ARCHIMEDES

in the Middle Ages

VOLUME THREE

THE FATE OF THE MEDIEVAL ARCHIMEDES

1300 to 1565

PART III: The Medieval Archimedes in
the Renaissance, 1450–1565

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The *Archimedis Liber de sphaera et cylindro*
of Francesco Maurolico

40 /Archimedis liber de sphaera et cylindro, ex traditione
Eutocii per Franciscum Maurolycum Mamertinum
Mathematicae disciplinae studiosissimum emendati, et
ad optimum ordinem restituti et adaucti.

5 Propositio I.

PYRAMIDIS SUPER BASIM AEQUILATERAM ET AEQUI-
ANGULAM ERECTAE SUPERFICIES (QUAE CONGERIES EST
TRIGONORUM AD VERTICEM PYRAMIDIS COEUNTIUM)
AEQUALIS EST TRIGONO RECTANGULO CUIUS UNUM
10 EORUM QUAE CIRCA RECTUM ANGULUM AEQUALE EST
PERPENDICULARI QUAE A VERTICE AD LATUS BASIS, RE-
LIQUUM VERO PERIMETRO BASIS.

Sit, exempli gratia, super basim pentagonam $ABGD$ aequilateram
et aequiangulam pyramis, cuius vertex Z, ita ut recta quae a puncto Z ad
15 centrum circuli circumscribentis pentagonum ABG sit ipsi pentagono
perpendicularis [see Fig. III.5C.1]. Unde fit ut triangula quae ad verticem
Z concurrunt sint invicem aequilatera et aequiangula; cadat autem a
puncto Z ad unum laterum pentagoni, utpote AB, perpendicularis ZH,
ponaturque trigonum TKL rectum qui apud K angulum habens, cuius
20 latus TK ipsi ZH, latus autem KL universo perimetro pentagoni ABG sit
aequale.

Dico itaque totam superficiem pyramidis, quae congeries est trigonorum
ad verticem Z coniunctorum, aequalem esse trigono TKL.

Secetur KL in tot segmenta quot sunt latera basis ABG, hoc est in hoc
25 exemplo in quinque partes KM, MN, NX, XO, OL, quae singulae sint
aequales singulis lateribus basis ABG, et connectantur TM, TN, TX, TO.
Itaque quoniam trigonorum AZB, TKM bases AB, KM sunt aequales, et
perpendiculares ZH, TK aequales, ideo ipsa trigona per 38^{am} primi
aequalia; sed per primam sexti trigonum TKL ad trigonum TKM, sicut
41 basis KL ad basim KM, et ideo quincuplum; ipsa quoque / superficies
31 pyramidis ABG, quae congeries est quinque trigonorum ad punctum Z
compactorum, quincupla est ad trigonum AZB. Ergo et dicta superficies
pyramidis ABG aequalis est trigono TKL, quod erat demonstrandum.

Propositio II.

CONI CURVA SUPERFICIES AEQUALIS EST TRIGONO REC-
TANGULO CUIUS UNUM LATERUM RECTUM ANGULUM CON-

TINENTIUM AEQUALE EST CONICO LATERI, RELIQUUM VERO
5 PERIPHERIAE BASIS.

Sit trigonum rectangulum *ABG*, rectum qui ad *B* angulum habens
[Fig. III.5C.2] quo super axem *AB* revoluto donec redeat ad locum
suum, describatur conus verticem habens *A*, basim vero circulum *GDE*,
sitque trigonum *ZHT* rectum apud *H* angulum habens, cuius latus *ZH*
10 ipsi hypothemisae (! hypothenusae) *AG* sit aequale; latus autem *HT*
peripheriae circulari *GDE* aequalis.

Dico quod conica superficies coni *AGDE*, quam describit linea *AG*,
aequalis est trigono *ZHT*.

Nam si trigonum *ZHT* non sit aequum conicae superficiei *AGDE*, erit
15 omnino aequum conicae superficiei coni habentis basim minorem mai-
oremve circulo *GD*, et altitudinem eandem cum cono *AGD*. Sit ergo
primum trigonum *ZHT* aequum conicae superficiei coni, cuius vertex *A*
basisque circulus *KL* minor ipso circulo *EG*, et latus *AL*, qui videlicet
conus ab ipso trigono *ALB* describitur. Et per 13^{am} 12^i intra circulum
20 *EG* describatur polygonium rectilineum aequilaterum et aequiangulum
cuius latera minime contingant circulum *KL*, et a vertice *A* descendant
hypothenusae ad angulos polygonii, quae cum lateribus polygonii con-
tinebunt pyramidem habentem cum cono *AGD* verticem eundem, et in-
clusam eidem cono. Itaque ducatur a vertice *A* perpendicularis ad unum
25 laterum polygonii *EG*, ut pote ad latus *GD*, quae perpendicularis sit
AM, eritque per praecedentem superficies pyramidis *AGD*, quae congeries
est trigonorum ad verticem *A* coeuntium, aequalis trigono rectangulo
cuius unum latus eorum quae circa rectum aequale est perpendiculari
AM, reliquum perimetro polygonii *GDE*. Huiusmodi ergo trigonum
30 minus est trigono *ZHT*, cuius latera *ZH*, *HT*, quae circa rectum,
maiora sunt, quippe quae aequalia sunt lateri *AG* et peripheriae circuli
GDE, quae maiora sunt perpendiculari *AM* et perimetro polygonii *GDE*.
Sed trigonum *ZHT* aequum fuit superficiei coni *ALK*. Igitur superficies
dicta pyramidis *AGD* minor est superficie coni *ALK*, quod est impos-
35 sibile, cum superficies pyramidis maior sit superficiei (!) coni *ALK*
inclusa. Vel sic, quoniam trigonum *ZHT* maius est trigono cui aequalis
est superficies pyramidis, superficies autem pyramidis maior est super-
ficie coni *ALK* inclusa, ergo trigonum *ZHT* maius est superficie coni
ALK; itaque non est ei aequale, sicut supponebatur.

40 Esto nunc trigonum *ZHT* aequum, si possibile est, conicae super-
42 ficiei coni habentis basim maio/rem circulo *GD* et eandem altitudinem
cum cono *AGD*; sitque, brevitatis causa, conus suppositus *ALK*, et trigoni
ZHT latus *ZH* aequum hypothenusae *AL*, at latus *HT* aequum peripheriae
circuli *KL*.

45 Aio iam quod non est possibile trigonum *ZHT* esse aequum super-
ficiei curvae alicuius coni habentis basim maiorem circulo *KL* et verticem *A*.
Nam, si possibile est, sit trigonum *ZHT* aequum superficiei coni cuius
basis circulus *EG* maior circulo *KL*, et describatur ut prius intra

circulum *EG* polygonium aequalium laterum non tangentium circulum
50   *KL*; et super poligonium pyramis *GD* vertice *A*, a quo ad unum laterum
polygonii *DG* cadat perpendicularis *AM*. Eritque per praecedentem
superficies pyramidis *AGD*, quae congeries est trigonorum ad verticem,
aequalis trigono rectangulo cuius laterum quae circa rectum unum aequale
est perpendiculari *AM*, reliquum perimetro polygonii *GDE*. Hoc itaque
55   trigonum maius est trigono *ZHT*, cum habeat latera quae circa rectum
maiora. Sed trigonum *ZHT* aequum fuit superficiei coni *AGDE*. Ergo
dicta pyramidis superficies maior est superficie coni *AGDE*, inclusa
claudente, quod est impossibile. Similiter ergo si trigoni *ZHT* latus *ZH*
hypothenusae *AG*, et latus *HT* peripheriae circuli *EG* sit aequale, non
60   esset trigonum *ZHT* aequale superficie (*!*) coni cuiuspiam cuius basis sit
maior circulo *EG* et vertex *A*. Fuitque ostensum quod nec idem trigonum
aequum est superficiei coni cuiuslibet cuius basis sit minor circulo *EG*,
et vertice *A*. Superest ergo ut idem trigonum *ZHT* aequum sit conicae
superficiei coni *AGDE*, quod erat demonstrandum.

65   Corollarium I.

Manifestum est ergo quod ex ductu conici lateris in dimidium
periphaeriae basis producitur conica superficies, quemadmodum ex ductu
unius laterum quae circa rectum in trigono rectangulo in reliqui dimidium
per 41^{am} primi consurgit area trigoni.

70   Corollarium II.

Rursum ex ductu semidiametri conicae basis in semiperipheriam circuli
cuius semidiameter est latus conicum producitur conica superficies, quod
sequitur ex tertio corollario primae (*!* sextae).

                            Corollarium III.

75   Demum circulus cuius semidiameter est media proportionalis inter
latus conicum ac semidiametrum conicae basis aequalis est conicae
superficiei, quod sequitur ex prostremo corollario primae (*!* sextae).

43   /Propositio III.

CONICA SUPERFICIES AD BASIM EST SICUT CONICUM LATUS
AD SEMIDIAMETRUM BASIS.
Sit conus *ADG* descriptus a trigono *ABG* angulum *B* rectum habente
5   et revoluto circa axem *AB*, cuius vertex *A*, basis *DG*, eiusque semi-
diameter *BG* [Fig. III.5C.3].
Aio quod curva superficies coni *ABG* ad circulum *DG* est sicut
hypothenusae *AG* ad semidiametrum *BG*.
Sit enim trigonum *ZHT* rectum angulum *Z* habens, cuius latus *ZH*
10   sit aequum hypothenusae *AG*, ipsum vero *ZT* aequum peripheriae (!)

circuli *DG*. Item de *ZH* (quae maior est ipsa *BG*, quoniam *AG* maior eadem) abscindatur ipsi *BG* aequalis *ZN*, et connectatur *NT*. Per 4ᵃᵐ libelli de dimensione circuli trigonum *ZNT* aequum circulo *DG*, trigonum vero *ZHT* per praemissam aequum curvae superficiei coni *DAG*. Sed per
15 primam 6ⁱ et coniunctam proportionem, sicuti *ZH* ad *ZN*, sic trigonum *ZTH* ad trigonum *ZTN*. Ergo superficies curva coni *ADG* ad circulum *DG*, sicut *ZH* ad *ZN*, et ideo sicut *AG* ad *BG*, quod erat demonstrandum.

Vel sic per corollarium quartae de dimensione circuli, area circuli *DG* aequalis est ei quod fit ex semidiametro *BG* in semiperipheriam
20 *DG*. Per corollarium praemissae, conica superficies *ADG* aequalis est ei quod fit ex *AG* latere in semiperipheriam *DG*. Ergo adhuc per primam 6ⁱ sicut *AG* ad *BG*, sic conica superficies *ADG* ad circulum *DG*, quod est propositum.

Propositio IV.

CYLINDRI CURVA SUPERFICIES AEQUALIS EST REC-
TANGULO SUB LATERIBUS CONTENTO QUORUM UNUM AXI
CYLINDRICO, RELIQUUM VERO PERIPHERIAE BASIS EST
5 AEQUALE.

Sit parallelogrammum rectangulum *ABGD*, quo semel revoluto circa latus *AB* fixum, describatur cylindrus axem habens *AB*, basim vero circulum *GE* [Fig. III.5C.4]. Sitque rectangulum *ZHT*, cuius latus quidem *ZH* axi *AB*, latus vero *HT* peripheriae circuli *EG* sit aequale.
10 Aio quod cylindrica superficies quam describit latus *GD* aequalis est rectangulo *ZT*.

Nam si rectangulum *ZHT* non sit aequum cylindricae superficiei quam describit linea *DG*, erit aequum cylindricae superficiei alicuius cylindri
44 ha/bentis basim maiorem, minoremve circulo *EG* circa eundem axem *AB*.
15 Sit ergo primum rectangulum *ZHT* aequum curvae superficiei cylindri cuius basis sit circulus *KL* minor circulo *EG*, et axis idem *AB*, qui scilicet cylindrus a rectangulo *ABLM* circa axem *AB* circumducto describitur, et cuius curva superficies describitur a linea *LM*. Et per 13ᵃᵐ 12ⁱ inscribitur circulo *EG* polygonium *EG* laterum aequalium minime
20 contingentium circulum *KL*, super quod polygonium erigatur prisma eandem cum cylindro habens celsitudinem, ductis lateribus cylindri super angulos polygonii perpendicularibus. Eruntque parallelogrammata pris-matis quorum bases sunt latera polygonii simul aequalia per 36ᵃᵐ primi vel per primam 6ⁱ rectangulo quod fit ex axe *AB* in perimetrum
25 polygonii *EG*. Sed hoc rectangulum minus est rectangulo *ZT*, quod fit ex axe *AB* in peripheriam circuli *EG*, quae maior est perimetro polygonii. Rectangulum autem *ZT* aequum fuit superficiei cylindricae quam describit linea *LM*. Igitur parallelogrammum prismatis cuius basis polygonium *EG* minus est quam superficies cylindrica quam describit
30 linea *LM*; superficies, inquam, claudens minor est inclusa, quod est impossibile.

Vel sic quoniam rectangulum ZT maius est rectangulis prismatis praedictis, rectangula vero prismatis maiora quam superficies cylindrica quam describit linea LM, quandoquidem includens inclusa maior, ideo
35 rectangulum ZT maius superficie cylindrica quam describit linea LM. Quare non ei aequale sicut proponebatur.

Esto nunc rectangulum ZHT aequum, si possibile est, cylindricae superficiei cylindri habentis basim maiorem circulo EG et axem AB. Sitque brevitatis causa, cylindrus suppositus cuius axis AB et basis circulus KL,
40 qui scilicet describitur a rectangulo $ABLM$ circumducto semel circa axem AB, et rectanguli ZT latus ZH aequum axi AB, et latus HT aequum peripheriae circuli KL.

Aio iam quod non est possibile rectangulum ZT esse aequum superficiei curvae alicuius cylindri circum axem AB descripti, et habentis basim
45 maiorem circulo KL.

Nam, si possibile est, sit rectangulum ZT aequum superficiei curvae cylindri cuius basis circulus EG maior circulo KL, qui scilicet a rectangulo $ABGD$ circum axem AB revoluto, et cuius superficies curva a linea GD describitur. Et inscribatur ut prius circulo EG polygonium aequalium
50 laterum non tangentium circulum KL, et super polygonium prisma inclusum cylindricae superficiei quam describit linea GD. Eruntque prismatis rectangula quorum bases sunt latera polygonii simul aequalia per 36^{am} primi vel primam 6^i rectangulo quod fit ex axe AB in perimetrum polygonii EG. Hoc itaque rectangulum maius est rectangulo ZT, quoniam
55 illius unum latus aequale uni lateri huius, et reliquum reliquo maius (maius enim perimeter polygonii EG quam peripheria KL). Sed rectangulum ZT aequum fuit superficiei cylindricae quam describit linea GD. Ergo dicta prismatis rectangula simul maiora sunt superficie cylindrica quam describit linea GD. Superficies itaque inclusa prismatis maior super-
60 ficie cylindrica claudente, quod est impossibile: Non est igitur rectangulum ZT aequum alicui curvae superficiei cylindri circum axem AB cuius basis sit maior circulo KL.

45 / Similiter ergo si rectanguli ZT latus ZH axi AB, et latus HT peripheriae circuli EG sit aequale, non erit rectangulum ZT aequum superficiei curvae
65 cylindri cuiuspiam circum axem AB cuius basis sit maior circulo EG. Fuitque ostensum quod nec idem rectangulum aequum est superficiei curvae cylindri cuiuspiam circum axem AB cuius basis sit minor circulo EG. Superest ergo ut idem rectangulum ZT aequale sit superficiei curvae cylindri cuius axis AB basisque circulus EG, quam scilicet superficiem
70 describit linea GD, quod est propositum.

