, whatever the height may be, always give equal straight lines or equal surfaces respectively.) And Democritus would of course see that the three pyramids into which a prism on the same base and of equal height with the original pyramid is divided (as in Eucl. XII. 7) satisfy this test of equality, so that the pyramid would be one third part of the prism. The extension to a pyramid with a polygonal base would be easy. And Democritus may have stated the proposition for the cone (of course without an absolute proof) as a natural inference from the result of increasing indefinitely the number of sides in a regular polygon forming the base of a pyramid.
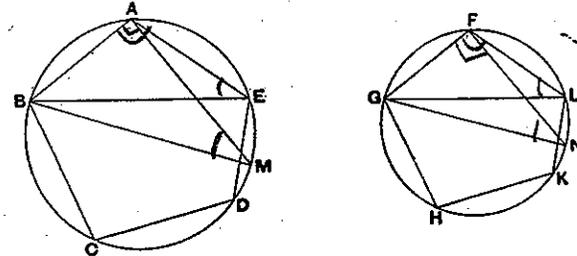
BOOK XII. PROPOSITIONS.

PROPOSITION I.

Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let ABC, FGH be circles,
let $ABCDE$, $FGHKL$ be similar polygons inscribed in them,
and let BM, GN be diameters of the circles;
I say that, as the square on BM is to the square on GN, so is the polygon $ABCDE$ to the polygon $FGHKL$.

For let BE, AM, GL, FN be joined.
Now, since the polygon $ABCDE$ is similar to the polygon $FGHKL$,
the angle BAE is equal to the angle GFL,
and, as BA is to AE, so is GF to FL.   [VI. Def. 1]
Thus BAE, GFL are two triangles which have one angle equal to one angle, namely the angle BAE to the angle GFL, and the sides about the equal angles proportional;
therefore the triangle ABE is equiangular with the triangle FGL.   [VI. 6]
Therefore the angle AEB is equal to the angle FLG.

But the angle AEB is equal to the angle AMB,

for they stand on the same circumference;   [III. 27]

and the angle FLG to the angle FNG;

therefore the angle AMB is also equal to the angle FNG.

But the right angle BAM is also equal to the right angle GFN;   [III. 31]

therefore the remaining angle is equal to the remaining angle.
[I. 32]

Therefore the triangle ABM is equiangular with the triangle FGN.

Therefore, proportionally, as BM is to GN, so is BA to GF.   [VI. 4]

But the ratio of the square on BM to the square on GN is duplicate of the ratio of BM to GN,

and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is duplicate of the ratio of BA to GF;   [VI. 20]

therefore also, as the square on BM is to the square on GN, so is the polygon $ABCDE$ to the polygon $FGHKL$.

Therefore etc.

                                        Q. E. D.

As, from this point onward, the text of each proposition usually occupies considerable space, I shall generally give in the notes a summary of the argument, to enable it to be followed more easily.

Here we have to prove that a pair of corresponding sides are in the ratio of the corresponding diameters.

Since $\angle$s BAE, GFL are equal, and the sides about those angles proportional,

$$\triangle s\ ABE,\ FGL \text{ are equiangular,}$$

so that $$\angle AEB = \angle FLG$$

Hence their equals in the same segments, $\angle$s AMB, FNG, are equal.
And the right angles BAM, GFN are equal.
Therefore $\triangle$s ABM, FGN are equiangular, so that

$$BM : GN = BA : GF.$$

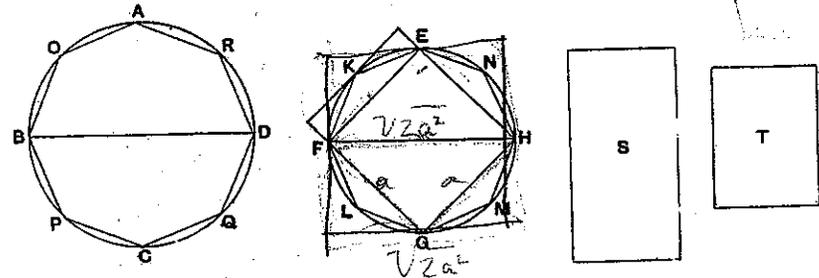The duplicates of these ratios are therefore equal,

whence   (polygon $ABCDE$) : (polygon $FGHKL$)

$$= \text{duplicate ratio of } BA \text{ to } GF$$
$$= \text{duplicate ratio of } BM \text{ to } GN$$
$$= BM^2 : GN^2.$$

PROPOSITION 2.

