THE WORKS OF
ARCHIMEDES

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THE METHOD OF ARCHIMEDES TREATING OF MECHANICAL PROBLEMS— TO ERATOSTHENES

"Archimedes to Eratosthenes greeting.

I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows.

1. If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms*, and its sides [i.e. four generators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one sixth part of the whole prism.

2. If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms† and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube.

Now these theorems differ in character from those communicated before; for we compared the figures then in question,

* The parallelograms are apparently *squares*.

† i.e. squares.

conoids and spheroids and segments of them, in respect of size, with figures of cones and cylinders: but none of those figures have yet been found to be equal to a solid figure bounded by planes; whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes. The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure* though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it† and I do not want to be thought to have uttered vain words, but

* *περὶ τοῦ εἰρημένου σχήματος*, in the singular. Possibly Archimedes may have thought of the case of the pyramid as being the more fundamental and as really involving that of the cone. Or perhaps "figure" may be intended for "type of figure."

† Cf. Preface to *Quadrature of Parabola*.

equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e. a parabola) is four-thirds of the triangle which has the same base and equal height,

and after this I will give each of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions]...

[I premise the following propositions which I shall use in the course of the work.]

1. If from [one magnitude another magnitude be subtracted which has not the same centre of gravity, the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder. [*On the Equilibrium of Planes*, I. 8]

2. If the centres of gravity of any number of magnitudes whatever be on the same straight line, the centre of gravity of the magnitude made up of all of them will be on the same straight line. [Cf. *Ibid.* I. 5]

3. The centre of gravity of any straight line is the point of bisection of the straight line. [Cf. *Ibid.* I. 4]

4. The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another. [*Ibid.* I. 13, 14]

5. The centre of gravity of any parallelogram is the point in which the diagonals meet. [*Ibid.* I. 10]

6. The centre of gravity of a circle is the point which is also the centre [of the circle].

7. The centre of gravity of any cylinder is the point of bisection of the axis.

8. The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base].

[All these propositions have already been] proved*. [Besides these I require also the following proposition, which is easily proved:

If in two series of magnitudes those of the first series are, in order, proportional to those of the second series and further] the magnitudes [of the first series], either all or some of them, are in any ratio whatever [to those of a third series], and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series. [*On Conoids and Spheroids*, Prop. 1.]"

Proposition 1.

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.

Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC.

From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.

* The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes. It may have been solved either in a separate treatise, such as the περὶ ζυγῶν, which is lost, or perhaps in a larger mechanical work of which the extant books *On the Equilibrium of Planes* formed only a part.

Consider CH as the bar of a balance, K being its middle point.

Let MO be any straight line parallel to ED, and let it meet CF, CK, AC in M, N, O and the curve in P.

Now, since CE is a tangent to the parabola and CD the semi-ordinate,

$$EB = BD;$$

"for this is proved in the Elements [of Conics]*."

Since FA, MO are parallel to ED, it follows that

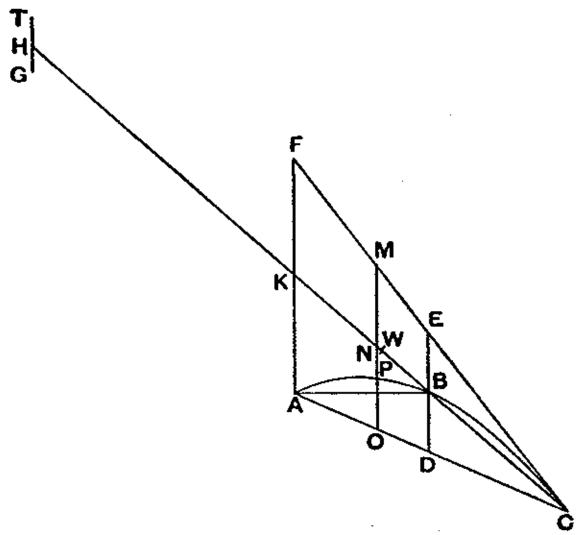
$$FK = KA, \quad MN = NO.$$

Now, by the property of the parabola, "proved in a lemma,"

$MO : OP = CA : AO$ [Cf. *Quadrature of Parabola*, Prop. 5]

$= CK : KN$ [Eucl. VI. 2]

$= HK : KN.$



Take a straight line TG equal to OP, and place it with its centre of gravity at H, so that $TH = HG$; then, since N is the centre of gravity of the straight line MO,

and $$MO : TG = HK : KN,$$

* i.e. the works on conics by Aristaeus and Euclid. Cf. the similar expression in *On Conoids and Spheroids*, Prop. 3, and *Quadrature of Parabola*, Prop. 3.

it follows that TG at H and MO at N will be in equilibrium about K. [*On the Equilibrium of Planes*, I. 6, 7]

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point on KC and (2) a length equal to the intercept between the curve and AC placed with its centre of gravity at H will be in equilibrium about K.

Therefore K is the centre of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO,

and the segment CBA is made up of all the straight lines like PO within the curve,

it follows that the triangle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H.

Divide KC at W so that $CK = 3KW$;

then W is the centre of gravity of the triangle ACF; "for this is proved in the books on equilibrium" (ἐν τοῖς ἰσορροπικοῖς).

[Cf. *On the Equilibrium of Planes* I. 15]

Therefore $\triangle ACF : (\text{segment } ABC) = HK : KW$

$= 3 : 1.$

Therefore segment $ABC = \frac{1}{3} \triangle ACF$.