Corollarium I.

Manifestum est ergo quod cylindrica superficies producitur ex axe cylindrico in peripheriam basis.

Corollarium II.

75   Rursum ex ductu diametri cylindricae basis in peripheriam circuli cuius diameter est cylindricus axis producitur cylindrica superficies, quod sequitur ex tertio corollario primae (! sextae).

Demum circulus cuius semidiameter est media proportionalis inter latus seu axem cylindricum ac diametrum cylindricae basis superficiei
80   cylindricae aequalis est, quod sequitur ex postremo corollario primae (! sextae).

Propositio V.

CYLINDRICA SUPERFICIES AD CONICAM EUNDEM AXEM EANDEMQUE BASIM HABENTEM EST SICUT AXIS AD DIMIDIUM CONICI LATERIS; CYLINDRICA QUOQUE SUPERFICIES AD
5   BASIM EST SICUT AXIS AD DIMIDIUM SEMIDIAMETRI.

Circumducto rectangulo $ABGD$ circum axem AB semel describatur cylindrus, et circum eundem axem revoluto triangulo ABG fiat conus, eritque tam cylindri quam coni basis circulus quem describit linea BG, qui sit EG [Fig. III.5C.5].
10   Aio quod cylindrica superficies quam describit linea DG ad conicam superficiem quam describit linea AG est sic axis AB ad dimidium lineae AG, ad circulum vero EG sicut axis AB ad dimidium lineae BG.

Nam per praecedentem cylindrica superficies quam describit linea GD aequalis est ei quod fit ex AB in peripheriam circuli EG. Per 2^{am} autem
15   huius conica superficies quam describit linea AG aequalis est ei quod fit ex AG in dimidium peripheriae EG; ergo et ei quod fit ex dimidio ipsius AG in totam peripheriam EG. Per corollarium vero quartae de circuli dimensione, circulus EG aequalis est ei quod fit ex BG in dimidium peripheriae EG; ergo et ei quod ex dimidio ipsius BG in totam peripheriam
20   EG. Igitur cum horum trium rectangulorum una sit altitudo, quae est aequalis peripheriae circuli EG, iam per primam 6^i erunt ad invicem sicut bases. Itaque cylindrica superficies quam describit linea DG ad conicam quam describit linea AG erit sicut DG ad dimidium AG; ad circulum vero EG sicut linea AB ad dimidium ipsius BG, quod est propositum.

46   /Propositio VI.

CIRCULORUM PERIPHERIAE SUNT DIAMETRIS PROPOR-TIONALES

[Demonstratio huius propositionis habetur in 8va praeambuli Maurolyci.]
5   Sunto enim duo circuli AB, GD, quorum diametri AB, GD [Fig. III.5C.6].

Aio quod diameter AB ad diametrum GD, sicut peripheria AB ad peripheriam GD.

Sit enim sicut diameter *AB* ad diametrum *GD*, sic iam peripheria
10 *AB* ad peripheriam *EZ*. Et erit peripheria *EZ* aequalis peripheriae *GD*,
secus enim erit minor aut maior. Si minor, sit ipsi *GD* concentrica, et
per 13^am 12^i inscribatur peripheriae *GD* polygonium aequilaterum quod
peripheriam *EZ* non tangat; et aliud ipsi simile intra peripheriam *AB*.
Eritque, propter figurarum similitudinem, sicut diameter *AB* ad diametrum
15 *GD*, sic perimeter figurae *AB* ad perimetrum figurae *GD*. Quare sicut
perimeter figurae *AB* ad perimetrum figurae *GD*, sic erit peripheria *AB*
ad peripheriam *EZ*. Maior autem perimeter figurae *GD* quam peripheria
EZ, nimirum continens contento. Ergo per 14^am 5^i maior erit perimeter
figurae *AB* quam peripheria *AB*: comprehensum comprehendente, quod est
20 imposibile (*!*). Si autem peripheria *EZ* sit maior quam peripheria *GD*, tunc
conversim erit peripheria *EZ* ad peripheriam *AB*, sicut diameter *GD* ad
diametrum *AB*, et sic sit iam peripheria *GD* ad peripheriam *TH*; eritque
per 14^am, 5^i minor peripheria *TH* quam peripheria *AB*. Itaque erit sicut
diameter *GD* ad diametrum *AB*, sic nunc peripheria *GD* ad peripheriam
25 *TH* minorem peripheria ipsius diametri *AB*, quod ducit ad primum im-
possibile. Quamobrem non erit peripheria *EZ* maior quam pripheria (*!*)
GD. Fuitque ostensum quod nec minor. Omnino igitur erit ei aequalis; fuit
autem peripheria *AB* ad peripheriam *EZ*, sicut diameter *AB* ad diametrum
GD. Ergo et peripheria *AB* ad peripheriam *GD* erit item sicut diameter
30 *AB* ad diametrum *GD*, quod erat demonstrandum.

Manifestum est ergo quod sicut est diameter propositi circuli ad ag-
gregatum ex diametris quorumcumque circulorum, sic est peripheria pro-
positi circuli ad aggregatum ex peripheriis omnium illorum circulorum,
hoc enim sequitur ex praemissa, et 13^a 5^i.
35 Quare si diameter propositi circuli sit aequalis aggregato ex diametris
quotcumque circulorum, et peripheria propositi circuli aequalis erit ag-
gregato ex peripheriis omnium illorum circulorum.

Item quod fit ex diametro primi circuli in peripheriam secundi aequale
est ei quod fit ex diametro secundi in peripheriam primi; idemque de
40 semidiametris ac peripheriis dicendum. Id enim sequitur ex praemissa, et
15^a 6^i.

Denique si trium circulorum diametri sint continue proportionales, tunc
quoniam et eorum peripheriae sunt in eadem ratione proportionales,
44 estque ob id, sicut semidiametrum primi ad semidiametrum secundi, sic
47 semiperipheria secundi ad semiperi/pheriam tertii. Propterea per 15^am 6^i
erit quod fit ex [semi]diametro primi in semiperipheriam tertii, aequale ei
quod fit ex semidiametro secundi in semiperipheriam ipsius secundi; et
perinde aequale ipsi circulo secundo, quin ex semidiametro in semi-
peripheriam producitur area circuli.

50   Alia demonstratio eiusdem sextae.

Sunto duo circuli *AB*, *GD*, quorum diametri *AB*, *GD*.
Aio quod peripheria *AB* ad peripheriam *GD* est sicut diameter *AB* ad
diametrum *GD*.

Nam si sic non fuerit, erit sicut diameter *AB ad* diametrum *GD*, sic
55   peripheria *AB* ad aliquam peripheriam minorem, maioremve peripheria
GD. Sit ergo primum sicut diameter *AB* ad diametrum *GD*, sic peripheria
AB ad peripheriam *EZ* minorem peripheria *GD*, et ei concentricam. Et per
13^{am} 12^i inscribatur circulo *GD* poligonium aequalium laterum non tan-
gentium circulum *EZ*, quod sit polygonium *GD*; et simile ipsi circulo *AB*
60   inscribatur. Eritque, propter polygoniorum similitudinem, perimeter poly-
gonii *AB* ad perimetrum polygonii *GD*, sicut diameter *AB* ad diametrum
GD, et ideo sicut peripheria circuli *GB* (*! AB*) ad peripheriam circuli
EZ. Et permutatim, erit sicut peripheria circuli *AB* ad perimetrum poly-
gonii *AB*, sic peripheria circuli *EZ* ad perimetrum polygonii *GD*. Sed
65   peripheria circuli *AB* maior est perimetro polygonii *AB*; ergo peripheria
circuli *EZ* maior est perimetro polygonii *GD*: inclusa includente, quod
est impossibile.

Sit deinde sicut diameter *AB* ad diametrum *GD*, sic peripheria *AB* ad
peripheriam *HT* maiorem peripheria *GD* eritque conversim, sicut diameter
70   *GD* ad diametrum *AB*, sicut peripheria *HT* ad peripheriam *AB*, et sic sit
peripheria *GD* ad quampiam peripheriam *EZ*; eritque permutatim, sicut
peripheria *HT* ad peripheriam *GD*, sic peripheria *AB* ad peripheriam *EZ*.
Maior autem peripheria *HT* quam peripheria *GD*. Ergo et maior peripheria
AB quam peripheria *EZ*. Quare fiet sicut diameter *GD* ad diametrum
75   *AB*, sic peripheria *GD* ad peripheriam *EZ* minorem peripheria *AB*. Unde
sequitur idem impossibile quod prius. Itaque ratio diametri *AB* ad
diametrum *GD* non est sicut peripheria *AB* ad peripheriam aliquam
maiorem, minoremve peripheria *GD*. Erit ergo sicut ipsa peripheria *AB*
ad peripheriam *GD*, quod erat demonstrandum, quamquam idem in quinta
80   libelli de dimensione circuli abstractive demonstratur.

Corollarium.

Manifestum est ergo quod sicut est diameter dati circuli ad aggregatum
diametrorum quotcunque circulorum, sic peripheria dati circuli ad ag-
gregatum ex peripheriis omnium illorum circulorum. Hoc enim patet ex
85   praemissa, et 13^a 5^i. Quapropter, si dimeter (*!*) sit aequalis aggregato
diametrorum, peripheria erit aequalis aggregato peripheriarum.

Corollarium.

Item quod fit ex diametro circuli in peripheriam alterius circuli,
aequum ei quod ex diametro huius in peripherium illius. Hoc patet ex
90   praesenti, et 15^a 6^i.

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/Propositio VII.

CONI-COLURI CURVA SUPERFICIES AEQUALIS EST EI QUOD
FIT EX LATERE IPSIUS CONI IN DIMIDIAS BASIUM PERIPHERIAS.
Sit trigonum *ABG* rectum qui apud *B* angulum habens, et *DE* ipsi *BG*
5   parallelus [Fig. III.5C.7]; et circumducto triangulo *ABG* semel circum

axem *AB*, sit a trigono *ABG* descriptus conus *AGZ* basim habens circulum *GZ*, qui vero a trigono *ADE* conus describitur, sit *AHE* basim habens circulum *HE*. Horum autem conorum differentia, quae scilicet describitur a trapezio *DBGE*, vocatur colurus-conus; habetque duas bases in-

10   aequales circulos scilicet *GZ*, *EH*.

Aio itaque quod curva superficies coni-coluri, quae scilicet describitur a linea *EG*, aequalis est ei quod fit ex ipsa linea *EG* in dimidias peripherias circulorum *GZ*, *EH*.

Exponatur enim trigonum *TKL* rectum qui apud *K* angulum habens,

15   cuius latus *TK* sit ipsi *AG* hypotenusae, latus vero *KL* ipsi *GZ* peripheriae aequale, eritque per 2ᵃᵐ huius coni *AGZ* curva superficies aequalis trigono *TKL*. Ponatur ipsi *GE* aequalis *KM*, et ducatur ipsi *KL* parellelus *MN*, eritque, propter triangulorum similitudinem, sicut *TK* ad *TM*, et ideo sicut *GA* ad *AE*, et ideo sicut semidiametrum *BG* ad semidiametrum *DE*, sic *KL* ad

20   *MN*. Sed per praemissam, sicut *BG* semidiameter ad *DE* semidiametrum, sic *GZ* peripheria ad *EH* peripheriam. Igitur sicut *GZ* peripheria ad *EH* peripheriam, sic *KL* ad *MN*; et permutatim, sicut peripheria *GZ* ad *KL*, sic peripheria *EH* ad *MN*; aequalis autem est peripheria *GZ* lineae *KL*. Ergo et peripheria *HE* aequalis lineae *MN*. Et quoniam *TM* aequalis ipsi

25   *AE*, ideo per 2ᵃᵐ huius superficies coni *AHE* aequalis est trigono *TMN*; fuit autem curva superficies coni *AZG* aequalis trigono *TKL*. Ergo curva superficies coni-coluri *EZ* aequalis erit trapezio *KMNL*.

Connectantur itaque *KN*, *ML*, eruntque per 7ᵃᵐ primi triangula *KML*, *KNL* aequalia invicem. Sed quod fit ex *KM* in *KL* est duplum trianguli

30   *KML*, ergo et duplum trianguli *KNL*. Quod autem fit ex *KM* in *MN* duplum est trianguli *KMN*. Igitur quod fit ex *KM* in ipsas *KL*, *MN* duplum est totius trapezii *KMNL*. Itaque quod fit ex *KM* in dimidium ipsarum *KL*, *MN* aequum est trapezio *KMNL*. Verum *KM* fuit aequalis ipsi *EG*, ipsa autem *KL* aequalis peripheriae *ZG*, ipsa vero *MN* aequalis peripheriae

35   *EH*. Ergo quod fit ex *EG* in dimidias peripherias circulorum *GZ*, *EH* aequum est trapezio *KMNL*, et ideo curvae superficiei coni-coluri *EZ* cui fuit trapezium aequale, et hoc erat demonstrandum.

Corollaria.

Itaque quod fit ex aggregato semidiametrorum basium coni-coluri in

40   peripheriam cuius diameter est latus conicum aequale est superficiei conicae, quod sequitur ex 2° et 3° corollario secundae.

49   /Adhuc et circulus cuius semidiameter est media proportionalis inter latus coni-coluri et aggregatum ex semidiametris basium aequalis est conicae superficiei, quod sequitur ex postremo corollario primae (! sextae)

Propositio VIII.

SI IN CIRCULO DESCRIPTI POLYGONII AEQUILATERI DI-
MIDIUM, AD TERMINOS DIAMETRI TERMINATUM, DIAMETRO
STANTE MOVEATUR PERFECTA REVOLUTIONE, DESCRIPTI

5 SOLIDI CONICAE SUPERFICIES CONIUNCTAE AEQUALES
ERUNT EI QUOD FIT EX DUCTU LATERIS POLIGONII IN
OMNES PERIPHERIAS CIRCULORUM AB ANGULIS POLYGONII
DESCRIPTORUM.

Intra circulum *AB*, cuius diameter *AB*, describatur polygonium
10 aequalium laterum [Fig. III.5C.8], ut puta decagonum *AGDEZBHTKL*,
et ducantur *GL*, *DK*, *ET*, *ZH*, quae etiam ad rectos angulos secabunt
diametrum *AB* in punctis *M*, *N*, *X*, *O*; circumvolvatur autem dimidium
polygonii utrumlibet, ut puta dimidium *ADB*, stante diametro *AB* completa
revolutione, qua circumductione ipsum polygonii dimidium describet
15 quodam solidum tornatile *ADB* compositum ex diversis solidis, scilicet
ex conis quos describunt triangula *AGM*, *BZO*, ex conis-coluris quos de-
scribunt trapezia *MGDN*, *OZEX*, et ex cylindro quem describit rec-
tangulum *NDEX*; describunt, inquam, in ipsa semipolygonii revolutione.
Si numerus laterum ipsius semipolygonii sit par, tunc ad compositionem
20 descripti solidi non interveniet cylindrus, non enim erit latus medium,
quod parallelum sit diametro, sicut hic *ED* parallelus est ipsi *AB*, et in
revolutione describat cylindrum.

Aio itaque quod solidi tornatilis *AB* conicae superficies, hoc est tota
solidi superficies, aequalis est ei quod fit ex ductu lateris *AG* in peripherias
25 circulorum quos describunt lineae *MG*, *ND*, *XE*, *OZ*.