Circles are to one another as the squares on the diameters.

Let $ABCD$, $EFGH$ be circles, and BD, FH their diameters;

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the square on BD to the square on FH.

For, if the square on BD is not to the square on FH as the circle $ABCD$ is to the circle $EFGH$,

then, as the square on BD is to the square on FH, so will the circle $ABCD$ be either to some less area than the circle $EFGH$, or to a greater.

First, let it be in that ratio to a less area S.

Let the square $EFGH$ be inscribed in the circle $EFGH$;

then the inscribed square is greater than the half of the circle $EFGH$, inasmuch as, if through the points E, F, G, H we draw tangents to the circle, the square $EFGH$ is half the square circumscribed about the circle, and the circle is less than the circumscribed square;

hence the inscribed square $EFGH$ is greater than the half of the circle $EFGH$.

Let the circumferences EF, FG, GH, HE be bisected at the points K, L, M, N,

and let EK, KF, FL, LG, GM, MH, HN, NE be joined;

therefore each of the triangles EKF, FLG, GMH, HNE is also greater than the half of the segment of the circle about it, inasmuch as, if through the points K, L, M, N we draw tangents to the circle and complete the parallelograms on the straight lines EF, FG, GH, HE, each of the triangles EKF,

FLG, GMH, HNE will be half of the parallelogram about it,

while the segment about it is less than the parallelogram; hence each of the triangles *EKF, FLG, GMH, HNE* is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK, KF, FL, LG, GM, MH, HN, NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN*, is greater than the area *S*.

Let there be inscribed, also, in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN*; therefore, as the square on *BD* is to the square on *FH*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*.

[XII. 1]

But, as the square on *BD* is to the square on *FH*, so also is the circle *ABCD* to the area *S*;

therefore also, as the circle *ABCD* is to the area *S*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*;

[v. 11]

therefore, alternately, as the circle *ABCD* is to the polygon inscribed in it, so is the area *S* to the polygon *EKFLGMHN*.

[v. 16]

But the circle *ABCD* is greater than the polygon inscribed in it;

therefore the area *S* is also greater than the polygon *EKFLGMHN*.

But it is also less:
which is impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area less than the circle *EFGH*.

Similarly we can prove that neither is the circle *EFGH* to any area less than the circle *ABCD* as the square on *FH* is to the square on *BD*.

I say next that neither is the circle *ABCD* to any area greater than the circle *EFGH* as the square on *BD* is to the square on *FH*.

For, if possible, let it be in that ratio to a greater area *S*.

Therefore, inversely, as the square on *FH* is to the square on *DB*, so is the area *S* to the circle *ABCD*.

But, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*;

therefore also, as the square on *FH* is to the square on *BD*, so is the circle *EFGH* to some area less than the circle *ABCD*:

[v. 11]

which was proved impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area greater than the circle *EFGH*.

And it was proved that neither is it in that ratio to any area less than the circle *EFGH*;

therefore, as the square on *BD* is to the square on *FH*, so is the circle *ABCD* to the circle *EFGH*.

Therefore etc.

Q. E. D.

LEMMA.

I say that, the area *S* being greater than the circle *EFGH*, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*.

For let it be contrived that, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to the area *T*.

I say that the area *T* is less than the circle *ABCD*.

For since, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to the area *T*,

therefore, alternately, as the area S is to the circle $EFGH$, so is the circle $ABCD$ to the area T.   [v. 16]

But the area S is greater than the circle $EFGH$; therefore the circle $ABCD$ is also greater than the area T.

