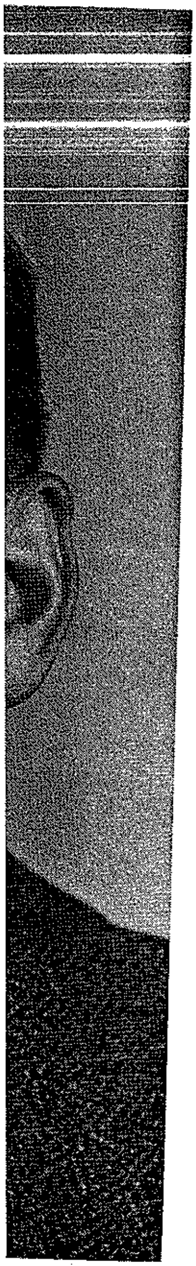

EXEGESIS AND ARGUMENT

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Was There a Special Epicurean Mathematics?

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In the preceding decade there have again been attempts to take seriously the Epicurean scientific system of the world. Gregory Vlastos did so in his "Minimal Parts in Epicurean Atomism," *Isis*, 56 (1965), 121–47, and David Furley in *Two Studies in the Greek Atomists* (Princeton, 1967). On the evidence of the passages in Proclus' commentaries on the first book of Euclid's *Elements*, Vlastos argued that there has never been something we could call a special Epicurean mathematical system, and that all we learn about the Epicurean criticism of Euclid is concerned with the question, whether Euclid's axiomatic system is complete or not. Furley takes a similar stand in his highly learned commentary and translation of the famous passage in the *Letter to Herodotus* about the atoms and their smallest parts, the *minimae partes*. Both of them are criticizing the earlier attempts of Hans von Arnim, Solomon Luria, and those of myself. To state the problem again: Did antiquity accept the mathematical system of Euclid, Archimedes, Apollonius of Perga, etc., as the only possible one, or is there at least one alternative? Did the ancients think that all mathematicians had to accept the axiom that any length can be divided into as many parts as one likes—or, as Archimedes (*De sph. et cyl.* I ass. 5 and prp. 2) puts it, that given two magnitudes in the proportion $a : b$, there are always two straight lines in the proportion $c : d$, where $c : d < a : b$?

Epicurus Letter to Herodotus 56 ff.

Let us scan first the two Epicurean passages that have induced scholars to raise the question. The first is Epicurus *Letter to Herodotus* § 56 (in Diogenes Laertius X.56 ff.). In §§ 42–45 Epicurus had explained the world system as consisting of atoms and void or empty space. From §§ 46–65 he likewise explains sense perception in atomist terms —viz. since atoms are always separated by void, and since void has no density, and thereby cannot resist the motion of the atoms, atoms are always in motion at an infinitely high speed. Even when they are packed into those conglomerates that we call visible bodies, they do not stop moving back and forth but are always hitting each other and rebounding. By this oscillating movement of the bodies, something like thin films is always projected from their surface and meets our organs of sense perception. The qualities of visible bodies we perceive are therefore caused by the way in which atoms have been packed together, not by special qualities of the atoms themselves. The atoms have no such special qualities. From this Epicurus returns to the atoms again in § 55. Here, in the second part of § 55, we learn:

> But we must not think that any size is to be found in the atoms, else the sensible facts would give evidence of the contrary. However, we have to think that there are certain differences in sizes, for on this hypothesis we can much better explain everything that concerns our feeling and sense perceptions. The assumption that every atomic magnitude exists would not be useful for the explanation of the differences of qualities either, since it could follow that eventually a visible atom would reach our eyes; but that has never been observed, and a visible atom is something that is quite inconceivable (we have already said that sensible qualities belong to conglomerates of atoms only, not to atoms themselves).
>
> Furthermore, one must not think that in a finite body there are infinitely many units of volume, even not those of any size you like (of course smaller and smaller ones are meant). Therefore one must not only reject the division *ad infinitum* in the direction of the ever and ever smaller, since otherwise we would make everything weak, and would be forced to destroy by grinding into dust the existing things in trying to conceive them, but we must also reject the possibility of passing from one volume unit to the other *ad infinitum* in finite bodies.[1]

None of the many attempts to explain the words μηδὲ τοὔλαττον, which follow, has succeeded, and I am now convinced that they have to be canceled as a gloss on the words five lines above.