Nam per 2[am] huius conica superficies quam describit linea *AG* aequalis
est ei quod fit ex *AG* in dimidium peripheriae descriptae ab angulo
G, et per eandem conica superficies quam describit linea *BZ* aequalis est
ei quod fit ex *BZ*, vel *AG*, in dimidium peripheriae descriptae ab angulo *Z*.
30 Item per praecedentem curva superficies coni-coluri quam describit linea
GD aequalis est ei quod fit ex ductu lateris *GD*, vel *AG* in dimidias
peripherias descriptas ab angulis *G*, *D*; et per eandem superficies coni-
coluri quam describit linea *EZ* aequalis est ei quod fit ex ductu lateris
EZ, vel *AG*, in dimidias peripherias descriptas ab angulis *Z*, *E*. Adhuc
35 cylindrica superficies quam describit linea *DE* per primam (! quartam)
huius aequalis est ei quod fit ex ductu lateris *DE* in peripheriam
descriptam ab angulo *D*, vel angulo *E*, cum uterque describat peripheriam
cylindricae basis. Et ideo dicta cylindrica superficies aequalis est ei quod
fit ex ductu lateris *DE* vel *AG* in dimidias peripherias descriptas ab angulis
40 *E*, *D*. Itaque in hos ductus concurrunt omnes et integrae peripheriae
descriptae ab angulis *G*, *D*, *E*, *Z*. Quare per primam 2[i] elementorum omnes
conicae superficies descriptae a semipolygonio *ADB*, quae tota est a semi-
poligonio descripti solidi superficies, aequalis est ei quod fit ex ductu
lateris *AG* in peripherias omnes et integras descriptas ab angulis *G*, *D*,
45 *E*, *Z*.

/Hoc idem ostendam de solido descripto a semipolygonio cuius laterum
numerus erit par. Sed quoniam ibi non contingit aliquis cylindrus, non
opus erit citare 4[am] huius sed solum 2[am] propter conos qui semper fiunt
apud extrema diametri stantis, et per praecedentem, propter conos-coluros
50 intermedios. Verum ergo, quod proponitur.

Propositio IX.

SI IN CIRCULO DESCRIPTI POLYGONII AEQUILATERI DIMIDIUM AD TERMINOS DIAMETRI TERMINATUM DIAMETRO STANTE, DONEC AD LOCUM SUUM REDEAT, CIRCUM-
5 DUCATUR, DESCRIPTI SOLIDI TOTA SUPERFICIES AEQUALIS ERIT EI QUOD FIT EX DUCTU PERIPHERIAE CONTINENTIS POLYGONIUM IN LINEAM QUAE CUM DIAMETRO EIUSDEM CIRCULI ET LATERE POLYGONII IN IPSO CIRCULO CON- STITUIT TRIGONUM RECTANGULUM.

10   Intra circulum AB, cuius diameter AB, sit descriptum aequilaterum polygonium [Fig. III.5C.9], quodvis ut puta dodecagonum, cuius anguli sint apud puncta $A, G, D, E, Z, H, B, T, K, L, M, N$, et ducantur lineae GN, DM, EL, ZK, HT secantes orthogonaliter diametrum AB apud puncta X, O, P, R, S. Item lineae DN, EM, ZL, HK, secantes dictum (!)
15 diametrum apud puncta Y, F, C, Q. Item recta BG faciens cum GA latus rectum angulum per 29^{am} 3^{ii} et circumducto semipolygonio AEB super axem AB completo ambitu describatur solidum tornatile conicarum superficierum, quemadmodum in praecedenti.

Aio quod huiusmodi solidi tota superficies, quae scilicet aggregatum est
20 conicarum superficierum a lateribus semipolygonii descriptarum, aequalis est ei quod fit ex ductu lineae BG in peripheriam circuli AB.

Nam per praecedentem quod fit ex AG in omnes peripherias descriptas ab angulis G, D, E, Z, H aequum est toti superficiei solidi a semipoly- gonio descripti, quam demonstrationem, si placet, sic repete.

25

Superficies conica quam describit latus	AG aequale est ei quod fit ex latere AG in dimidium peripheriae cuius diameter			GN
	GD .... AG .... GN			DM
	DE .... AG .... DM			EL
	EZ .... AG .... EL			ZK
	ZH .... AG .... ZK			HT
	HB .... AG .... HT			

30

Et hoc per 7^{am} praemissam et 2^{am}. Quare per primam 2^{i} Elementorum tota superficies solidi a semipolygonio descripti aequalis est ei quod fit ex latere AG in omnes peripherias, quarum diametri GN, DM, EL, ZK,
51 HT. Sed triangula $GAX, NGX, DQO, MCO, / ECP, LFP, ZFR, KRY$
35 HXS (! HYS), TBS sunt similia triangulo BGA quoniam scilicet sunt orthogonia, et anguli assumentes aequales arcus sunt aequales per 26^{am} 3^{ii}; quare et reliqui aequales.

Itaque per 4^{am} sexti: sicut BG ad GA, sic

$$\begin{array}{l} GX - \text{ad} - XA \\ XN \ldots\ldots XQ \\ DO \ldots\ldots QO \\ OM \ldots\ldots OC \\ EP \ldots\ldots CP \\ PL \ldots\ldots PF \\ ZR \ldots\ldots FR \\ RK \ldots\ldots RY \\ HS \ldots\ldots YS \\ ST \ldots\ldots SB. \end{array}$$

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45

Quare per 13^{am} 5^i sicut *BG* ad *GA*, sic aggregatum *GN*, *DM*, *EL*, *ZK*, *HT* ad totam *AB*, et ideo per corollarium 6^{ae} huius, sicut *BG* ad *GA*, sic peripheria cuius diameter est aggregatum ex *GN*, *DM*, *EL*, *ZK*, *HT* ad peripheriam cuius diameter *AB*. Sed peripheria cuius diameter est aggregatum ex *GN*, *DM*, *EL*, *ZK*, *HT* per corollarium 6^{ae} huius aequalis est aggregato periperiarum (*!*) quarum diameteri *GN*, *DM*, *EL*, *ZK*, *HT*. Igitur sicut *BG* ad *GA*, sic aggregatum peripheriarum quarum diametri *GN*, *DM*, *EL*, *ZK*, *HT* ad peripheriam cuius diameter *AB*. Quare per 15^{am} sexti Elementorum, quod fit ex *BG* in peripheriam cuius diameter *AB* aequum est ei quod fit ex *GA* in aggregatum peripheriarum quarum diameteri (*!*) *GN*, *DM*, *EL*, *ZK*, *HT*. Sed per praecedentem, quod fit ex *GA* in aggregatum peripheriarum quarum diametri *GN*, *DM*, *EL*, *ZK*, *HT* aequale est superficiei solidi a semipolygonio *AEB* descripti. Ergo quod fit ex *BG* in peripheriam cuius diameter *AB* aequale est superficiei solidi a semipolygonio *AEB* descripti; et hoc erat demonstrandum.

Corollarium I.

Unde et quod fit ex diametro *AB* in peripheriam circuli cuius diameter est *BG* aequale erit eiusdem solidi tornatilis superficiei, quod quidem sequitur ex praesenti, et ex tertio corollario primae (*!* sextae). Item circulus cuius semidiameter est media proportionalis inter ipsas lineas *AB*, *BG* aequalis est superficiei memorati solidi tornatilis, quod sequitur ex ultimo corollario primae (*!* sextae).

Corollarium II.

Hinc manifestum est quod superficies solidi a semipolygonio *AEB* descripti aequalis est cylindricae superficiei cuius axis est linea *BG* et basis circulus *AB*; namque per 4^{am} huius cylindrica superficies aequalis est ei quod ex axe in peripheriam basis.

/ Corollarium III.

Item quod fit ex perpendiculari a centro circuli ad latus polygonii in peripheriam circuli continentis polygonium dimidium est totius superficiei solidi a semipolygonio super diametrum semel revoluto descripti. Nam perpendicularis quae a centro *P* ad latus *AG* dimidium est ipsius *BG*, sicut *AP* dimidium ipsius *AB*. Itaque quod ex dicta perpendiculari in peripheriam *AB* dimidium est eius quod ex *BG* in peripheriam *AB*, et ideo dimidium superficiei praedicti solidi a semipolygonio descripti.

Propositio X.

SPHAERAE SUPERFICIES AEQUALIS EST RECTANGULO QUOD FIT EX DIAMETRO SPHERAE (*!*) IN PERIPHERIAM MAXIMI SUI CIRCULI.

Sit circulus *ABGD*, cuius centrum *E*, diameter *AG*, et circumducto semel

altero semicirculorum, ut puta semicirculo ABG, stante diametro AG, describatur sphaera ABG [Fig. III.5C.10].

Aio quod superficies sphaerae ABG aequalis est ei quod fit ex diametro AG in peripheriam circuli ABG.

10 · Nam si quod fit ex AG in peripheriam circuli ABG non est aequale superficiei sphaerae ABG, erit aequale superficiei sphaerae alicuius maioris vel minoris sphaera ABG. Sit ergo primum aequale superficiei sphaerae ZHT minoris ipsa sphaera ABG et cum ea concentricae, cuius diameter ZT, et quam circa diametrum ZT revolutus describit semicirculus ZHT. Et per 13^{am} 12^{i} inscribatur circulo ABG polygonium
15 aequalium laterum minime tangentium circulum ZHT, quod sit polygonium $ABGD$, cuius laterum unum sit AK. Et connectatur GK, ac polygonii dimidio circa diametrum AG circumducto, describatur solidum tornatile conicarum superficierum iam minime tangentium sphaeram ZHT.

20 Eritque per praecedentem, quod fit ex GK in peripheriam ABG circuli aequum superficiei universae solidi ABG a semipolygonio descripti. Sed quod fit ex diametro AG in peripheriam circuli ABG maius est eo quod fit ex GK in peripheriam eandem. Ergo quod fit ex diametro AG in peripheriam circuli ABG, et ideo ipsa sphaerae ZHT superficies, maior
25 erit solidi a semipolygonio ABG descripti superficie: inclusa includente, quod est impossibile.

Vel sic, quoniam quod fit ex AG diametro in peripheriam ABG circuli maius est solidi a semipoligonio ABG descripti superficie, et haec maior superficie sphaerae ZHT inclusae, ideo quod fit ex AG diametro in
30 peripheriam ABG circuli maius est ipsa sphaerae ZHT superficie; non est ergo ei aequalis sicut supponebatur.

Sit deinde quod fit ex AG in peripheriam circuli ABG aequale superficiei sphaerae maioris ipsa sphaera ABG, sed, brevitatis causa, sit proposita sphaera ZHT.
35 Aio quod id quod fit ex diametro ZT in peripheriam circuli ZHT non erit aequale superficiei alicuius sphaerae maioris sphaera ZHT.

Nam, si possibile est, sit aequale superficiei sphaerae ABG ipsi ZHT concentricae, et quam circa diametrum AG describit semicirculus ABG, et inscribatur, ut prius, circulo ABG polygonium aequalium laterum non
40 tangentium circulum ZHT, cuius unum laterum fit AK, et ducetur GK, et EL perpendicularis ad AK. Et circumducto semipolygonio ABG super diametrum AG describatur solidum tornatile conicarum superficierum
53 / iam minime tangentium sphaeram ZHT. Est autem GK dupla ipsius EL, sicut AG dupla ipsius AE. Sed ZT dupla ipsius ZE minor ipsa EL. Ergo
45 GK maior quam ZT. Quare quod fit ex GK in peripheriam circuli ABG maius est eo quod fit ex diametro ZT in peripheriam circuli ZHT. Verum quod fit ex GK in peripheriam circuli ABG, per praecedentem, aequale est superficiei solidi a semipolygonio ABG descripti. Quod autem fit ex diametro ZT in peripheriam circuli ZHT aequum est (per hypothesim)
50 superficiei sphaerae ABG. Igitur superficies solidi a semipoligonio ABG descripti maior est superficie sphaerae ABG: inclusa includente, quod est

impossibile. Non est ergo quod fit ex *ZT* diametrum (*!* diametro) in peripheriam *ZHT* aequale superficiei alicuius sphaerae maioris sphaera *ZHT*. Similiter ostendam quod id quod sit ex *AG* diametrum (*!* diametro)
55 in peripheriam circuli *ABG* non est aequale superficiei sphaerae alicuius maioris sphaera *ABG*; sed nec minoris, ut fuit ostensum. Superest ergo ut id quod fit ex diametro *AG* in peripheriam *ABG* aequale sit superficiei sphaerae *GBG* (*!* *ABG*), quod erat demonstrandum.

Corollarium.

60 Manifestum est ergo quod sphaerae superficies aequalis est circulo cuius semidiameter aequalis est diametro sphaerae. Nam area talis circuli per 4^{am} de dimensione circuli producitur ex suo semidiametro, quae est diameter sphaerae, in dimidium suae peripheriae, quae per sextam huius aequalis est peripheriae circuli maximi in sphaera.

Propositio XI.

SPHAERAE SUPERFICIES QUADRUPLA EST AD SUUM MAX-
IMUM CIRCULUM, ESTQUE AEQUALIS CURVAE SUPERFICIEI
EIUS CYLINDRI CUIUS TAM AXIS QUAM BASIS DIAMETER
5 AEQUALIS EST SPHAERAE DIAMETRO.

Esto sphaera *ABG*, quam describat semicirculus *ABG*, stante diametro *AG* circumductus, cuius centrum *D* [Fig. III.5C.11].

Aio quod superficies sphaerae *ABG* quadrupla est ad circulum *ABG*.

Nam per praemissam sphaerae *ABG* superficies aequalis est ei quod
10 fit ex diametro *AG* in peripheriam circuli *ABG*. Per 4^{am} vero libelli de dimensione circuli, area circuli aequalis est ei quod fit ex semidiametro *DG* in dimidium periphaeriae totius circuli *ABG*. Estque per 18^{am} sexti quod fit ex diametro *AG* in peripheriam circuli *ABG* quadruplum ad id quod ex semidiametro *DG* in peripheriam semicirculi *ABG* (quandoquidem
15 latera singulorum laterum dupla). Igitur sphaerae *ABG* superficies quadrupla est ad circulum *ABG* maximum in sphaera, quod est primum ex propositis. Reliquum sic ostendo.

Sit cylindrus cuius tam axis quam basis diameter sit aequalis diametro *AG*.
20 Aio quod spherae *ABG* superficies aequalis est curvae superficiei huiusmodi cylindri.

Nam talis cylindri basis erit circulus *ABG*; quare per 4^{am} huius curva superficies ipsius cylindri aequalis erit ei quod ex diametro *AG* in peripheriam circuli *ABG*. Sed hoc aequale est sphaericae superficiei. Ergo
25 sphaerica superficies aequalis erit curvae superficiei talis cylindri; quod supererat demonstrandum.

Corollarium.

Manifestum est ergo quod cylindri cuius tam axis quam basis diameter aequalis est sphaerae diametro tota superficies est ad sphaerae super-

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ficiem sesquial/tera. Namque duae bases cylindri sunt dimidium sphaer-
icae superficiei, et cum sphaerica superficies sit quadrupla ad unam il-
larum, et curva superficies cylindri aequalis sphaericae superficiei, igitur
tota cylindri superficies, quae constat ex curva superficie et basibus, con-
tinet sphaericam superficiem semel et insuper eius dimidium.

Propositio XII.

5

SI CIRCULO DUAE AEQUILATERAE ET AEQUIANGULAE
FIGURAE UNA INSCRIBATUR, ALTERA CIRCUMSCRIBATUR
CORRESPONDENTIBUS ANGULIS, ET DIAMETRO STANTE, TAM
SEMICIRCULUS, QUAM FIGURARUM DIMIDIA, DONEC AD
LOCUM SUUM REDEANT, CIRCUMVOLVANTUR, DESCRIPTAE A
SEMICIRCULO SPHAERAE SUPERFICIES MEDIA PROPOR-
TIONALIS INTER SOLIDORUM A DIMIDIIS FIGURARUM DE-
SCRIPTORUM SUPERFICIES.