Hence, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

Q. E. D.

Though this theorem is said to have been proved by Hippocrates, we may with tolerable certainty attribute the proof of it given by Euclid to Eudoxus, to whom XII. 7 Por. and XII. 10 (which Euclid proves in exactly the same manner) are specifically attributed by Archimedes. As regards the lemma used herein (Eucl. X. 1) and the somewhat different lemma by means of which Archimedes says that the theorems of XII. 2, XII. 7 Por. and XII. 18 were proved, see my note on X. 1 above.

The first essential in this proposition is to prove that we can *exhaust* a circle, in the sense of X. 1, by successively inscribing in it regular polygons, each of which has twice as many sides as the preceding one. We take first an inscribed square, then bisect the arcs subtended by the sides and so form an equilateral polygon of eight sides, then do the same with the latter, forming a polygon of 16 sides, and so on. And we have to prove that what is left over when any one of these polygons is taken away from the circle is *more than half exhausted* when the next polygon is made and subtracted from the circle.

Euclid proves that the inscribed square is greater than half the circle and that the regular octagon when subtracted takes away more than half of what was left by the square. He then infers that the same thing will happen whenever the number of sides is doubled.

This can be seen generally by taking *any* arc of a circle cut off by a chord AB. Bisect the arc in C. Draw a tangent to the circle at C, and let AD, BE be drawn perpendicular to the tangent. Join AC, CB.

Then DE is parallel to AB, since

$$\angle ECB = \angle CAB, \text{ in alternate segment, } [III. 32]$$
$$= \angle CBA.   [III. 29, I. 5]$$

Thus $ABED$ is a $\square$;

and it is greater than the segment ACB.

Therefore its half, the $\triangle ACB$, is greater than half the segment.

Thus, by X. 1, Euclid's construction of successive regular polygons in a circle, if continued far enough, will at length leave segments which are together less than any given area.

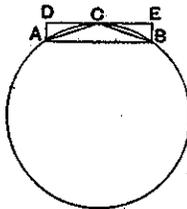Now let X, X' be the areas of the circles, d, d' their diameters, respectively. Then, if

$$X : X' \neq d^2 : d'^2,$$
$$d^2 : d'^2 = X : S,$$

where S is some area either greater or less than X'.

I.   Suppose $S < X'$.

Continue the construction of polygons in X' until we arrive at one which

---

leaves over segments together less than the excess of X' over S, i.e. a polygon such that

$$X' > (\text{polygon in } X') > S.$$

Inscribe in the circle X a polygon similar to that in X.

Then $(\text{polygon in } X) : (\text{polygon in } X') = d^2 : d'^2$ [XII. 1]
$$= X : S, \text{ by hypothesis;}$$

and, alternately,

$$(\text{polygon in } X) : X = (\text{polygon in } X') : S.$$

But $(\text{polygon in } X) < X$;
therefore $(\text{polygon in } X') < S$.
But, by construction. $(\text{polygon in } X') > S$:
which is impossible.

Hence S cannot be *less* than X' as supposed.

II.   Suppose $S > X'$.

Since $d^2 : d'^2 = X : S$,
we have, inversely, $d'^2 : d^2 = S : X$.
Suppose that $S : X = X' : T$,
whence, since $S > X'$, $X > T$.   [v. 14]
Consequently $d'^2 : d^2 = X' : T$,

where $T < X$.

This can be proved impossible in exactly the same way as shown in Part I.
Hence S cannot be *greater* than X' as supposed.
Since then S is neither greater nor less than X',

$$S = X',$$

and therefore $d^2 : d'^2 = X : X'$.

With reference to the assumption that there *is* some space S such that

$$d^2 : d'^2 = X : S,$$

i.e. that there is a fourth proportional to the areas d^2, d'^2, X, Simson observes that it is sufficient, in this and the like cases, that a thing made use of in the reasoning can possibly *exist*, though it cannot be exhibited by a geometrical construction. As regards the assumption see note on v. 18 above.

There is grave reason for suspecting the genuineness of the Lemma at the end of the proposition; though, if it be rejected, it will be necessary to delete the words "as was before proved" in corresponding places in XII. 5, 18.