As we see here, Epicurus explicitly rejects the so-called εἰς ἄπειρον τομή. This seems to be a technical term for the mathematical axiom that any size can be bisected again and again *ad infinitum*. We meet this expression as a heading of a passage of Lucretius (I.550) in the oldest manuscripts, and it was put into the text by Diels on the assumption that it originated at the time of the Roman emperors. The mathematicians and Aristotle mean by the word τομή usually a section, for instance a conic section, a cylinder section, or something

[1] Cf. Aristot. *De Lin. Insec.* 968a18, 968a26.

like that. We cannot expect to find the expression εἰς ἄπειρον τομή in mathematical writings, since it is one of the main efforts of mathematicians generally to avoid the concept infinity and to explain infinite procedures by finite steps.[2]

Excursus: *The Aristotelian* De Lineis Insecabilibus

In Aristotle *De Lin. Insec.* 970b3 there is an expression that contains the word τομή and resembles the usage in our Epicurean passage very much. This short treatise testifies that at the time of its appearance there was a tendency among mathematicians to argue against division *ad infinitum*. Consider 969b28:

> [The preceding reasons show that there cannot exist indivisible lines.] Further it becomes even more evident from reasons like this: The assumption of indivisible lines violates mathematical propositions and assumptions that must either be sustained or abandoned for stronger reasons.

There follows a long string of arguments as to which mathematical law would be violated, and how the violation is to be understood. I summarize:

> The definition of straight line will not hold for the indivisible line; all lines will be commensurable; let a be an indivisible line, then $((b>a)$ and $(bx=a^2)) \rightarrow (x<a)$, which is absurd. Further, there would be more constructions that imply lines shorter than an assumed indivisible line. Lines consisting of an even and of an odd number of indivisibles would be divisible in different ways, etc.

I cannot agree with Paul Gohlcke[3] who denies any relation between this passage and *Physics* 6.1–2, as well as similar contexts in the major works. Actually the difficulty of the problem we are concerned with seems to have inspired the medieval copyist to mistakes of all sorts in all sources. In *De Lin. Insec.* 971b27–31 Bonitz felt obliged to supply γραμμὴν <μὴ> συνεχῆ. However, the sentence is still mutilated.

[2] E.g. Eucl. *El.* X prp. 1: "Given two unequal magnitudes, subtract from the greater more than its half, from the rest more than its half, and so on. You will reach a magnitude smaller than the given smaller one." That it is always possible to do so was proved, and how it is done was shown, in I.10. In X.1 we may take any line as the greater, and the hypothetical indivisible line as the smaller magnitude. The law of division *ad infinitum* has thus been stated without mentioning infinity.

[3] *Aristotel. kleine Schriften zur Physik und Metaphysik* (Paderborn, 1957), p. 10.

The key words are ἐφεξῆς, ἁπτόμενον, συνεχές, the same words as in *Phys.* 6.1–2, a relation to which was denied by Gohlcke. The *Physics* chapter contains the proof that a continuum cannot consist of indivisibles. It proves this by using the definitions συνεχῆ μὲν ὧν τὰ ἔσχατα ἕν, ἁπτόμενα δ' ὧν ἅμα, ἐφεξῆς δ' ὧν μηδὲν μεταξὺ συγγενές. The proof was condensed to a mathematical form and used as entry point into the *Elementatio Physica* by Proclus.[4]

Simplicius (*In Ar. Phys.* VI.1.231a21, p. 925 H.) tells us that Aristotle is opposing Democritus and Leucippus, who regarded matter as consisting of atoms and atoms as unchangeable due to their being partless (ἀμερῆ). Epicurus afterwards had to consider Aristotle's proofs and abandoned the "partless." This is exactly what we read in our passage of Epicurus. I am afraid the sentence in *De Lin. Insec.* which we started from is beyond repair. If we adopt Bonitz's conjecture, we are still left with a ὥστε c. inf. in the apodosis (our passage is listed among the few instances for ὥστε in apodosis). An attempt to restore the text has to start from fixing a syllogism, e.g. Baroco, εἰ ἐνδέχεται ἐφεξῆς τι εἶναι μὴ ἁπτόμενον (2d prem.), τὸ δὲ συνεχὲς <ἁπτόμενόν τι> (1st prem.), οὐδὲν ἄλλο λέγομεν ἢ τό· ἐξ ὧν ἐστιν <ἡ γραμμὴ στιγμαί τινες οὐ συνεχεῖς εἰσίν, πάντων τῶν συνεχῶν> ἁπτομένων· ὥστε καὶ οὕτως.... (I am using an additional axiom not explicitly referred to by Aristotle: If two things are συνεχῆ, they are also ἁπτόμενα and ἐφεξῆς, but not vice versa. If they are ἁπτόμενα, they are also συνεχῆ, but not vice versa.) This critical remark is to introduce evidence for the close relation between *De Lin. Insec.* and *Physics* VI. The many allusions to professional mathematics and the fact that sometimes mathematical statements are referred to that do not come from Euclid show that Heath[5] is right in attributing the *De Lin. Insec.* to one of the first successors of Aristotle and to the time when Euclid was well known but was not yet the only Elementarist. It is well known that Plato developed his own atomic theory in the *Timaeus*. That Xenocrates tried to introduce indivisible lines into mathematics we learn from Proclus (*In Eucl.* I.279.5 F). Our treatise *De Lin. Insec.* 969a21 closely connects the controversy about elementary bodies with that about indivisible lines; it is fighting against the first successor of Plato, who lived when Platonism was still a Pythagorean dogmatic school. We know from Sextus Empiricus that the discussion was continued by the sceptical Middle Academy.