10

Circulo ABG, cuius diameter AG centrumque D, inscribatur figura
aequilatera cuius laterum unum sit AB [Fig. III.5C.12], et secto arcu
AB bifariam in signo E, et ducta semidiametro DE secante chordam AB
in puncto Z bifariam, et orthogonaliter ducatur HET ad rectos ipsi DE.
Et ideo contingens circulum in puncto E per 15^{am} 3^{ii}, quae ipsis DA, DB

15

semidiametris productis occurrat ad signa H, T, eritque TH latus figurae
aequilaterae et aequiangulae circumscribentis circulum ABG et similis ipsi
figurae ABG inscriptae. Compleatur itaque figura circumscripta, sitque
THK. Et circumducatur semel tam semicurculus (!) ABG, quam semi-
polygonium ABG quamque semipolygonium THK super axem TK

20

stantem.
Aio itaque quod superficies sphaerae quam describit semicirculus ABG
media proportionalis est inter superficiem solidi quod describit semi-
polygonium ABG et inter superficiem solidi quod describit semipoly-
gonium THK, quod sic ostendo.

25

Cadat a signo E ad DB perpendicularis EL, eruntque triangula DEL,
DBZ invicem aequilatera; quare ipsae DZ, DL aequales. Per corollarium
autem 3^{um} nonae huius ex DZ, et ideo ex DL, in peripheriam ABG fit
dimidium superficiei solidi descripti a semipolygonio ABG. Per 10^{am} autem
huius ex DB semidiametro in peripheriam circuli ABG fit dimidium

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superficiei sphaerae descriptae a semicirculo ABG. Adhuc per corollarium
3^{um} nonae ex DE in peripheriam circuli THK circumscribentis poly-
gonum (!), et ideo ex DH in peripheriam circuli ABG, (sunt enim haec
duo aequalia per secundum corollarium 6^e huius) fit dimidium super-
ficiei descriptae a semipolygonio THK. Igitur haec tria producta, quae

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sunt dimidia dictarum superficierum, sunt per primam 6^i ad invicem sicut
lineae DL, DE, DH, quae sunt bases productorum, nam altitudo
est aequalis periphaeriae (!) circuli ABG. Sed DL, DE, DH sunt continuae
proportionales, propter similitudinem triangulorum DEL, DHE. Ergo tria
producta, quae sunt dimidia dictarum superficierum, sunt continue pro-

40 portionalia. Quare et earum dupla, totae scilicet superficies, sunt continuae proportionales. Itaque superficies solidi descripti a semipolygonio *ABG*, superficies sphaerae descriptc a semicirculo *ABG*, superficies solidi descripti a semipolygonio *THK* sunt continuae proportionales in proportione scilicet ipsarum *DL*, *DE*, *DH* linearum, quod est propositum.

55 /Scholium.

46 Quod si aequilaterae et aequiangulae figurae rectilineae duo circuli unus circumscribatur, aliter (! alter) inscribatur, et diametro manente tam semipoligonium, quam duo semicirculi circumducantur, descripti quoque a semipoligonio solidi superficies media proportionalis est inter sphaer-
50 arum a semicirculis descriptarum superficies.

Descriptioni praecedentis addatur circulus *THK* circumscriptus figurae rectilineae *THK*, et sphaera per revolutionem semicirculi *THK* descripta.

Aio iam quod solidi *THK* superficies media proportionalis est inter sphaerarum *ABG*, *THK* superficies.

55 Nam per 10^{am} ex recta *HD* in peripheriam circuli *THK* fit dimidium superficiei sphaerae *THK*, atque, ut in praemissa, ostensum est, ex recta *HD* in peripheriam circuli *ABG* fit dimidium superficiei solidi *THK*. Quare per primam sexti superficies sphaerae *THK* ad superficiem solidi *THK* erit sicut peripheria circuli *THK* ad peripheriam circuli *ABG*, et ideo per
60 sextam sicut semidiameter *DH* ad semidiametrum *DE*. Sed fuit in praemissa, sicut *DH* ad ipsam *DE*, sic superficies solidi *THK* ad superficiem sphaerae *ABG*. Ergo et sicut superficies solidi *THK* ad superficiem sphaerae *ABG* sic superficies sphaerae *THK* ad superficiem solidi *THK*. Itaque superficies solidi *THK* media proportionalis est inter
65 sphaerarum *THK*, *ABG* superficies; quod erat demonstrandum.

Corollarium.

Manifestum est ergo quod si circulo polygonium aequilaterum, et polygonio rursum circulus; et circulo adhuc polygonium respondentibus angulis: et ita deinceps, quotics lubet, inscribatur, et diametro manente
70 tam semicirculi quam semipolygonia circumducantur: descriptarum sphaerarum et tornatilium solidorum superficies sunt continuae proportionales secundum inscriptionis ordinem.

Hactenus de solidi tornatilis ac sphaerae superficie, nunc de segmentorum a dictis solidis abscissorum superficiebus verba faciemus.

Propositio XIII.

CONICAE SUPERFICIES SEGMENTI QUOD SUMITUR A VERTICE SOLIDI DESCRIPTI A SEMIPOLYGONIO AEQUI-
LATERO AD UNUM CIRCULORUM AB ANGULIS DESCRIP-
5 TORUM CONIUNCTAE SUNT AEQUALES EI QUOD FIT EX DUCTU LATERIS CIRCUMDUCTAE IN PERIPHERIAS DE-

SCRIPTAS AB ANGULIS MINUS DIMIDIO PERIPHAERIAE CIRCULI SEGMENTUM [DE] SOLIDO ABSCINDENTIS.

Intra circulum AB, cuius diameter AB, describatur polygonium
10 aequalium laterum AGB, cuius dimidio semel circa diametrum AB stantem circumducto describatur solidum tornatile, de quo sumatur segmentum a vertice A ad unum circulorum ab angulis descriptorum, ut puta ad circulum descriptum ab angulo E, quod sit segmentum AEZ [Fig. III.5C.13].

15   Aio quod conicae superficies segmenti AEZ, quae scilicet a lateribus AG, GD, DE describuntur, aequales sunt simul ei quod fit ex latere AG
56 in aggregatum ex peripheriis ab angulis G, D, E minus dimidio / peripheriae descriptae ab angulo E. Ducantur enim GT, DH, EZ, secantes diametrum AB apud K, L, N, secabunt autem ad angulos rectos. Itaque
20 per 2^{am} huius conica superficies quam describit linea AG aequalis est ei quod ex AG in dimidium peripheriae quam describit punctum G. Item per 7^{am} conica superficies coni-coluri quam describit linea GD aequalis est ei quod ex GD in dimidias peripherias descriptas a punctis G, D. Item per 7^{am} conica superficies quam describit linea DE, sive per 4^{am} si $LDEM$ sit
25 parallelogrammum, ac ideo superficies descripta cylindrica aequalis est ei quod ex DE, vel AG, in dimidias peripherias descriptas a punctis D, E. Huc ergo concurrunt integrae peripheriae descriptae a punctis G, D, et dimidium eius quae describitur a puncto ultimo E. Quare per primam 2^i Elementorum omnes conicae superficies descriptae a lineiis AG, GD, DE,
30 quae est superficies segmenti AEZ, aequalis est ei quod fit ex ductu lateris AG in omnes peripherias descriptas ab angulis, G, D, E minus dimidio periphaeriae descriptae ab angulo E infimo, quae periphaeria est circuli abscindentis segmentum AEZ de toto solido per semipolygonium descripto, quod erat demonstrandum.
35   Hoc idem ostenditur etiam si chorda AG non fuerit latus polygonii aequilateri circulo inscripti, dum arcus AG, GD, DE sint aequales.

Propositio XIV.

CONICAE SUPERFICIES SEGMENTI QUOD SUMITUR A VERTICE SOLIDI DESCRIPTI A SEMIPOLYGONIO AEQUILATERO AD UNUM CIRCULORUM AB ANGULIS DESCRIPTORUM
5 CONIUNCTAE SUNT AEQUALES EI QUOD FIT EX DUCTU PERIPHERIAE CIRCULI CUIUS DIAMETER EST AXIS IPSIUS SEGMENTI IN LINEAM QUAE CUM DIAMETRO CIRCULI CONTINENTIS POLYGONIUM ET LATERE POLYGONII IN IPSO CIRCULO CONSTITUIT TRIANGULUM ORTHOGONIUM.
10   Quid verbis opus erit? Assumo totam nonae descriptionem, sed de solido quod describit semipolygonium AEB assumo segmentum cuius vertex est A, basis vero aliquis circulorum ab angulis polygonii descriptorum, utpote circulus cuius diameter DM, quod segmentum vocetur DAM [Fig. III.5C.14].
15   Aio itaque quod conicae superficies segmenti DAM, quae scilicet a

lineis AG, GD describuntur, aequales simul sunt ei quod fit ex linea BG in peripheriam circuli cuius diameter AO, quem voco axem segmenti DAM.

Que demonstratio fere eadem est cum demonstratione nonae: nam, sicut ibi, triangula GAX, NQX, DQO similia sunt triangulo BGA, hoc est aequiangula. Quare per 4am sexti sicut BG ad GA, sic GX ad AX, sic etiam XN ad XQ, sic et DO ad QO. Et ideo per 13am 5i sicut BG ad GA, sic aggregatum ex GN, DO ad totam AO. Itaque per corollarium sextae huius sicut BG ad GA, sic peripheria cuius diameter / est aggregatum ex GN, DO ad peripheriam cuius diameter AO. Sed peripheria cuius diameter est aggregatum ex GN, DO per corollarium sextae aequalis est aggregato peripheriarum quarum diametri GN, DO. Igitur sicut BG ad GA, sic aggregatum peripheriarum quarum diametri GN, DO ad peripheriam cuius diameter est AO. Peripheria autem cuius diameter DO per corollarium sextae aequalis est dimidio peripheriae cuius diameter DM, cum diameter DO sit dimidium diametri DM. Ergo sicut BG ad GA, sic aggregatum ex peripheria cuius diameter GN et ex dimidio peripheriae cuius diameter DM ad peripheriam cuius diameter AO. Unde per 15am sexti elementorum quod fit ex BG in peripheriam cuius diameter AO aequale est ei quod fit ex GA in aggregatum ex peripheria cuius diameter GN et ex dimidio peripheriae cuius diameter DM. Sed per praemissam, quod fit ex GA in aggregatum ex peripheria cuius diameter GN et ex dimidio peripheriae cuius diameter DM aequale est conicis superficiebus segmenti solidi DAM. Igitur quod fit ex BG in peripheriam cuius diameter AO aequum est conicis superficiebus segmenti solidi DAM, quod erat demonstrandum.

Non aliter ostendam quod id quod fit ex BG linea in peripheriam cuius diameter AP aequale est conicis superficiebus segmenti solidi EAL. Nec secus concludam id quod fit ex linea BG in peripheriam cuius diameter AR aequale esse conicis superficiebus segmenti ZAK, quandoquidem segmenti EAL axis est ipsa AP, segmenti vero ZAK ipsa AR linea. Verum est ergo quod proponitur.

Hoc idem ostendetur etiam si chorda AG non sit latus polygonii aequilateri circulo AB inscripti, modo latera segmenti sint aequalia.

Corollaria.

Igitur et quod fit ex linea AO in peripheriam circuli cuius diameter BG aequale est conicis superficiebus DAM, quod sequitur ex corollario 3° 1e (! 6ae). Nec non circulus cuius semidiameter est media proportionalis inter lineas AO, BG aequalis est iisdem conicis superficiebus segmenti DAM, quod sequitur ex 4° corollario primae (! sextae).

Propositio XV.

SI SPHAERA PLANO SECETUR, SEGMENTI UTRIUSLIBET SPHAERICA SUPERFICIES AEQUALIS EST RECTANGULO QUOD

FIT EX DIAMETRO SPHAERAE IN PERIPHERIAM EIUS CIRCULI
5 CUIUS DIAMETER EST AXIS SEGMENTI.

Sit circulus *ABGD*, cuius diameter *AG*, centrum *E*, secetur autem diameter *AG* recta *BD* ad angulos rectos in signo *Z* quocumque, et circumducto altero semicirculo, ut puta *ABG*, semel super axem *AG* stantem, describatur sphaera *ABGD* [Fig. III.5C.15]. Unde in tali ambitu
10 linea *ZB* describit circulum qui secabit sphaeram in duo segmenta *BAD*, *BGD*.

Aio itaque quod sphaerica superficies utriuslibet segmenti, ut puta segmenti *BAD*, aequalis est ei quod fit ex diametro *AG* in peripheriam circuli cuius diameter est *AZ*, quem voco axem segmenti *BAD*, quae
15 demonstratio similis est demonstrationi decimae.

Nam si quod fit ex *AG* diametro in peripheriam circuli cuius diameter *AZ* non est aequale superficiei sphaericae segmenti *ABD*, sit aequalis
58 / superficiei sphaericae alicuius segmenti maiori vel minori superficie segmenti *BAD*. Et primum minori scilicet superficiei sphaericae segmenti
20 *THL* abscissi per planum dicti circuli de sphaera *HTKL*, quam sphaerae *ABG* concentricam describit semicirculus *HTK* super diametrum *HK*, et secentur arcus *AB*, *AD* iterum, atque iterum donec per 13^{am} 12^{i} arcuum chordae non contingant peripheriam *THL*. Sitque una chordarum *AM*, et connectatur *GM*. Eritque maius quod ex diametro *AG* in peripheriam
25 circuli cuius diameter *AZ* quam id quod ex *GM*, quae minor diametro, in peripheriam circuli cuius diameter *AZ*. Sed quod ex diametro *AG* in peripheriam circuli cuius diameter *AZ* per hypothesim aequale est superficiei segmenti sphaerici *THL*; quod autem ex *GM* in peripheriam circuli cuius diameter *AZ* per praecedentem, aequale est superficiebus
30 conicis descriptis a chordis arcuum in quos secatur arcus *AB* maior. Ergo superficies sphaerica segmenti *THL* maior est quam superficies conicae descriptae a chordis portionum arcus *AB*. Superficies ergo inclusa maior includente, quod est impossibile.

Vel sic, quoniam quod fit ex *AG* diametro in peripheriam circuli
35 cuius diameter *AZ* maius est eo quod ex *GM* in peripheriam eandem, et ideo maius superficiebus conicis descriptis per chordas portionum arcus *AB*, haeque superficies maiores sunt superficie sphaerica segmenti *THL* inclusa, ideo quod fit ex diametro *AG* in peripheriam circuli cuius diameter *AZ* maius est superficie sphaerica segmenti *THL*. Ergo non est ei aequalis,
40 sicut supponebatur.

Sit deinde quod ex *AG* diametro in peripheriam circuli cuius diameter *AZ* aequalis sphaericae superficiei maiori superficie sphaerica segmenti *BAD*. Sed brevitatis causa sit suppositum segmentum *THL*.

Aio quod id quod ex *HK* diametro in peripheriam circuli cuius diameter
45 *HZ* non est aequale alicuius segmenti sphaericae superficiei maiori superficie segmenti *THL* sphaerici.

