THE WORKS OF

Archimedes

EDITED IN MODERN NOTATION
WITH INTRODUCTORY CHAPTERS BY

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WITH A SUPPLEMENT
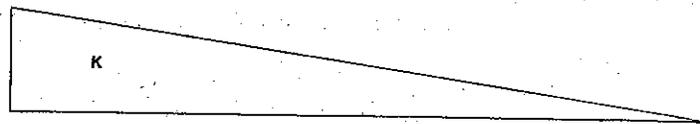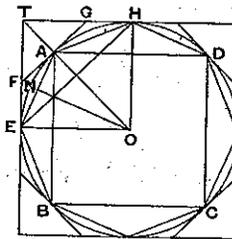
The Method of Archimedes

RECENTLY DISCOVERED BY HEIBERG

MEASUREMENT OF A CIRCLE.

Proposition 1.

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.

Then, if the circle is not equal to K, it must be either greater or less.

I.   If possible, let the circle be greater than K.

Inscribe a square $ABCD$, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K.

Thus the area of the polygon is greater than K.

Let AE be any side of it, and ON the perpendicular on AE from the centre O.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in K.

Therefore the area of the polygon is less than K; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.

II. If possible, let the circle be less than K.

Circumscribe a square, and let two adjacent sides, touching the circle in E, H, meet in T. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let A be the middle point of the arc EH, and FAG the tangent at A.

Then the angle TAG is a right angle.

Therefore $TG > GA$

$$> GH.$$

It follows that the triangle FTG is greater than half the area $TEAH$.

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of K over the area of the circle.

Thus the area of the polygon will be less than K.

Now, since the perpendicular from O on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle K; which is impossible.

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Therefore the area of the circle is not less than K.

Since then the area of the circle is neither greater nor less than K, it is equal to it.

Proposition 2.

The area of a circle is to the square on its diameter as 11 *to* 14.

[The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.]

Proposition 3.

[*The ratio of the circumference of any circle to its diameter is less than* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.]

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. These various approximations and the machinery of Greek arithmetic in general will be found discussed in the Introduction, Chapter IV.]

I. Let AB be the diameter of any circle, O its centre, AC the tangent at A; and let the angle AOC be one-third of a right angle.

Then $OA : AC\,[=\sqrt{3} : 1] > 265 : 153\,$............... (1),

and $OC : CA\,[=2 : 1]=306 : 153$............... (2).

First, draw OD bisecting the angle AOC and meeting AC in D.

Now $CO : OA = CD : DA$,   [Eucl. VI. 3]

so that $[CO+OA : OA = CA : DA,\ \text{or}]$
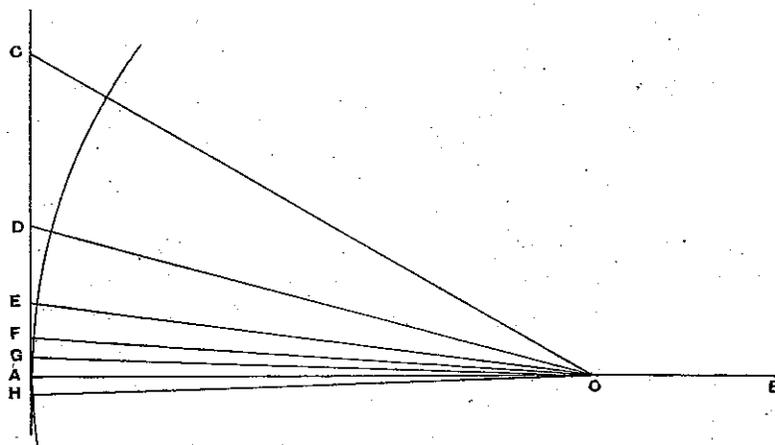
$CO+OA : CA = OA : AD$.

Therefore [by (1) and (2)]

$OA : AD > 571 : 153$ .... (3).

Hence $OD^2 : AD^2\,[=(OA^2+AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409,$

so that $OD : DA > 591\tfrac{1}{8} : 153$ .... (4).

Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then $DO : OA = DE : EA$,

so that $DO+OA : DA = OA : AE$.]

Therefore $OA : AE\,[> (591\tfrac{1}{8}+571) : 153,\ \text{by (3) and (4)}]$

$> 1162\tfrac{1}{8} : 153$....................(5).

$: 153 \ldots \ldots \ldots (1),$

$: 153 \ldots \ldots \ldots (2).$

AOC and meeting AC

[Eucl. VI. 3]

, or]

.

$3 \ldots \ldots \ldots (3).$

$^2) : AD^2$

$^2) : 153^2]$

$3409,$

$\ldots \ldots \ldots (4).$

O B

D, meeting AD in E.

$) : 153,$ by (3) and (4)]

$\ldots \ldots \ldots (5).$

[It follows that
$$OE^2 : EA^2 > \{(1162\tfrac{1}{8})^2 + 153^2\} : 153^2$$
$$> (1350534\tfrac{33}{64} + 23409) : 23409$$
$$> 1373943\tfrac{33}{64} : 23409.]$$

Thus $\qquad OE : EA > 1172\tfrac{1}{8} : 153 \ldots \ldots \ldots (6).$

Thirdly, let OF bisect the angle AOE and meet AE in F.

We thus obtain the result [corresponding to (3) and (5) above] that
$$OA : AF [> (1162\tfrac{1}{8} + 1172\tfrac{1}{8}) : 153]$$
$$> 2334\tfrac{1}{4} : 153 \ldots \ldots \ldots (7).$$

[Therefore $\quad OF^2 : FA^2 > \{(2334\tfrac{1}{4})^2 + 153^2\} : 153^2$
$$> 5472132\tfrac{1}{16} : 23409.]$$

Thus $\qquad OF : FA > 2339\tfrac{1}{4} : 153 \ldots \ldots \ldots (8).$

Fourthly, let OG bisect the angle AOF, meeting AF in G.