[4] *E.P.*, ed. H. Boese (Berlin, 1958).

[5] T. Heath, *A History of Greek Mathematics* (Oxford, 1921), vol. 1, p. 347.

From all this it follows that the concept of indivisible mathematical magnitudes was still alive when Epicurus wrote his letter, and that in writing the letter he took a stand in a discussion between mathematicians about the axiomatic system itself, not only about its logical completeness.

Μετάβασις

Another question: What is the difference between division *ad infinitum* and μετάβασις? The lexicon does not help us, as Furley already pointed out. According to Ettore Bignone[6] Epicurus tried to revive the atomist system of Democritus and Leucippus by considering the objections made by Aristotle and his school. Aristotle, in *De Gener. et Corr.* I.2 advocates, for the sake of discussion, the atomist theory that in magnitudes there must be a lower limit. Here and in *De Lin. Insec.* the arguments are very similar. In *De Gener. et Corr.* he says that given a wood log, if one tries to saw it apart again and again, then, finally, what turns up is something like sawdust; and if one tries to cut up the single grains of sawdust again, there will be left nothing. Those nothings put together will never make a magnitude. So the εἰς ἄπειρον τομή has been rejected. But Aristotle states another alternative. Suppose the actual cutting up of physical magnitudes is impossible, one could still ask oneself whether it is possible to assume certain mathematical points all over a given magnitude, and to say that the magnitude can be bisected in each of these points. But a magnitude cannot consist of points, since a point is by definition the touch between one part of the length and the other, so that two points can never touch each other, and, as we can add, it is impossible to pass over a given length in stepping from one point to the next one. I do not find another explanation for the term μετάβασις, and, as all the commentators have done, I would like to recall the famous dilemma of Zeno of Elea, where μετάβασις is illustrated by the race between Achilles and the tortoise, and the flying arrow (cf. *De Lin. Insec.* 968a18).

Minimae Partes

The Epicurus text goes on to show that the atom, in spite of the fact that it cannot be mechanically cut up into parts or be changed at all,

[6] *Epicuro* (Bari, 1920), p. 95.

must, in an abstract sense, have parts, because atoms have different forms and different sizes. The question is, of course, whether this abstract sense points to two different minima, i.e. physical and mathematical.

Declinatio

The other passage that made me suppose that the Epicurean school rejected the main axioms of Greek mathematics is the one about the cosmogonic theory of "the swerve" (*declinatio*). Our most important testimony is Lucretius II.240–245. We will see that the passages containing this theory in Cicero and Plutarch can help us, although they do not try to understand Epicurus' reason, but are only refuting it on the basis of sceptical Platonism. Lucretius had explained that *in statu nascendi* of a cosmos, atoms are falling down by their weight in parallel lines and with equal speed. A cosmos—that is a conglomeration of atoms—can only come to be by atoms hitting each other, rebounding, and entangling themselves with each other. That cannot happen, given the parallel downward motion. One could think that atoms hit each other because the heavier ones are falling faster than the lighter ones. This, however, is not the case, because the atomic motion through the void takes place always at the same speed, because there is nothing that could give them any resistance and so slow them down:

> Haud igitur poterunt levioribus incidere unquam ex supero graviora, neque ictus gignere per se, qui varient motus, per quos natura gerat res. Qua re etiam atque etiam paulum inclinare necesse est corpora; nec plus quam minimum, ne fingere motus obliquos videamur, et id res vera refutet.