Sit enim, si possibile est, aequale superficiei sphaericae segmenti *BAD* maiori superficie *THL* abscissi per planum circuli secantis sphaeram *BD* de sphera *ABGD*, quam describit semicirculus *ABG* super diametrum

50   *AG*. Et secentur arcus *AB*, *AD* per 13^am 12^i donec portionum chordae
non contingant arcum *THL*. Sitque ex chordis una *AM*, et connectatur
GM, et ducatur ad *AM* perpendicularis *EN*. Et quoniam *AG* dupla ipsius
AE, ideo et *GM* dupla ipsius *EN*, sed *HK* dupla ipsius *HE* minoris
ipsa *EN*; minor ergo *HK* quam *GM*. Quare quod fit ex *GM* in peripheriam

55   circuli cuius diameter *HZ* maius est eo quod fit ex *HK* in peripheriam
circuli cuius diameter *HZ*. Sed quod fit ex *GM* in peripheriam circuli cuius
diameter *AZ* per praecedentem aequale est superficiebus conicis de-
scriptis a chordis arcuum in quos secatur arcus *AB*. Quod autem fit ex
HK in peripheriam circuli cuius diameter *HZ* per hypothesim aequale est

60   superficiei sphaericae segmenti *BAD*. Ergo superficies conicae descriptae
a chordis portionum arcus *AB* maius sunt superficie sphaerica segmenti
BAD. Superficies itaque inclusa maior includente, quod est impossibile.
Non est ergo quod ex *HK* in peripheriam circuli cuius diameter *HZ* aequale
superficiei sphaericae segmenti alicuius maiori superficie sphaerica seg-

65   menti *THL*. Similiter ostendam quod id quod ex *AG* in peripheriam
circuli cuius diameter *AZ* non est aequale alicuius segmenti sphaericae
superficiei maiori superficie sphaerica segmenti *BAD*. Fuitque ostensum,
quod nec minori. Omnino igitur id quod fit ex *AG* in peripheriam
circuli cuius diameter *AZ* aequale est segmenti *BAD* sphaericae super-

70   ficiei, quod erat demonstrandum.

59   Suppo/suimus autem segmentum sphaericum *BAD* minus emispherio.
Et similiter ostendi poterit quod id quod fit ex *AG* in peripheriam
circuli cuius diameter *GZ* aequale est sphaericae superficiei segmenti
BGD. Vel si lubet pro linea *BZ* ducatur linea *BE*, et pro *ZD* linea

75   *ED*, secantes peripheriam circuli *HTK* apud puncta *X*, *O*. Et agatur demon-
stratio per segmenta sphaerica descripta per arcus *BG*, *XK* similes et per
superficies conicas descriptas a chordis portionum arcus *BG* non tangenti-
bus peripheriam *XK*. In secunda autem parte demonstrationis ducatur *XO*
secans ipsam *EZ* apud *P*, eritque *GZ* axis segmenti *BGD* maior quam

80   *KP* axis segmenti *XKO*, quod est necessarium in secunda parte demon-
strationis. Unde concludens eodem modo quo prius id quod fit ex *AG*
in peripheriam circuli cuius diameter *GZ* esse maius omni superficie
sphaerica segmenti descripti per arcum minorem ipso *BG*, et ei similem,
et minus omni sphaerica superficie segmenti descripti per arcum maiorem

85   ipso *BG*, et ei similem, et ideo aequalem esse sphaericae superficiei
segmenti ab arcu *BG* descripti.

         Verum postquam ostenderis hoc de uno segmentorum sphaericorum,
potest ex hoc idem facilius ostendi de reliquo sic.

         Quoniam ostensum est quod id quod fit ex *AG* in peripheriam circuli

90   cuius *AZ* est diameter aequum superficiei sphaericae segmenti *BAD*: ex
hoc ego ostendam quod id quod fit ex *AG* in peripheriam circuli cuius
diameter *GZ* aequum erit superficiei sphaericae segmenti *BGD*. Nam per
10^am huius superficies tota sphaerae *ABG*, quod est aggregatum ex super-
ficiebus sphaericis duorum segmentorum *BAD*, *BGD*, aequalis est ei quod

95   fit ex *AG* in peripheriam circuli *ABG*. Sed peripheria circuli *ABG* per

primum corollarium sextae aequalis est peripheriis circulorum quorum
diametri *AZ*, *ZG* coniunctis. Ergo aggregatum ex superficiebus sphaericis
segmentorum *BAD*, *BDG* aequale est ei quod fit ex *AG* in peripherias
circulorum quorum diametri *AZ*, *ZG*. Auferatur inde segmenti *BAD*
100 sphaerica superficies, hinc vero quod fit ex *AG* in peripheriam circuli
cuius diameter *AZ*, quae, ut fuit ostensum, sunt aequalia, et supererit
segmenti *BDG* superficies sphaerica aequalis ei quod fit ex *AG* in
peripheriam circuli cuius diameter *ZG*, et hoc erat propositum.

Corollarium I.

105   Manifestum est ergo quod superficies sphaerica segmenti sphaerici
aequalis est ei quod fit ex axe segmenti in peripheriam circuli maximi in
sphaera, et ideo curvae superficiei cylindri cuius basis est circulus max-
imus sphaerae, celsitudo vero axis segmenti. Hoc patet ex 2° (*!* 3°)
corollario sextae.

110   ## Corollarium II.

Item et circulus cuius semidiameter media proportionalis est inter
axem sphaerici segmenti ac sphaerae diametrum aequalis est sphaericae
superficiei ipsius segmenti, quod liquet ex quarto corollario primae
(*!* sextae). Ex quo corollario facillime demonstrabitur sequens propositio
115 decima sexta.

60   ## /Propositio XVI.

SI SPHAERA PLANO SECETUR, UTRIUSLIBET SEGMENTI
SUPERFICIES AEQUALIS EST EI QUOD FIT EX LINEA CADENTE
A VERTICE IPSIUS SEGMENTI AD PERIPHERIAM CIRCULI
5 SECANTIS IN PERIPHERIAM CIRCULI CUIUS DICTA LINEA EST
DIAMETER. ET IDEO IPSA SPHAERICA SUPERFICIES SEGMENTI
AEQUALIS EST CIRCULO CUIUS DICTA LINEA EST SEMI-
DIAMETER.

Circuli *ABGD* diametrum *AG* secet ad rectos angulos linea *BD* apud *Z*
10 punctum quodcumque; et circumducto semel circa *AG* diametrum
semicirculo *ABG*, describatur sphaera *ABGD*; in qua revolutione linea
ZB describet circulum qui terminus est communis duorum sphaericorum
segmentorum *BAD*, *BGD* [cf. Fig. III.5C.15]. Protrahantur itaque rectae
AB, *BG*, quae sunt a verticibus sphaericorum segmentorum ad
15 peripheriam circuli cuius diameter *BD* secantis.

Aio iam quod sphaerica superficies segmenti *BAD* aequalis est ei quod
fit ex *AB* in peripheriam circuli cuius diameter *AB*. Nam per 8^am sexti
sicut *AG* ad *BA*, ita *BA* ad *AZ*. Ergo per 6^am huius sicut *GA* ad *AB*, sic
peripheria cuius diameter *AB* ad peripheriam cuius diameter *AZ*. Quare
20 per 15^am sexti quod fit ex *AB* in peripheriam cuius diameter *AB* aequum
est ei quod ex *AG* in peripheriam cuius diameter *AZ*. Sed per praemis-

sam quod ex AG in peripheriam cuius diameter AZ aequale est segmenti
BAD sphaericae superficiei. Igitur segmenti BAD sphaerica superficies
aequalis est ei quod ex AB in peripheriam cuius diameter AB, et hoc est
25 primum ex propositis.

Cumque peripheria cuius diameter AB sit per sextam dimidium
periphaeriae (!) cuius AB semidiameter, ideo ex AB in dimidium
peripheriae cuius AB semidiameter fit etiam sphaerica superficies seg-
menti BAD. Sed ex AB in dimidium peripheriae cuius AB est semidia-
30 meter fit circulus ipse cuius AB semidiameter per 4^{am} de dimensione
circuli. Itaque circulus cuius AB semidiameter aequalis est sphaericae
superficiei segmenti BAD, quod erat alterum ex propositis.

Iisdem penitus mediis ostendam quod sphaerica superficies segmenti
BGD aequalis est ei quod fit ex BG in peripheriam circuli cuius diameter
35 BG, vel circulo cuius semidiameter BG.

Verum hoc apud unum segmentorum ut puta BAD demonstrato, potest
idem ostendi apud reliquum sic. Quoniam angulus ABG rectus per 29^{am}
(! 30^{am}) 3^{ii}, ideo per 9^{am} de dimensione circuli duo circuli quorum semi-
diameteri AB, BG simul aequales sunt circulo cuius semidiameter AG.
40 Hic autem per 10^{am} huius aequalis superficiei sphaericae, et ideo sphaericis
superficiebus segmentorum BAD, BDG. Quare circuli quorum semi-
diametri AB, BG aequales sunt sphaericis superficiebus segmentorum
BAD, BDG. Aequalis autem fuit circulus cuius semidiameter AB segmenti
BAD sphaericae superficiei. Superest ergo circulus cuius semidiameter
45 BG aequalis sphaericae superficiei segmenti BDG, quod erat propositum.

/Corollarium.

Manifestum est igitur quod sphaerica superficies segmenti sphaerici
aequalis est circulo cuius semidiameter est linea quae a vertice segmenti
ad peripheriam circuli secantis spaeram (!). Nam per sextam peripheria
50 cuius AB diameter aequalis est dimidio peripheriae cuius AB semidiameter.
Aequale est igitur quod ex AB in peripheriam cuius AB diameter ei quod
ex AB in dimidium peripheriae cuius AB semidiameter. Sed quod ex AB
in peripheriam cuius AB diameter aequum est sphaericae superficiei
segmenti BAD. Quod autem ex AB in dimidium peripheriae cuius AB
55 semidiameter per 4^{am} de dimensione circuli aequum est circulo cuius AB
semidiameter. Ergo sphaerica superficies segmenti BAD aequalis est
circulo cuius AB semidiameter, quod est propositum. Non aliter ostendam
quod superficies sphaerica segmenti BGD aequalis est circulo cuius BG
semidiameter.

Propositio XVII.

SI SPHAERA PLANO SECETUR, SEGMENTORUM SPHAERICAE
SUPERFICIES SUNT AD INVICEM SICUT AXES SEGMENTORUM.
Repetens proximae descriptionem, aio quod superficies sphaerica seg-

5 menti *BAD* ad superficiem sphaericam segmenti *BGD* est sicut axis *AZ* ad axem *ZG* [Fig. III.5C.15A].

Nam per 8^{am} 6^i ratio *AZ* ad *ZG* dupla est eius que *AZ* ad *ZB*; sicut *AZ* ad *ZB*, sic *AB* ad *BG* (propter similitudinem triangulorum *AZB*, *ABG*). Ergo ratio *AZ* ad *ZG* dupla est eius quae *AB* ad *BG* et eius quae *AB*

10 ad *BG* dupla est per 18^{am} sexti ratio quadrati *AB* ad quadratum *BG*. Quare quadratum *AB* ad quadratum *BG*, sicut *AZ* ad *ZG*. Sed ex 2^a 12^i circulus cuius semidiameter *AB* ad circulum cuius semidiameter *BG*, sicut quadratum ex duplo *AB* ad quadratum ex duplo *BG*; et ideo sicut quadratum *AB* ad quadratum *BG*. Igitur circulus cuius semidiameter *AB* ad circulum

15 cuius semidiameter *BG*, sicut *AZ* ad *ZG*. Verum circulus cuius semidiameter *AB* aequalis est per praemissae corollarium sphaericae superficiei segmenti *BAD*. Ergo sphaerica superficies segmenti *BAD* ad sphericam superficiem segmenti *BDG* est sicut axis *AZ* ad axem *ZG*, quod est propositum.

Corollarium.

20 Manifestum est ergo quod sphaerae superficies ad sui segmenti sphaericam superficiem est sicut diameter sphaerae ad axem segmenti; patet ex coniuncta proportione.

/Propositio XVIII.

62

SI SPHAERA PLANO SECETUR, UTRIUSLIBET SEGMENTI SPHAERICA SUPERFICIES AD CIRCULUM SECANTEM EST SICUT DIAMETER SPHAERAE AD AXEM RELIQUI SEGMENTI.

5 Adhuc manebo in eadem descriptione [cf. Fig. III.5C.15A] dicens quod sphaerica superficies segmenti *BAD* ad circulum cuius diameter *BD* secantem est sicut diameter *AG* ad axem *GZ*. Item quod sphaerica superficies segmenti *BGD* ad circulum cuius diameter *BD* est sicut diameter *AG* ad axem *AZ*.

10 Nam per 8^{am} sexti ratio *AG* ad *GZ* est dupla eius quae *AG* ad *GB*; sed *AG* ad *GB*, sicut *AB* ad *BZ*, propter similitudinem triangulorum *AGB*, *ABZ*. Ergo ratio *AG* ad *GZ* dupla est eius quae *AB* ad *BZ*. Sed et eius quae *AB* ad *BZ* dupla est per 2^{am} 12^i quae circuli cuius semidiameter *AB* ad circulum cuius semidiameter *BZ*. Igitur circulus cuius semi-

15 diameter *AB* ad circulum cuius semidiameter *BZ* est sicut *AG* ad *ZG*. Fuit autem per 16^{am} circulus cuius semidiameter *AB* aequalis superficiei sphaericae segmenti *BAD*, circulus vero cuius semidiameter *BZ* aequalis est circulo *BD* sphaeram secante. Ergo sphaerica superficies segmenti *BAD* ad circulum cuius semidiameter *BZ* (qui segmenta disterminat) est

20 sicut *AG* diameter ad axem *GZ* reliqui segmenti.

Non aliter ostendam quod sphaerica superficies segmenti *BGD* ad circulum cuius semidiameter *BZ* est sicut diameter *AG* ad axem reliqui segmenti *AZ*, quod est propositum.

Propositio XIX.

SI SPHAERA DUOBUS PARALLELIS PLANIS SECETUR, INTER-
CEPTI SEGMENTI SPHAERICA SUPERFICIES, ZONA SCILICET
SPHAERICA, AEQUALIS EST EI QUOD FIT EX DIAMETRO
SPHAERAE IN PERIPHERIAM CIRCULI CUIUS DIAMETER EST
AXIS SEGMENTI.

Praeteritae descriptioni addo rectam RS ad rectos angulos secantem
diametrum AG in puncto Y [Fig. III.5C.16]. Itaque RY in circum-
ductione semicirculi describet circulum parallelum circulo cuius diameter
BD.

Aio itaque quod segmenti sphaerici BS, quod inter duos circulos
parallelos quorum diametri BD, RS intercipitur, sphaerica superficies,
quae quasi zona est ab arcu BR descripta, aequalis est ei quod fit ex
diametro AG in peripheriam circuli cuius diameter est axis ZY.

Nam per 15^{am} praemissam quod fit ex AG in peripheriam circuli
cuius diameter AY aequum est sphaericae superficiei segmenti RAS,
peripheria autem cir/culi cuius diameter AY aequalis est per primum
corollarium sextae peripheriis circulorum quorum diametri AZ, ZY. Ergo
quod fit ex AG in peripherias circulorum quorum diametri AZ, ZY aequum
est sphaericae superficiei segmenti RAS. Verum per 15^{am} quod fit ex AG
in peripheriam circuli cuius diameter AZ aequum est sphaericae super-
ficiei segmenti BAD. Itaque de eo quod ex AG in peripherias circulorum
quorum diametri AZ, ZY dematur id quod ex AG in peripheriam circuli
cuius diameter AZ. Item de superficie sphaerica segmenti RAS dematur
superficies sphaerica segmenti BAD (duo scilicet aequalia de duobus
aequalibus). Et supererit, inde, id quod ex AG in peripheriam circuli
cuius diameter ZY; hinc autem id quod superficies sphaerica segmenti
intercepti BS. Et per consequentiam aequalia, quod est propositum.