It will be observed that Euclid proves the impossibility in the second case by reducing it to the first. If it is desired to prove the second case independently, we must *circumscribe* successive polygons to the circles instead of inscribing them, in the way shown by Archimedes in his first proposition on the *Measurement of a circle*. Of course we require, as a preliminary, the proposition corresponding to XII. 1, that *Similar polygons circumscribed about circles are to one another as the squares on the diameters.*

Let AB, $A'B'$ be corresponding sides of the two similar polygons. Then $\angle$ s OAB, $O'A'B'$ are equal, since AO, $A'O'$ bisect equal angles.
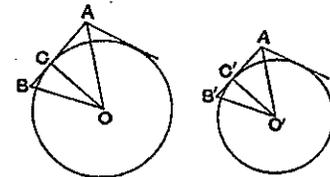
Similarly $\angle ABO = \angle A'B'O'$.

Therefore $\triangle$s AOB, $A'O'B'$ are similar, so that their areas are in the duplicate ratio of AB to $A'B'$.

The radii OC, $O'C'$ drawn to the points of contact are perpendicular to AB, $A'B'$, and it follows that

$$AB : A'B' = CO : C'O'.$$

Thus the polygons are to one another in the duplicate ratio of the radii, and therefore of the diameters.

Now suppose a square $ABCD$ described about a circle.

Make an octagon described about the circle by drawing tangents at the points E etc., where OA etc. meet the circle.

Then shall the tangent at E cut off more than half of the area between AK, AH and the arc HEK.

For the angle AEG is right, and is therefore $> \angle EAG$.

Therefore $AG > EG$

                                    $> GK$.

Therefore $\triangle AGE > \triangle EGK$.

Similarly $\triangle AFE > \triangle EFH$.

Hence $\triangle AFG > \frac{1}{2}$ (re-entrant quadrilateral $AHEK$),

and *a fortiori*, $\triangle AFG > \frac{1}{2}$ (area between AH, AK and the arc).

Thus the octagon takes from the square more than half the space between the square and the circle.

Similarly, if a figure of 16 equal sides be circumscribed by cutting off symmetrically the corners of the octagon, it will take away more than half of the space between the octagon and circle.

Suppose now, with the original notation, that

$$d^2 : d'^2 = X : S,$$

where S is greater than X'.

Continue the construction of circumscribed polygons about X' until the total area between the polygon and the circle is less than the difference between S and X', i.e. till

$$S > (\text{polygon about } X') > X'.$$

Circumscribe a similar polygon about X.

Then   (polygon about X) : (polygon about X') $= d^2 : d'^2$

                                                $= X : S$, by hypothesis,

and, alternately,

            (polygon about X) : X = (polygon about X') : S.

But   (polygon about X) $> X$.

Therefore   (polygon about X') $> S$.
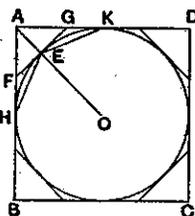
But $S > (\text{polygon about } X')$:   [above]

which is impossible.

Hence S cannot be *greater* than X'.

Legendre proves this proposition by a method equally rigorous but not, I think, possessing any advantages over Euclid's. It depends on a lemma corresponding to Eucl. XII. 16, but with another part added to it.

Two concentric circles being given, we can always inscribe in the greater a regular polygon such that its sides do not meet the circumference of the lesser, and we can also circumscribe about the lesser a regular polygon such that its sides do not meet the circumference of the greater.

Let CA, CB be the radii of the circles.

I. At A on the inner circle draw the tangent DE meeting the outer circle in D, E.

Inscribe in the outer circle any of the regular polygons which we can inscribe, e.g. a square.

Bisect the arc subtended by a side, bisect the half, bisect that again, and so on, until we arrive at an arc less than the arc DBE.

Let this arc be MN, and suppose it so placed that B is its middle point.

Then the chord MN is clearly more distant from the centre C than DE is; and the regular polygon, of which MN is a side, does not anywhere meet the circumference of the inner circle.

II. Join CM, CN, meeting DE in P, Q.

Then PQ will be the side of a polygon circumscribed about the inner circle and similar to the polygon inscribed in the outer;

and the circumscribed polygon of which PQ is a side will not anywhere meet the outer circle.