But $\triangle ACF = 4 \triangle ABC$.

Therefore segment $ABC = \frac{4}{3} \triangle ABC$.

"Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published*."

Proposition 2.

We can investigate by the same method the propositions that

(1) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*

(2) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is* $1\frac{1}{2}$ *times the sphere.*

(1) Let $ABCD$ be a great circle of a sphere, and AC, BD diameters at right angles to one another.

Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to CA.

Let CH be regarded as the bar of a balance, A being its middle point.

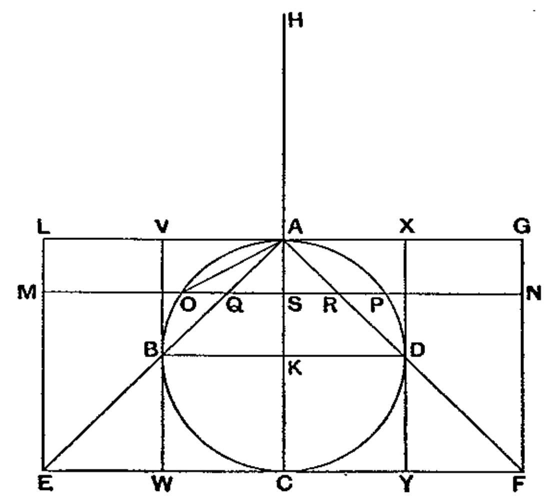
Draw any straight line MN in the plane of the circle $ABCD$ and parallel to BD. Let MN meet the circle in O, P, the diameter AC in S, and the straight lines AE, AF in Q, R respectively. Join AO.

* The word governing τὴν γεωμετρουμένην ἀπόδειξιν in the Greek text is τάξομεν, a reading which seems to be doubtful and is certainly difficult to translate. Heiberg translates as if τάξομεν meant "we shall give lower down" or "later on," but I agree with Th. Reinach (*Revue générale des sciences pures et appliquées*, 30 November 1907, p. 918) that it is questionable whether Archimedes would really have written out in full once more, as an appendix, a proof which, as he says, had already been published (i.e. presumably in the *Quadrature of a Parabola*). τάξομεν, if correct, should apparently mean "we shall appoint," "prescribe" or "assign."

Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.

Now, since $MS = AC$, and $QS = AS$,

$$MS \cdot SQ = CA \cdot AS$$
$$= AO^2$$
$$= OS^2 + SQ^2.$$



And, since $HA = AC$,

$$HA : AS = CA : AS$$
$$= MS : SQ$$
$$= MS^2 : MS \cdot SQ$$
$$= MS^2 : (OS^2 + SQ^2), \text{ from above,}$$
$$= MN^2 : (OP^2 + QR^2)$$
$$= (\text{circle, diam. } MN) : (\text{circle, diam. } OP + \text{circle, diam. } QR).$$

That is,

$HA : AS =$ (circle in cylinder) : (circle in sphere + circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centres of gravity at H.

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder, the sphere and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centres of gravity at H.

Therefore, since K is the centre of gravity of the cylinder,

$$HA : AK = \text{(cylinder)} : \text{(sphere + cone } AEF\text{)}.$$

But $HA = 2AK$;

therefore cylinder $= 2$ (sphere + cone AEF).

Now cylinder $= 3$ (cone AEF); [Eucl. XII. 10]

therefore cone $AEF = 2$ (sphere).

But, since $EF = 2BD$,

cone $AEF = 8$ (cone ABD);

therefore sphere $= 4$ (cone ABD).

(2) Through B, D draw VBW, XDY parallel to AC; and imagine a cylinder which has AC for axis and the circles on VX, WY as diameters for bases.

Then cylinder $VY = 2$ (cylinder VD)

$= 6$ (cone ABD) [Eucl. XII. 10]

$= \frac{3}{2}$ (sphere), from above.

Q.E.D.

"From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended

that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius*."

Proposition 3.

By this method we can also investigate the theorem that

A cylinder with base equal to the greatest circle in a spheroid and height equal to the axis of the spheroid is $1\frac{1}{2}$ times the spheroid;

and, when this is established, it is plain that

If any spheroid be cut by a plane through the centre and at right angles to the axis, the half of the spheroid is double of the cone which has the same base and the same axis as the segment (i.e. the half of the spheroid).

Let a plane through the axis of a spheroid cut its surface in the ellipse $ABCD$, the diameters (i.e. axes) of which are AC, BD; and let K be the centre.

Draw a circle about BD as diameter and in a plane perpendicular to AC;

imagine a cone with this circle as base and A as vertex produced and cut by a plane through C parallel to its base; the section will be a circle in a plane at right angles to AC and about EF as diameter.

Imagine a cylinder with the latter circle as base and axis AC; produce CA to H, making AH equal to CA.

Let HC be regarded as the bar of a balance, A being its middle point.

In the parallelogram LF draw any straight line MN parallel to EF meeting the ellipse in O, P and AE, AF, AC in Q, R, S respectively.

* That is to say, Archimedes originally solved the problem of finding the solid content of a sphere before that of finding its surface, and he inferred the result of the latter problem from that of the former. Yet in *On the Sphere and Cylinder* I. the surface is independently found (Prop. 33) and *before* the volume, which is found in Prop. 34: another illustration of the fact that the order of propositions in the treatises of the Greek geometers as finally elaborated does not necessarily follow the order of discovery.