Idem ostendes per corollarium 15^{ae}. Nam per illud ex linea AY in
peripheriam circuli ABG fit superficies sphaerica segmenti RAS, ex linea
autem AZ in peripheriam eandem fit superficies sphaerica segmenti
BAD. Unde sequitur ex prima 2^{i} elementorum ut ex ZY in dictam
peripheriam fiat superficies sphaerica segmenti BS intercepti, quod est
propositum.

Corollarium I.

Manifestum est ergo quod sphaerici segmenti parallelis circulis inter-
clusi sphaerica zona aequalis est ei quod fit ex axe segmenti in peripheriam
circuli maximi in sphaera; et ideo curvae superficiei cylindri cuius basis
est circulus maximus sphaerae, celsitudo autem axis segmenti.

Corollarium II.

Item manifestum est quod, parallelis planis quotcumque sphaeram
secantibus, sphaericae superficies planis interceptae sunt ad invicem sicut
ipsorum sphaeralium segmentorum axes. Patet hoc ex corollario praece-

dentis et prima sexti Elementorum. Poterat et hoc corollarium demon-
45   strari per corollarium 17^{ae} praeteritae, ex aequa, et disiuncta proportione
facillime.

Propositio XX.

SI SPHAERA ET CYLINDRUS DIAMETRUM COMMUNEM ET
AXEM COMMUNEM HABEANT, SECENTUR AUTEM PARAL-
LELIS QUOTLIBET PLANIS QUIBUS AXIS PERPENDICU-
5   LARITER INSTET, ERUNT SPHAERICAE SUPERFICIES
CYLINDRICIS SUPERFICIEBUS INTER EADEM SECANTIA
PLANA INTERPOSITIS SINGULAE SINGULIS AEQUALES.

Circulo $ABGD$ circumscribatur quadratum $EZHT$, sintque contactuum
puncta A, B, G, D, et connectatur AG [Fig. III.5C.17]. Item ducantur inter
10   latera ZH, ET ipsisque EZ, HT paralleli KL, MN ad placitum secantes
axem, quidem AG, apud X, O, peripheriam autem ABG apud P, R, S, Y.
Et circum AG diametrum immotam circumducatur, donec ad locum suum
red[e]at; parallelogrammum cylindrum, semicirculus vero sphaeram de-
scribet, eritque AG axis tam sphaerae quam cylindri. Item EZ, HT diametri
64   basium cylindri / sphaerae diametro aequales, et cylindrus sphaeram
16   circumscribet: Contingent namque cylindricae bases sphaericam super-
ficiem in punctis A, G; cylindrica quoque superficies sphaericam con-
tinget, eritque contactus circuli peripheria per punctum contactus B
descripta. Ipsae quoque lineae KX, MO circumductae describent circulos
20   parallelos secantes cylindrum, ipsa autem puncta P, S describent
circulorum peripherias, quae communes sectiones erunt dictorum cir-
culorum secantium cylindrum et sphaericae superficiei.

Aio itaque quod sphaerica superficies interiacens circulis quorum
diametri ZE, KL, quae scilicet describitur ab arcu AP, aequalis est
25   cylindricae superficiei iisdem circulis interpositae, quae scilicet de-
scribitur a linea ZK. Item quod sphaerica superficies quae inter circulos
quorum diametri KL, MN, quam scilicet describit arcus PS, aequalis est
cylindricae superficiei quae inter eosdem circulos, quae scilicet a linea
KM describitur. Item quod sphaerica superficies quae inter circulos
30   quorum diametri MN, TH, quam scilicet describit arcus SBG, aequalis
est cylindricae superficiei quae inter eosdem clauditur circulos, quam
scilicet describit linea MH.

Primum horum et tertium patet per corollarium 15^{ae} praemissae, cum
cylindrus ZL habeat altitudinem aequalem ipsi AX, quae est axis segmenti
35   sphaerici PAR, et basim circulum sphaerae maximum, atque etiam cum
cylindrus MT habeat altitudinem OG, qui est axis segmenti sphaerici SGY,
et basim circulum in sphaera maximum. Secundum autem patet per
primum corollarium praecedentis, cum cylindrus KN habeat altitudinem
XO, qui est axis segmenti sphaerici $SPRY$, et basim circulum maximum
40   sphaerae. Hoc idem ostendam de superficiebus sphaericis ac cylindricis
quos intercipiant quaecunque duo plana parallela basibus cylindri, quibus

scilicet axis cylindri rectus insistat, et hoc erat quod demonstrandum proponebatur.

 Ostendam hoc idem aliter. Cum per quartam huius ex AG in peripheriam
45 circuli ABG, quae basis est cylindri ZT, fiat superficies curva cylindri ZT, itemque ex AX in peripheriam eiusdem circuli fiat superficies curva cylindri ZL, propterea per primam sexti et coniunctam proportionem erit curva cylindri ZT superficies ad curvam cylindri ZL superficiem sicut AG ad AX. Sed per corollarium 17^{ae} praeteritae, sicut AG ad AX, sic
50 sphaerae ABG superficies ad segmenti sphaerici PAR sphaericam superficiem. Igitur et sicut sphaerae ABG superficies ad segmenti PAR sphaericam superficiem, sic curva cylindri ZT superficies ad curvam cylindri ZL superficiem. Et permutatim sicut sphaerae ABG superficies ad curvam cylindri ZT superficiem, sic segmenti PAR sphaerica super-
55 ficies ad curvam cylindri ZL superficiem; aequalis autem est, per 11^{am} huius, sphaerae ABG superficies curvae cylindri ZT superficiei. Ergo et segmenti PAR sphaerica superficies aequalis est curvae cylindri ZL superficiei. Similiter ostendam quod segmenti SAY sphaerica superficies aequalis est curvae cylindri ZN superficiei. Unde per consequentiam
60 supererit sphaerica [superficies] segmenti PY, quae zona est sphaerica ab arcu PS descripta, aequalis curvae superficiei cylindri KN. Itemque sphaerici segmenti SGY superficies aequalis est curvae MT cylindri superficiei. Hoc idem ostendam quibuscumque planis cylindricae basi parallelis tam cylindrum quam sphaeram cylindro circumscriptam secantibus,
65 et hoc est quod propositio nostra significat.

 Haec de superficiebus tornatilium, et sphaericorum segmentorum. Postulat ordo, ut iam de soliditate primum quidem tornatilis corporis, et exinde sphaerae disputemus.

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/Propositio XXI.

CONUS CUIUS BASIS AEQUALIS EST AGGREGATO BASIUM QUOTLIBET CONORUM SUB EODEM FASTIGIO AEQUALIS EST AGGREGATO OMNIUM ILLORUM.

5 Exempli gratia, basis coni A aequalis basibus simul sumptis aliquot conorum, utputa trium B, G, D [Fig. III.5C.18]. Sitque omnium celsitudo una.

 Aio quod conus A aequalis est aggregato conorum B, G, D.
 Cum enim eorum altitudines sint aequales, erit per 11^{am} 12^{i} conus B
10 ad conum A, sicut basis coni B ad basim coni A. Item conus G ad conum A, sicut basis coni G ad basim coni A. Quare per 24^{am} 5^{i} coni B, G simul sumpti ad conum A, sicut bases conorum B, G simul sumptae ad basim coni A. Adhuc conus D ad conum A, sicut basis coni D ad basim coni A. Rursum ergo per 24^{am} 5^{i} coni B, G, D simul sumpti ad conum A, sicut
15 bases conorum B, G, D simul sumptae ad basim coni A. Sed bases conorum B, G, D, simul sumptae, per hypothesim, sunt aequales basi

coni A. Ergo et coni B, G, D simul sumpti sunt aequales cono A, quod est propositum. Idem demonstrabimus de quotcumque conis.

Idem concludere potes de cylindris quorum celsitudines aequales.

Propositio XXII.

CONUS CUIUS CELSITUDO AEQUALIS EST AGGREGATO CELSITUDINUM QUOTLIBET CONORUM SUPER AEQUAS BASES EST AEQUALIS AGGREGATO ILLORUM.

5   Exempli causa, celsitudo coni A sit aequalis celsitudinibus aliquot conorum, ut puta trium B, G, D [Fig. III.5C.19]. Sintque omnium bases aequales.

Aio quod conus A aequalis est conis B, G, D simul sumptis.

Nam cum eorum bases sint aequales, erit per 11^{am} 12^i conus B ad
10   conum A, sicut altitudo coni B ad altitudinem coni A. Item conus G ad conum A, sicut altitudo coni G ad altitudinem coni A. Ergo per 24^{am} 5^i coni B, G, D simul ad conum A, sicut altitudines conorum B, G, D simul sumptae ad [altitudinem] conum (! coni) A. Adhuc conus D ad conum A, sicut altitudo coni D ad altitudinem coni A. Rursum ergo per 24^{am} 5^i
15   coni B, G, D simul ad conum A, sicut altitudines conorum B, G, D simul sumptae ad coni A celsitudinem: sed celsitudines conorum B, G, D simul sumptae aequales sunt celsitudini coni A. Ergo et coni B, G, D simul sumpti aequales sunt cono A, quod est propositum. Idem de quotcumque conis demonstrabimus repetita quoties opus fuerit 24^a 5^i, quod propositum est.

66   /Propositio XXIII.

SOLIDUM QUOD A TRIANGULO CIRCA UNUM LATERUM FIXUM, DONEC AD LOCUM SUUM REDEAT, CIRCUMDUCTO DESCRIBITUR, AEQUALE EST EI CONO CUIUS BASIS AEQUALIS
5   EST CONICAE SUPERFICIEI AB ALTERO LATERUM TRIANGULI CIRCUMDUCTORUM DESCRIPTAE, ALTITUDO AUTEM AEQUALIS PERPENDICULARI AD IDEM LATUS CADENTI AB ANGULO OPPOSITO. QUIN ETIAM SI AB ALTERO TERMINORUM LATERIS FIXI AD OPPOSITUM LATUS LINEA UTCUNQUE
10   DUCATUR, SOLIDUM A TRIANGULO ABSCISSO DESCRIPTUM AEQUALE EST EI CONO CUIUS BASIS AEQUALIS EST CONICAE SUPERFICIEI DESCRIPTAE A SEGMENTO LATERIS ABSCISSI ALTITUDO AUTEM AEQUALIS EST PERPENDICULARI AD IDEM LATUS A DICTO TERMINO LATERIS FIXI. ADHUC SI
15   LATUS TRIANGULI AEQUIDISTET AXI CIRCA QUEM CIRCUMDUCITUR TRIANGULUM, SOLUSQUE TRIGONI ANGULUS IN AXE TERMINETUR, DESCRIPTUM A TRIANGULO SOLIDUM AEQUALE ERIT EI CONO CUIUS BASIS AEQUALIS EST CYLINDRICAE SUPERFICIEI DESCRIPTAE A LATERE AXI
20   AEQUIDISTANTE, CELSITUDO AUTEM AEQUALIS PERPENDICULARI AD IDEM LATUS AB AXE IPSO DELAPSAE.

Brevissime hanc trifariam propositionem ostendam. Sit namque trigonum *ABG* quod circa unum laterum, ut puta circa latus *AB*, tanquam axem semel circumductum describat solidum *AGB* [Fig. III.5C.20(a)].

25 Ab angulo autem opposito *AG*, ut puta *B*, ad latus oppositum *AG* cadat perpendicularis *BD*. Sitque conus *H*, cuius basis sit aequalis conicae superficiei quam describit latus *AG*, celsitudo vero aequalis perpendiculari *BD*.

Aio quod conus *H* aequalis est solido *AGB*.

30 Ponatur enim primo trigonum *ABG* rectum angulum habens, qui apud *B*; et tunc per definitionem coni solidum *ABG* a triangulo descriptum est conus cuius axis *AB* basisque semidiameter *BG*. Quare per 3^{am} huius superficies conica quam describit linea *AG*, et ideo basis coni *H* illi aequalis, ad basim coni *ABG*, sicut linea *AG* ad semidiametrum *BG*; et

35 ideo propter triangulorum *AGB*, *ABD* similitudinem, sicut axis *AB* ad perpendicularem *BD*, quae celsitudo est coni *H*. Itaque conorum *AGB* et *H* bases sunt celsitudinibus mutuae; aequales ergo sunt per 12^{am} 12^{i} coni *AGB* et *H*, quod est propositum.

Ponantur nunc anguli trigoni *ABG* qui apud *A*, *B* acuti [Fig. III.5C.20(b)],

40 et demittatur a puncto *G* perpendicularis ad *AB* axem linea *GE*; quae in revolutione trianguli *ABG* describet circulum cuius semidiameter *GE*. Eritque solidum *AGB* compositum ex duobus conis *AGE*, *BGE* communem basem habentes circulum cuius semidiameter *GE*. Sit itaque conus *K* basim dicto circulo aequalem habens, celsitudinem vero aequalem

45 ipsi *AB*; eritque per praecedentem aequalis conus *K* conis *AGE*, *BGE* simul sumptis, et proinde solido *AGB*. Itaque quoniam per 3^{am} superficies conica quam describit linea *AG*, et ideo basis coni *H* illi aequalis ad basim coni *AGE*, et ideo ad basim coni *K* illi aequalem, est sicut

67 *AG* ad *GE*; et ideo propter triangulorum *AGE*, /*ABD* similitudinem, sicut

50 *AB* celsitudo coni *K* ad *BD* celsitudinem coni *H*, ideo per 12^{am} 12^{i} aequalis est conus *H* cono *K*, et ideo solido *AGB*, quod est propositum.

Demum in tertia descriptione [Fig. III.5C.20(c)] ponatur ex angulis trianguli *ABG*, qui apud *B* obtusus, et tunc cadat a puncto *G* perpendicularis ad axem *AB* productum, qui sit *GE*, ut in ambitu trianguli linea

55 *GE* circulum describat, qui basis erit tam coni *AGE* a triangulo *AGE* descripti, quam coni *BGE* a triangulo *BGE* descripti. Sit itaque conus *K* basim habens dicto circulo aequalem, et celsitudinem rectae *AB*; eruntque per praemissam conus *K* et conus *BGE* simul sumpti aequales cono *AGE*; communis utrinque auferatur conus *BGE*, et supererit conus *K* aequalis

60 solido quod in ambitu describitur per triangulum *ABG*. Itaque quoniam per 3^{am} superficies conica quam describit linea *AG*, et ideo basis coni *H* illi aequalis, ad basim coni *AGE*, et ideo ad basim coni *K* illi aequalem, est sicut *AG* ad *GE*, et ideo propter triangulorum *AGE*, *ABD* similitudinem, sicut *AB* celsitudo coni *K* ad *BD* celsitudinem coni *H*, per 12^{am} 12^{i}

65 aequalis est conus *H* cono *K*, et ideo solido *AGB*, quod est propositum. Hactenus demonstratum est primum ex propositis.

Alterum sic se habet [Fig. III.5C.21]. In ipso triangulo *ABG* ab altero

terminorum AB, qui sit B, ducatur ad oppositum latus linea BZ; sitque conus N habens basim aequalem conicae superficiei quam in circum-
70 ductione describit linea GZ, et celsitudinem aequalem perpendiculari BD.

Aio quod conus N aequalis est solido quod in ambitu describitur a trigono BGZ.