> Therefore it is impossible that the heavier atoms should ever hit the lighter ones from above, and that they by themselves should create collisions that could alter the motions, the motions by which Nature could operate things. Therefore we must say again and again: The bodies swerve a little bit, and this *not more than the minimum*, otherwise one could say that we are imagining oblique motions, and actual facts would refute that. (Lucr. II.240 ff.)

When Lucretius says, "not more than the minimum," he must have in mind a certain magnitude of the minimum size. That this word *minimum* was of great importance within the Epicurean theory we learn from Cicero *De Fato* 10.22, where Cicero tells us, quoting the Greek term in his text, that Epicurus called this minimum value the ἐλάχιστον. Lucretius himself says implicitly that a motion swerving from the straight line by the minimum is still a straight motion. Of course the Greek word ἐλάχιστον cannot be pressed as a technical term, because there is no other word to express a very small value. However, the quotation from Cicero and the context in which we have the word *minimum* in Lucretius show that in this case it is really used terminologically. Consequently it becomes most likely that in *Letter to Herodotus* § 59 line 2 the word ἐλάχιστον also means a certain constant which is part of Epicurean physics and mathematics.

Epicurean Mathematics

The next question: What do we know about Epicurean efforts in the field of mathematics? On papyrus 144 of Herculaneum we have the life of the Epicurean philosopher Philonides who served the Seleucid kings Antiochus Epiphanes (175–164) and Demetrios Soter (162–150) as a royal philosopher. The papyrus tells us that he was concerned with geometry, dialectic, and rhetoric. Without pressing the mutilated text too much, we can say that he wrote something geometrical about ΠΕΡΙ ΕΛΑ[ΧΙΣ]ΤΩΝ in connection with the 8th book on physics of Epicurus. As a young boy, he had been in contact with the famous mathematician Apollonius of Perga (I 192 H.). If we accept Croenert's restoration—and it is very plausible—we have here the proof that ἐλάχιστα is really a term of Epicurean mathematics. Another Epicurean mathematician was Polyaenus, who according to Cicero *Acad. Pr.* II.33.106 was a great mathematician indeed before he turned Epicurean and thus came to believe that the entire geometry was false. The third one is Zeno of Sidon to whom we have to return later.

Since Aristotle it has become a habit to say that everybody who does not accept even one of the Euclidean axioms is not a mathematician at all. But we will see that even Archimedes got inspirations from the side of some sort of atomist geometry. It is well known that Greek mathematics started from operations with integers. The old Pythagoreans divided them into two sets, the even and odd numbers, and developed the system of polygonal numbers. What we now call fractions were then handled as relations between two integers. The musical theory of harmony supplied the observational material that had to be expressed in this way. In Plato's *Timaeus* this reasoning still played a predominant role, but here and in the *Theaetetus* we already have to do with geometrical means and square and cubic roots, which cannot be expressed by the relations of integers but are irrational. It was the generation of Plato and of his immediate followers like Eudoxus that developed the theory of, and operations with, irrationals, such as the solution of the Delic problem, conic sections, the quadrature of curves, etc. By the time of Euclid's *Elements* it could be said that geometry stands or falls with the axiom of the infinite divisibility of magnitudes, but Euclid's theory of numbers is still influenced by the old Pythagorean theory, as we can see from the fact that he applies the term ἀριθμοί to the integers from two onwards, but calls the number one μονάς.

Are Geometry and Finite Divisibility Compatible?

The already quoted papyrus about the Epicurean philosopher and mathematician Philonides tells us that there was some contact at the royal court of Pergamon between this Epicurean and great mathematicians like Archimedes and Apollonius of Perga. At the start of our century a book of Archimedes called *Ephodos* was discovered in a monastery on Mt. Athos. In this book Archimedes solves some problems about area and volume which cannot be solved by elementary geometry. He starts from the hypothesis that a plane consists of a great many parallel lines or stripes, and that a body consists of parallel planes or very thin slices. This, however, had already been refuted by Aristotle *De Caelo* 300a1, and Archimedes knows that it is mathematically impossible. Nevertheless he slices up stereometrical and geometrical figures into areas or lines, respectively, weighs each of these slices or stripes on a sort of balance, and proves that, when that has been done with all the slices or stripes contained in the figure, the area or volume has been found. Of course he knows, and says so explicitly, that this is not a mathematical proof but that this way of reasoning is useful for finding theorems which later can be proved by strict mathematical methods. Further, and this is very important, in the letter to Eratosthenes which forms part of the *Ephodos* he praises Democritus as the inventor of the theorem that, given a pyramid and a prism with same height and same base, the volume of the pyramid is the third part of that of the prism. It is not too daring to state, as Luria did, that Archimedes owed this heuristic, not properly mathematical, method to some members of the atomist school.