Legendre now proves XII. 2 after the following manner.

For brevity, let us denote the area of the circle with radius CA by (circ. CA).

Then it is required to prove that, if OB be the radius of a second circle,

$$(\text{circ. } CA) : (\text{circ. } OB) = CA^2 : OB^2.$$
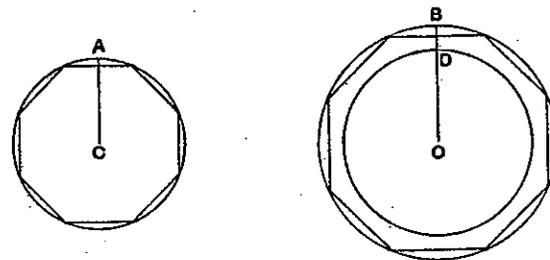
Suppose, if possible, that this relation is not true. Then CA^2 will be to OB^2 as (circ. CA) is to an area greater or less than (circ. OB).

I. Suppose, first, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OD),$$

where OD is less than OB.

Inscribe in the circle with radius OB a regular polygon such that its sides do not anywhere meet the circumference of the circle with centre OD;

[Lemma]

and inscribe a similar polygon in the other circle.

The areas of the polygons will then be in the duplicate ratio of CA to OB, or

[XII. 1]

(polygon in circ. CA) : (polygon in circ. OB)

$$= CA^2 : OB^2$$

$$= (\text{circ. } CA) : (\text{circ. } OD), \text{ by hypothesis.}$$

But this is impossible, because the polygon in (circ. CA) is *less* than (circ. CA), but the polygon in (circ. OB) is *greater* than (circ. OD).

Therefore CA^2 cannot be to OB^2 as (circ. CA) is to a *less* circle than (circ. OB).

II. Suppose, if possible, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{some circle} > \text{circ. } OB).$$

Then inversely

$$OB^2 : CA^2 = (\text{circ. } OB) : (\text{some circle} < \text{circ. } CA),$$

and this is proved impossible exactly as in Part I.

Therefore $CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OB).$

PROPOSITION 3.

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

Let there be a pyramid of which the triangle ABC is the base and the point D the vertex;

I say that the pyramid $ABCD$ is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

For let AB, BC, CA, AD, DB, DC be bisected at the points E, F, G, H, K, L, and let HE, EG, GH, HK, KL, LH, KF, FG be joined.

Since AE is equal to EB, and AH to DH, therefore EH is parallel to DB.

[VI. 2]
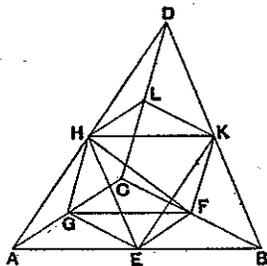
For the same reason

HK is also parallel to AB.

Therefore $HEBK$ is a parallelogram;

therefore HK is equal to EB.

[I. 34]

But EB is equal to EA;

therefore AE is also equal to HK.

But AH is also equal to HD;

therefore the two sides EA, AH are equal to the two sides KH, HD respectively,

and the angle EAH is equal to the angle KHD;

therefore the base EH is equal to the base KD.   [I. 4]

Therefore the triangle AEH is equal and similar to the triangle HKD.

For the same reason

the triangle AHG is also equal and similar to the triangle HLD.

Now, since two straight lines EH, HG meeting one another are parallel to two straight lines KD, DL meeting one another, and are not in the same plane, they will contain equal angles.   [XI. 10]

Therefore the angle EHG is equal to the angle KDL.

And, since the two straight lines EH, HG are equal to the two KD, DL respectively,

and the angle EHG is equal to the angle KDL,

therefore the base EG is equal to the base KL;   [I. 4]

therefore the triangle EHG is equal and similar to the triangle KDL.

For the same reason

the triangle AEG is also equal and similar to the triangle HKL.

Therefore the pyramid of which the triangle AEG is the base and the point H the vertex is equal and similar to the pyramid of which the triangle HKL is the base and the point D the vertex.   [XI. Def. 10]

And, since HK has been drawn parallel to AB, one of the sides of the triangle ADB,