Sit enim conus H basim habens aequalem conicae superficiei quam describit linea AG, et celsitudinem BD. Item conus M basim habens
75 aequalem conicae superficiei quam describit linea AZ, et celsitudinem BD. Eritque, ut ostensum est, conus H aequalis solido descripto per trigonum ABG; conus autem M aequalis solido descripto per rrigonum (! trigonum) ABZ. Item quoniam conica superficies quam describit linea AG aequalis est conicis superficiebus simul sumptis quas describunt
68 linea AZ et linea ZG, ideo basis coni H, / quae fuit aequalis illi conicae,
81 aequalis erit basibus conorum, M, N simul sumptis, quorum scilicet bases his conicis fuerunt aequales; cumque conorum H, M, N sit una celsitudo, erit per antepraemissam conus H aequalis conis M, N simul sumptis. Itaque de cono H et solido AGB invicem aequalibus, auferatur conus
85 M et solidum AZB invicem aequalia: et supererunt conus N et solidum descriptum a triangulo BGZ invicem aequalia, quod est propositum.

Superest tertia propositionis pars: circumducatur ut prius trigonum BGZ circum axem AB, sitque GZ latus ipsi AB parallelum, et conus H habeat basim aequalem cylindricae superficiei quam describit linea
90 GZ, celsitudinem vero aequalem perpendiculari BD ab angulo B ad latus GZ [Fig. III.5C.22].

Aio quod conus H aequalis est solido quod describitur a triangulo BZG circumducto.

Compleatur enim rectangulum $AZGX$, quod in revolutione describet
95 cylindrum cuius axis AX et basium semidiametri AZ, GX. Unde et BD circulum describet, qui terminus erit cylindrorum AD, DX; quin etiam triangula ABZ, BGX describent conos, quorum bases, quae et cylindri, et quorum vertex B. Sit itaque conus K basim quidem habens aequalem basi cylindri AG, celsitudinem vero duplam ipsius AX; eritque per
100 11^{am} 12^{i} conus K duplus coni habentis basim eandem, et axem AX. Sed hic conus per praemissam aequalis est conis AZB, BGX simul sumptis. Ergo conus K duplus est ad conos AZB, BGX simul sumptos. Sed eorumdem conorum AZB, BGX simul sumptorum duplum est solidum a triangulo BGZ descriptum, quando per 9^{am} 12^{i} totus cylindrus AG
105 triplus est eorundem conorum. Itaque aequalis est conus K solido per triangulum BGZ descripto.

At cum cylindrica superficies quam describit linea GZ, et ideo basis coni H ei aequalis, ad basim cylindri AG, et ideo ad basim coni K, sit per 5^{am} huius sicut axis AX ad dimidium semidiametri GX: et ideo sicut
110 celsitudo coni K, quae dupla est axis AX, ad semidiametrum GX, quae dupla est sui dimidii, et ideo ad BD celsitudinem coni H, propterea per 12^{am} 12^{i} aequalis est conus H cono K, et ideo solido per triangulum BGZ descripto, quod est propositum.

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Hoc autem cum perpendicularis *DB* cadit inter puncta *G*, *Z*; quod si perpendicularis a puncto *B* ad *GZ* sit altera ipsarum *BZ*, *BG*, demonstratio adhuc erit eadem, sed intra cylindrum *AG* unus describetur conus non duo. Si vero perpendicularis *DB* cadat extra puncta *G*, *Z* [Fig. III.5C.23], tunc / sic ostendam hanc propositionis partem. Producta *ZD* ad libitum ad signum *O* connectatur *BO*. Sitque *M* conus basim habens aequam cylindricae superficiei quam describit linea *OG*, celsitudo vero *BD*: item conus *N* basim habens aequalem cylindricae superficiei quam describit linea *OZ* et celsitudinem *BD*. Eritque, sicut dudum fuit demonstratum, conus *M* aequalis solido descripto per triangulum *OBG*, conus autem *N* aequalis solido descripto per triangulum *OBZ*. Et quoniam cylindrica superficies quam describit linea *OG* aequalis est cylindricis superficiebus simul sumptis quas describunt lineae *OZ*, *ZG*, ideo basis coni *M* aequalis est basibus conorum *N*, *H* simul sumptis; quorum, cum sit una celsitudo, erit per antepraemissam conus *M* aequalis conis *N*, *H* simul sumptis. Itaque de cono *M* conus *N*, et de solido per triangulum *OBG* descripto solidum per triangulum *OBZ* descriptum, ab aequalibus scilicet aequalia, subtrahantur, et relinquitur conus *H* aequalis solido per triangulum *BZG* descripto, quod est propositum.

Concluditur ergo quod solidum a triangulo lateraliter vel angulariter axi in eodem plano existenti applicato et circa eumdem axem perfecta revolutione circumducto descriptum, aequale est ei cono cuius basis aequalis est conicae superficiei vel cylindricae descriptae a latere trianguli quod opponitur angulo applicato ad axem, celsitudo vero aequalis perpendiculari quae a dicto angulo ad dictum latus egreditur ubicumque occurrat. Et haec est tota propositi summa, quae sequentibus inservit demonstrationibus.

Propositio XXIV.

SOLIDUM A DIMIDIO POLYGONII AEQUILATERI CIRCULO INSCRIPTI SUPER DIAMETRUM STANTEM PERFECTA REVOLUTIONE CIRCUMDUCTO DESCRIPTUM, AEQUALE EST EI CONO

5 CUIUS BASIS AEQUALIS EST UNIVERSAE SOLIDI SUPERFICIEI, AXIS VERO PERPENDICULARI QUAE A CIRCULI CENTRO AD QUODLIBET POLYGONII LATUS EGREDITUR.

In circulo *AB*, cuius diameter *AB*, inscriptum sit polygonium aequilaterum, ut puta decagonum *AB*, a cuius dimidio *AGDEZB* super

10 axem *AB* semel revoluto describatur solidum *AB* conicarum superficierum [Fig. III.5C.24]. Ad unum autem laterum polygonii, quod sit *AG*, ducatur a centro sphaerae *H* perpendicularis *HT*; sitque conus *K*, cuius basis sit aequalis superficiei totae solidi *AB*, celsitudo vero aequalis perpendiculari *HT*.

15 Aio quod aequalis est conus *K* solido *AB*.

Connectantur enim anguli *G*, *D*, *E*, *Z* cum centro *H*, et constituantur

coni sub celsitudine HT: videlicet conus L basim habens aequam conicae
superficiei quam describit linea AG, conus M ei quam describit linea
BZ, conus N ei / quam describit linea GD, conus X ei quam describit
linea ZE, conus O ei quam describit linea DE, quae cylindrica est. Unde
conorum L, M, N, X, O bases simul sumptae aequales erunt universae
superficiei solidi AB, et ideo aequales erunt basi coni K. Quare per 21^{am}
huius omnes hi coni L, M, N, X, O simul aequales erunt cono K. Per
primam autem partem praecedentis solidum quod per triangulum HAG
circumductum describitur aequum est cono L, quod autem per triangulum
HBZ aequum est cono M. Item quod per triangulum HGD aequum est
per secundam partem praecedentis cono N; quodque per triangulum
HZE aequum est cono X. Quod tandem per triangulum HDE aequum
est, per tertiam partem praecedentis, cono O. Itaque solidum AB, quod
a toto semipolygonio ADB describitur, aequum est aggregato conorum
L, M, N, X, O; fuit autem hoc aggregatum cono K aequale. Ergo et
AB solidum cono K aequale erit. Et hoc fuerat demonstrandum.

Manifestum est ergo quod solidum AB aequum est cylindro cuius basis
est aequalis superficiei ipsius solidi AB, axis vero aequalis tertiae parti
perpendicularis HT, namque talis cylindrus per 11^{am} et 9^{am} 12^{i} aequalis
est cono K, et ideo solido AB etc.

Propositio XXV.

SPHAERA AEQUALIS EST CONO CUIUS BASIS SPHAERICAE
SUPERFICIEI, CELSITUDO VERO SPHAERAE SEMIDIAMETRO
AEQUALIS EST.

Sit circulus $ABGD$, cuius centrum E, diameter AG, et circumacto
semel semicirculo ABG super immotam diametron AG describatur
sphaera ABG, sitque conus M, cuius basis sphaerae ABG superficiei,
celsitudo autem semidiametro AE sit aequalis [Fig. III.5C.25].

Aio quod aequalis est conus M sphaerae ABG.

Nam si conus M non sit aequalis sphaerae ABG, erit aequalis
sphaerae alteri alicui maiori vel minori quam est ipsa sphaera ABG.
Sit ergo primum, si possibile est, conus M aequalis sphaerae ZHT minori
quam est sphaera ABG, et cum ea concentricae, cuius diameter ZT, et
quam circa diametrum ZT revolutus describit semicirculus ZHT; et per
13^{am} 12^{i} inscribatur circulo ABG polygonium aequalium laterum minime
tangentium circulum ZT, quod sit $ABGD$, cuius laterum unum sit AK,
ad quod a centro perpendicularis exeat EL; et polygonii dimidio semel
circa diametrum AG circumacto describatur solidum co/nicarum super-
ficierum minime iam tangentium sphaeram ZHT, eritque per prae-
cedentem solidum ABG aequum cono cuius basis aequalis est superficiei
solidi ABG, celsitudo vero perpendiculari EL; huiusmodi itaque cono
maior est conus M, quandoquidem et basi et celsitudine superior est,
quoniam scilicet sphaerae ABG superficies maior est solidi ABG inclusi
superficie et linea AE maior quam perpendicularis EL. Et ideo sphaera

25 *ZHT* ipsi cono *M* aequalis maior erit solido *ABG*: pars toto, quod est impossibile.

Vel sic, quoniam conus *M* maior fuit cono dicto aequali ipsi solido *ABG*, et ideo maior ipso solido *ABG*, et solidum *ABG* maius sphaera *ZHT*; propterea conus *M* maior erit sphaera *ZHT*; non est ergo ei aequalis
30 sicut supponebatur.

Sit deinde, si possibile est, conus *M* aequalis sphaerae maiori quam est sphaera *ABG*. Sed, brevitatis causa, sit supposita ipsa *ZHT*, et conus *M* basim habeat aequalem superficiei sphaerae *ZHT*, celsitudinem vero aequalem semidiametro *EZ*.
35 Aio quod conus *M* non erit aequalis sphaerae alicui maiori quam est sphaera *ZHT*.

Nam si possibile est, sit aequalis sphaerae *ABG* maiori quam est sphaera *ZHT*, et cum ea concentricae; describatur intra sphaeram *ABG* solidum *ABG* ut prius, cuius superficies non tangat sphaeram *ZHT* etc.
40 Eritque per praecedentem solidum *ABG* aequum cono cuius basis aequalis est superficiei solidi *ABG* et celsitudo perpendiculari *EL*. Talis ergo conus maior cono *M*, quem et basi et celsitudine superat. Quare solidum *ABG* maius cono *M*. Sed conus *M*, per hypothesim, aequalis fuit sphaerae *ABG*. Ergo solidum *ABG* maius quam sphaera *ABG*: pars toto, quod est
45 impossibile. Non est ergo conus *M* aequalis alicui sphaerae maiori quam est sphaera *ZHT*. Similiter si conus *M* [habeat] basim superficiei sphaerae *ABG* et celsitudinem aequam semidiametro *EA*, non erit aequalis sphaerae cuiquam maiori quam est sphaera *ABG*; fuitque ostensum quod nec minori. Superest ergo ut aequalis omnino sit conus *M* sphaerae *ABG*,
50 quod erat demonstrandum.

Manifestum est ergo quod sphaera aequalis est cono cuius basis semidiameter aequalis est diametro sphaerae, axis vero semidiametro sphaerae. Patet, nam per corollarium 10^{ae} circulus cuius semidiameter aequalis diametro sphaerae est aequalis superficiei sphaerae.

Propositio XXVI.

SPHAERA DUPLA EST CONI CUIUS BASIS CIRCULO SPHAERAE MAXIMO, CELSITUDO VERO SPHAERAE DIAMETRO FUERIT AEQUALIS.

5 Sit sphaera *A*, cuius circulus maximus *BCD* [Fig. III.5C.26], conus autem *N* basim habeat aequalem circulo *BCD*, celsitudinem vero aequam diametro sphaerae *BD*.

Aio quod sphaera *A* dupla est coni *N*.

Ponatur enim conus *M* basim habens aequalem superficiei sphaerae *A* et
10 celsitudinem aequalem semidiametro sphaerae *AD*; eritque per praecedentem conus *M* aequalis sphaerae *A*. Ponatur item conus *X* basim habens duplam circuli *BCD* et celsitu/dinem aequalem diametro sphaerae
12 *BD*. Itaque basis coni *M* aequalis superficiei sphaerae *A* per 11^{am} est quadrupla ad circulum *BCD*, et basis coni *X* dupla circuli *BCD*. Igitur

15 basis coni M dupla ad basim coni X; celsitudo autem coni X dupla ad
celsitudinem coni M. Quare conorum M, X reciprocae sunt bases
fastigiis; et ideo per 12^{am} 12^{i} aequales sunt coni. Sed conus M aequalis
sphaerae A. Ergo et conus X aequalis sphaerae A. Verum conus X duplus
est coni N, est enim eiusdem celsitudinis et habet basim duplam, nam
20 basis coni X dupla ad circulum BCD et basis coni N aequalis circulo
eidem. Igitur sphaera A dupla est coni N, quod est propositum.

Corollarium.

Manifestum est ergo quod cylindrus cuius basis aequalis est circulo
sphaerae maximo, celsitudo autem aequalis diametro sphaerae, est ad
25 sphaeram sesquialter. Namque cylindrus per 9^{am} 12^{i} triplus est sui coni,
cuius sphaera dupla est, ut fuit ostensum. Unde fiet cylindrus ad sphaeram
sesquialter.

Propositio XXVII.

SI FUERIT AXIS CYLINDRI AEQUALIS DIAMETRO BASIS,
CONI AUTEM BASIS AEQUALIS UNIVERSAE SUPERFICIEI
CYLINDRI, CELSITUDO VERO AEQUALIS SEMIDIAMETRO
5 CYLINDRICAE BASIS, AEQUALIS ERIT CYLINDRUS CONO.

Rectanguli $ABGD$ latus AB duplum sit lateris BG [Fig. III.5C.27], et
circumducto rectangulo semel manente latere AB, describatur cylindrus
AG. Itaque cum BG, quae semidiameter est basis, sit dimidium axis
AB, erit tota diameter aequalis axi. Item sit conus E, cuius basis sit
10 aequalis universae cylindri AG superficiei, hoc est aggregato ex curva
superficie quam describit linea GD et ex basibus quas describunt lineae
AD, BG, celsitudo vero semidiametro AD.

Aio quod cylindrus AG et conus E sunt aequales.

Secetur enim axis AB bifariam apud Z, et connectantur DZ, ZG quo
15 fiet ut in circumductu rectanguli AG ipsa triangula ZDA, ZGB describant
conos quorum bases quae et cylindri et quarum (!) vertex punctum Z.
Cadat autem perpendicularis a puncto Z ad DG, quae sit ZH; sitque
conus T, cuius basis sit aequalis cylindricae superficiei quam describit
73 / linea GD, celsitudo vero aequalis perpendiculari ZH, et ideo semi
20 diametro AD, quae celsitudo est coni E; eritque per tertiam partem 23^{ae}
praecedentis conus T aequalis solido descripto per triangulum DZG.
Quamobrem conus T cum conis descriptis per triangula ZDA, ZGB
simul aequalis erit toti cylindro AG; basis quoque coni T cum basibus
cylindri aequalis erit universae superficiei cylindri; et ideo basis coni T
25 cum basibus conorum ADZ, BGZ, quae sunt bases cylindri, simul
aequales erunt basi coni E, quae fuit aequalis universae superficiei
cylindri; estque horum quatuor conorum celsitudo una, quoniam lineae
AD, AZ, ZB, ZH, quae fuerunt conorum celsitudines, sunt aequales.
Igitur per 21^{am} praedictam conus E aequalis est aggregato conorum T

30   *ADZ*, *BGZ*; fuit autem hoc aggregatum aequale cylindro *AG*. Ergo cylindrus *AG* aequalis erit cono *E*, quod est propositum.