This has answered another question already, viz.: Is mathematical reasoning without the axiom of division *ad infinitum* reasonable at all? From the Epicurean standpoint of course Yes, for from the reliable testimony of Archimedes we know that it is possible to find new and true theorems without that axiom. It is an indispensable part of the Epicurean philosophical system that logically perfect proofs are neither necessary nor wanted. The only real criterion of truth is the ἀντιμαρτύρησις and ἐπιμαρτύρησις, and it is not very likely that a theorem found in the above described way will ever be refuted by sense perception. It is further to be considered that the old Pythagorean operations with integers did not suffer any loss of interest in later times, whereas the axiomatic system of mathematics has always been the target of strong attacks from the side of the sceptical school, The arguments brought forward have been collected by Sextus Empiricus in his two books against the arithmeticians and against the geometricians.

The Attacks by Zeno of Sidon

Instead of going into further details, however, let us recall the best known Epicurean attacks against mathematics, namely the objections of Zeno of Sidon known from Proclus' commentary on the first book of Euclid.

General remarks we find on pages 199–200. In the preceding part of his commentary Proclus dealt with the definitions, postulates, and axioms of Euclid. Now he says that many objections have been raised against the system of geometry and divides them into three sets:

> (1) The sceptics try to do away with geometry by rejecting its principles because they had to do so with all dogmatic sciences.
>
> (2) The Epicureans did the same but only because they disliked mathematics.
>
> (3) Zeno of Sidon was one of those who, taking the principles of geometry for granted, tried to show that theorems cannot be logically deduced from them because certain axioms are lacking. (It is explicitly stated that he belonged to the school of Epicurus.)

Professor Vlastos takes (3) as proof for the fact that Zeno of Sidon did not reject the classical mathematical system but was merely trying to fill in some logical gaps. This does not seem probable to me. For if, as Proclus states, the Epicureans were trying hard to overthrow mathematics generally, then they would be one step ahead if, taking as much testimony as possible from the enemy, they managed to prove that he is contradicting himself. That would hit the opponent much harder than just not accepting his principles. Zeno's first actual objection is directed against proposition (1). The construction of the equilateral triangle cannot be deduced from the axioms since it is not necessary that the two sides projected from the endpoints of the basis should meet at one point only. It might be that they meet each other, run a short piece together, and then part again. This possibility could only be excluded by an additional axiom saying that two lines cannot have a common section. This however is closely connected with the theory of declination which we encountered while considering the atomic motion in the cosmogony. The principle underlying that theory we can formulate like this: A straight line never deviates from its original direction more than by the minimum, the ἐλάχιστον. Zeno's objection is taken quite seriously by Proclus, and he tries to show that the axiom lacking according to Zeno's opinion has been anticipated by Euclid in the definition for straight line and in the postulate that any straight line can be prolonged in the same direction. There is no need to go into further detail since it has already been done in the very elaborate paper of Prof. Vlastos. However, let me call attention to the fact that the great Poseidonius thought it worthwhile not only to take a stand in this discussion but even to write a whole book on the topic, a book aimed not only at Zeno but generally against the Epicurean school (Proclus 216.21 f.). The other objection, this time attributed by Proclus to the Epicurean school generally, is raised against proposition number 20, which says: In any triangle the sum of two sides is greater than the remaining side. It is typical of the Epicurean point of view that they say this proposition is superfluous, because they ridicule dialectical strictness and say, as we have seen already, that it is enough proof for a fact to have it testified by sense perceptions and not falsified by any of them. In this case, they say that even an ass knows that the straight way to a point is shorter than the one round a corner.

Let me sum up: The information available does not allow the conclusion that there was some special kind of atomist mathematics, or that there was none. However, a comparison of those passages that give us information about Epicurean attacks against mathematics with the Epicurean system shows that those attacks are in close connection with the central points of the Epicurean philosophy, and that later Epicureans, driven into a corner under the onslaught of Academic arguments, were at least trying to show that mathematics need not break down in an atomist system.