We have then
$$OA : AG [> (2334\tfrac{1}{4} + 2339\tfrac{1}{4}) : 153, \text{ by means of (7) and (8)}]$$
$$> 4673\tfrac{1}{2} : 153.$$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that
$$\angle AOG = \tfrac{1}{48} \text{ (a right angle).}$$

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then $\qquad \angle GOH = \tfrac{1}{24} \text{ (a right angle).}$

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

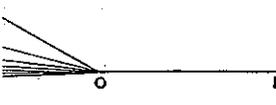And, since $\qquad OA : AG > 4673\tfrac{1}{2} : 153,$

while $\qquad AB = 2OA, \quad GH = 2AG,$

it follows that
$$AB : \text{(perimeter of polygon of 96 sides)} [> 4673\tfrac{1}{2} : 153 \times 96]$$
$$> 4673\tfrac{1}{2} : 14688.$$

But $$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$

$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} \right]$$

$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\tfrac{1}{7}$ times the diameter AB.

II.   Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle.   Join BC.
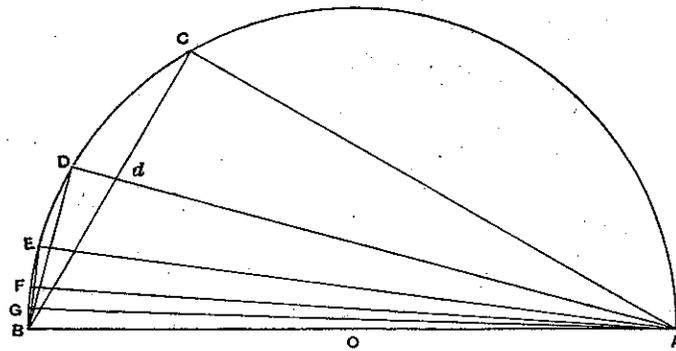
Then $AC : CB \, [= \sqrt{3} : 1] < 1351 : 780.$

First, let AD bisect the angle BAC and meet BC in d and the circle in D.   Join BD.

Then $\angle BAD = \angle dAC$

$$= \angle dBD,$$

and the angles at D, C are both right angles.

It follows that the triangles ADB, $[ACd]$, BDd are similar.

Therefore $AD : DB = BD : Dd$

$$[= AC : Cd]$$

$$= AB : Bd \qquad\qquad \text{[Eucl. VI. 3]}$$

$$= AB + AC : Bd + Cd$$

$$= AB + AC : BC$$

or $BA + AC : BC = AD : DB.$

[But $\qquad AC : CB < 1351 : 780$, from above,

while $\qquad\qquad BA : BC = 2 : 1$

$\qquad\qquad\qquad\quad = 1560 : 780.]$

Therefore $\qquad AD : DB < 2911 : 780$.................(1).

[Hence $\qquad AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$\qquad\qquad\qquad < 9082321 : 608400.]$

Thus $\qquad AB : BD < 3013\frac{3}{4} : 780$ .... (2).

Secondly, let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

Then we prove, in the same way as before, that

$\qquad AE : EB [= BA + AD : BD$

$\qquad\qquad < (3013\frac{3}{4} + 2911) : 780$, by (1) and (2)]

$\qquad\qquad < 5924\frac{3}{4} : 780$

$\qquad\qquad < 5924\frac{3}{4} \times \frac{4}{13} : 780 \times \frac{4}{13}$

$\qquad\qquad < 1823 : 240$ .... (3).

[Hence $\quad AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$\qquad\qquad\qquad < 3380929 : 57600.]$

Therefore $\qquad AB : BE < 1838\frac{9}{11} : 240$.................(4).

Thirdly, let AF bisect the angle BAE, meeting the circle in F.

Thus $\qquad\qquad AF : FB [= BA + AE : BE$

$\qquad\qquad < 3661\frac{9}{11} : 240$, by (3) and (4)]

$\qquad\qquad < 3661\frac{9}{11} \times \frac{11}{40} : 240 \times \frac{11}{40}$

$\qquad\qquad < 1007 : 66$ ....(5).

[It follows that

$\qquad\qquad AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$

$\qquad\qquad\qquad < 1018405 : 4356.]$

Therefore $\qquad AB : BF < 1009\frac{1}{6} : 66$......................(6).

Fourthly, let the angle BAF be bisected by AG meeting the circle in G.

Then $\qquad\qquad AG : GB [= BA + AF : BF]$

$\qquad\qquad < 2016\frac{1}{6} : 66$, by (5) and (6).
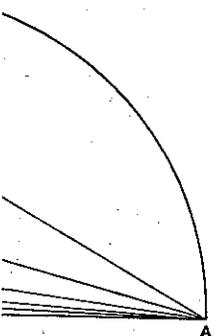
[And $AB^2 : BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} : 66^2$

                              $< 4069284\tfrac{1}{36} : 4356.]$

Therefore $AB : BG < 2017\tfrac{1}{4} : 66,$

whence $BG : AB > 66 : 2017\tfrac{1}{4}$ ....(7).

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre is

$$\tfrac{1}{24} \text{ (a right angle).]}$$

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

(perimeter of polygon) : $AB \; [> 96 \times 66 : 2017\tfrac{1}{4}]$

                              $> 6336 : 2017\tfrac{1}{4}.$

And $\dfrac{6336}{2017\tfrac{1}{4}} > 3\tfrac{10}{71}.$

Much more then is the circumference of the circle greater than $3\tfrac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$