Propositio XXVIII.

CYLINDRUS CUIUS TAM AXIS QUAM BASIS DIAMETER AEQUALIS EST SPHAERAE DIAMETRO SESQUIALTER EST AD SPHAERAM.

5   Quamvis haec propositio demonstrata sit in praemissa 26ª ut ipsius corollarium, placuit tamen hic eam aliter demonstrare.

Esto sphaera quidem *A*, cylindrus *Z*; sitque tam axis *Z* cylindri quam diameter aequalis sphaerae *A* diametro *CB* [Fig. III.5C.28].

Aio quod cylindrus *Z* sesquialter est ad sphaeram *A*.

10   Sit enim conus *M* basim habens aequalem superficiei sphaerae *A* et celsitudinem aequalem semidiametro sphaerae *AB*; eritque per 25^{am} praemissam conus *M* aequalis sphaerae *A*. Item sit conus *E* basim habens aequalem universae superficiei cylindri *Z* et celsitudinem aequalem semidiametro basis cylindri. Eritque per praecedentem conus *E* aequalis

15   cylindro *Z* (cum semidiameter sphaerae *A* et [semidiameter] cylindricae basis, quae sunt celsitudines conorum *M*, *E*, sint aequales per hypothesim) atque cylindri *Z* tota superficies, quae basis est coni *E*, sesquialtera sit per corollarium 11^{ae} superficiei sphaerae *A*, quae basis est coni *M*; et coni eiusdem altitudinis per 11^{am} 12^i sint ad invicem sicut bases. Iam et conus

20   *E* sesquialter erit ad conum *M*. Verum conus *M* sphaerae *A* conusque *E* cylindro *Z* fuit aequalis. Ergo et cylindrus *Z* sesquialter erit ad sphaeram *A*, quod fuit propositum.

Corollarium.

Rursum ergo hinc patet quod sphaera dupla est ad conum cuius tam

25   axis quam basis diameter aequalis est sphaericae diametro. Nam cum cylindrus huiusmodi sit per praemissam ad sphaeram sesquialter, ad conum vero per 9^{am} 12^i triplus, erit sphaera ad conum dupla.

Itaque sphaera sic se habet ad cylindrum et reliqua tornatilia. Nunc ad segmentorum soliditates transibimus.

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/ Propositio XXIX.

SI CIRCULI ARCUS QUISPIAM IN PORTIONES SECETUR AEQUALES, QUIBUS CHORDAE SUBTENDANTUR, ET ARCUS EXTREMA CUM CENTRO CIRCULI CONNECTANTUR, REC-

5   TILINEUM AUTEM SUB CHORDIS ET SEMIDIAMETRIS COM-PREHENSUM, ALTERA SEMIDIAMETRORUM STANTE, SEMEL CIRCUMDUCATUR: DESCRIPTUM SOLIDUM AEQUUM EST EI CONO CUIUS BASIS AEQUALIS EST CONICIS SUPERFICIEBUS PER CHORDAS DESCRIPTIS, CELSITUDO VERO AEQUALIS

10   PERPENDICULARI QUAE A CENTRO CIRCULI AD QUAMLIBET
CHORDARUM EGREDITUR.

In circulo AD [Fig. III.5C.29], cuius centrum H, arcus AD secetur
in quotvis partes aequales, ut puta duas AG, GD, quibus subtendantur
chordae AG, GD, et ductis semidiametris AH, DH circumducatur
15   rectilineum AD, stante semidiametro AH, ut describatur solidum AD.
Sitque conus K, cuius basis sit aequalis conicis superficiebus descriptis
per chordas AG, GD, celsitudo autem aequalis perpendiculari HT ad
latus AG.

Aio quod aequalis est conus K solido AD.

20   Connectatur enim GH, et constituantur sub celsitudine HT duo coni,
scilicet conus L basim habens aequam conicae superficiei quam describit
linea AG et conus N basim habens aequam conicae superficiei quam
describit linea GD. Unde conorum L, N bases simul aequales erunt basi
coni K, quae fuit dictis conicis superficiebus aequalis. Quare per 21$^{\text{am}}$
25   huius coni L, N simul aequales erunt cono K. Sed per 23$^{\text{am}}$ solidum a
triangulo HAG descriptum aequum est cono L, quodque per triangulum
GHD describitur aequum est cono N. Ergo solidum AD quod a toto
rectilineo AD describitur aequum est aggregato conorum L, N. Fuit autem
hoc aggregatum cono K aequale. Igitur et AD solidum cono K erit aequale,
30   quod demonstrandum proponitur.

Hoc idem ostendemus quotcunque fuerint chordae factis tot conis
quot fuerint triangula, qui singuli sunt singulis solidis per triangula
descriptis aequales, adhibitis semper 21$^{\text{a}}$ et 23$^{\text{a}}$ huius, estque similis
demonstratio 24$^{\text{ae}}$ huius.

Propositio XXX.

SI CONICA SUPERFICIES VERTICEM IN CENTRO SPHAERAE
HABENS SPHAERAM IN DUOS SECTORES DISTERMINET,
UTERLIBET SECTORUM AEQUALIS EST CONO CUIUS BASIS
5   AEQUALIS EST BASI SPHAERICAE IPSIUS SECTORIS, CEL-
SITUDO AUTEM AEQUALIS SEMIDIAMETRO SPHAERAE.

De circulo $ABGD$ [Fig. III.5C.30], cuius centrum E et diameter AG,
sumatur arcus AB, et ducatur semidiameter EB, circumducto autem
75   semicirculo ABG, stante diametro AG, / describetur sphaera, et in eodem
10   ambitu sector circuli ABE describet solidum sphaerae sectorem $ABED$,
ponatur conus M, cuius basis aequa sit basi sphaericae sectoris
ABD, quam scilicet describit arcus AB, celsitudo vero aequalis sit
semidiametro AE.

Aio quod aequalis est conus M sphaerico sectori ABD.

15   Nam secus erit aequalis alicui sectori maiori vel minori. Sit ergo
primum aequalis conus M sectori sphaerico ZHT minori quam est sector
ABD, et cum eo concentrico, quem describat arcus ZH ipsis lineis
AE, EB terminatus, et secetur arcus AB iterum atque iterum donec
per 13$^{\text{am}}$ 12$^{\text{i}}$ arcuum chordae non contingant peripheriam ZHT. Sitque

20 una chordarum AK, et ad eum (!) perpendicularis EL, et circumacto recti-
lineo $AKBE$ circa axem AE describatur solidum ABD, quod per praece-
dentem aequale erit cono cuius basis aequalis est conicis superficiebus
per chordas AK, KB descriptis, celsitudo vero perpendiculari EL. Hoc
itaque cono maior est conus M, qui et basi et celsitudine maior est.

25 Sed conus M aequalis fuit sphaerico sectori ZT. Ergo [sector] sphaericus
ZT maior est solido ABD, pars toto, quod est impossibile.

Vel sic, conus M maior est solido ABD, quandoquidem maior est cono
qui solido aequalis; ergo et maior sectore sphaerico ZT; non est ergo ei
aequalis, sicut supponebatur.

30 Sit deinde conus M aequalis sectori sphaerico maiori quam est sector
ABD. Sed brevitatis causa suppositus sector sit sector sphaericus $ZHTE$,
et conus M habeat basim aequam sphaericae sectoris ZT, quam scilicet
describit arcus ZH, et celsitudinem aequalem semidiametro EH.

Aio quod conus M non erit aequalis alicui sectori sphaerico maiori
35 quam est sector ZT.

Sit enim, si possibile est, aequalis sectori BD concentrico cum sectore
ZT et ipso maiori quem describit sector circuli ABE; et inscribantur
arcui AB chordae ut prius non tangentes circulum ZH, et circumducto
rectilineo ABE formetur solidum BD etc., quod per praecedentem
40 aequale erit cono cuius basis aequalis est conicis superficiebus per chordas
AK, KB descriptis, celsitudo vero perpendiculari EL. Hic ergo conus
maior erit cono M, quem et basi et celsitudine superat; sed conus M
aequalis fuit per hypothesim sectori BD; igitur et solidum BD maius est
sectore BD, pars toto, quod est impossibile. Non est ergo conus M aequalis
45 alicui sectori sphaerico maiori quam est sector ZT. Similiter si conus
M supponatur habere basim aequam sphaericae basi sectoris BD et celsi-
tudinem aequam semidiametro AE, non erit conus M aequalis alicui sectori
maiori quam est sector BD; sed nec minori, ut fuit ostensum. Superest
ergo ut aequalis sit omnino conus M sectori sphaerico $ABED$, quod est
50 propositum.

Et quoniam sphaera ABG dissecatur in duos sectores quos disterminat
conica superficies descripta per lineam EB, poterimus hoc idem ostendere
de sphaerico sectore BGD descripto per circuli secorem EBG. Sit ergo
conus N, cuius basis aequalis sit sphaericae basi sectoris BGD, quam
55 describit arcus BG, [et cuius celsitudo aequalis sit semidiametro AE], et
similiter ostendemus quod aequalis erit conus N sectori sphaerico BGD.

Vel hac via: sit conus X habens basim aequam superficiei toti sphaerae
ABG et celsitudinem aequam semidiametro AE; eritque per 25^{am} aequalis
76 conus X / sphaerae ABG; et basis coni X aequalis basibus conorum
60 M, N simul sumptis. Quapropter per 21^{am} et conus X conis M, N simul
sumptis aequalis erit. Igitur et coni M, N simul sumpti erunt aequales
sphaerae ABG. Itaque de sphaera ABG sector sphaericus ABD, et de
aggregato conorum M, N conus M auferatur, aequalia ab aequalibus;
et supererunt sector sphaericus BGD et conus N aequales, quod est
65 propositum.

Corollarium.

Manifestum est ergo quod scctor sphaericus aequalis est cono cuius basis semidiameter est aequalis lineae quae a vertice sphaerici segmenti ad peripheriam basis, celsitudo autem aequalis semidiametro sphaerae.

70 Nam per corollarium 16^{ae} circulus cuius semidiameter est linea quae a vertice segmenti sphaerici ad peripheriam basis aequalis est basi sphaericae ipsius sectoris. Quare conus cuius basis aequalis circulo cuius diameter est linea quae a vertice segmenti sphaerici ad peripheriam basis, celsitudo autem aequalis semidiametro sphaera[e]; iam aequalis

75 erit cono cuius basis aequalis est basi sphaericae sectoris, celsitudo autem semidiametro sphaerae. Sed hic conus aequalis fuit sphaerico sectori. Ergo et ille conus cuius basis est circulus habens semidiametrum aequalem lineae quae a vertice segmenti sphaerici ad peripheriam basis et [cuius] celsitudinem (!) aequalem (!) semidiametro sphaerae aequalis

80 erit sphaerico sectori, sicut infert corollarium.

Propositio XXXI.

SI CIRCULUS SPHAERAM SECET, SPHAERICORUM SEG-
MENTORUM UTRUMLIBET AEQUUM EST CONO CUIUS BASIS
EST SECANS CIRCULUS, CELSITUDO VERO EA RECTA QUAE
5 SIC SE HABET AD AXEM IPSIUS SEGMENTI SICUT AG-
GREGATUM EX SEMIDIAMETRO SPHAERAE ET EX AXE
RELIQUI SEGMENTI AD EUNDEM AXEM.

Est sphaera $ABGD$, quam circa axem AG manentem revolutus semel describit semicirculus ABG, quae secetur circulo cui recta sit diameter

10 AG, et cuius diameter sit BD secans AG apud P [Fig. III.5C.31]. Sintque segmenta ABD, BGD, quorum axes AP, PG. Sitque sicut aggregatum ex EA, PG ad ipsam PG, sic RP ad AP.

Aio quod conus cuius basis semidiameter BP et celsitudo RP aequalis est sphaerico segmento ABD.

15 Item sit sicut aggregatum ex EG, PA ad ipsam PA, sic SP ad GP.

Aio quod conus cuius basis semidiameter BP et celsitudo SP aequalis est sphaerico segmento BGD.

Primum sic ostendo. Cum EA, PG simul ad PG sit sicut RP ad AP, erit disiunctim sicut EA ad PG, sic RA ad AP; et conversim, sicut PG ad

20 EA, sic AP ad RA; et permutatim, sicut PG ad AP, sic EA ad RA; et conversim, sicut AP ad PG, sic RA ad AE; et coniunctim, sicut AG ad GP, sic RE ad EA. Sed AG ad GP ratio dupla est eius quae AG ad GB per 8^{am} sexti; AG autem ad GB sicut AB ad BP, propter similitudinem triangulorum AGB, ABP; ergo ratio AG ad GP dupla est eius quae

25 AB ad BP; fuit autem sicut AG ad GP, sic RE ad EA. Igitur ratio RE ad EA dupla est eius quae AB ad BP. Sed circulus cuius semidiameter AB ad circulum cuius semidiameter BP duplam habet rationem eius quam

77 AB ad BP (quandoquidem circuli ad invicem sunt / per 2^{am} 12^1 sicut quadrata diametrorum vel semidiametrorum). Itaque RE ad EA erit sicut

30 circulus cuius semidiameter *AB* ad circulum cuius semidiameter *BP*.
Quamobrem duo coni quorum unius quidem basis semidiameter *AB* et
celsitudo *EA*, alterius autem basis semidiameter *BP* et celsitudo *RE*, sunt
per 12^{am} 12^i aequales invicem, quando celsitudines sunt basibus recipro-
cae. Sed conus cuius basis semidiameter *BP* et celsitudo *RE* per 22^{am} huius

35 aequalis est duobus conis quorum basis semidiameter *BP* et celsitudines
RP, *PE*; conus autem cuius basis ex centro *AB* et celsitudo *AE* per corol-
larium praemissae aequalis est sectori sphaerico *ABED*. Igitur duo coni
quorum basis semidiameter *BP* et celsitudines *RP*, *PE* simul sumpti sunt
aequales sectori sphaerico *ABED*, quod est aggregatum ex segmento

40 sphaerico *ABD* et ex cono cuius basis semidiameter *BP* et celsitudo *PE*.
Auferatur ergo communis conus, cuius basis semidiameter *BP* et
celsitudo *PE*, et supererit conus cuius basis semidiameter *BP* et celsitudo
RP aequalis sphaerico segmento *ABD*, quod erat primum ex demon-
strandis. Reliquum sic ostemdo (!).

45   Cum *EG*, *PA* simul ad *PA* sit sicut *SP* ad *GP*, erit disiunctim sicut
EG ad *PA*, sic *SG* ad *PG*; et conversim, sicut *PA* ad *EG*, sic *GP* ad
SG; et permutatim, sicut *AP* ad *PG*, sic *EG* ad *GS*; et conversim, sicut
GP ad *PA*, sic *SG* ad *GE*; et coniunctim, sicut *GA* ad *AP*, sic *SE* ad *EG*.
Sed per 8^{am} sexti ratio *GA* ad *AP* dupla est eius quae *GA* ad *AB*;

50 estque sicut *GA* ad *AB*, sic *GB* ad *BP* propter similitudinem triangulorum
GAB, *GBP*. Igitur ratio *GA* ad *AP* dupla est eius quae *GB* ad *BP*; fuit
autem sicut *GA* ad *AP*, sic *SE* ad *EG*; quare et ratio *SE* ad *EG* dupla
eius est quae *GB* ad *BP*. Sed circulus cuius semidiameter *GB* ad circulum
cuius semidiameter *BP* duplam habet rationem eius quam *GB* ad *BP* ex

55 2^a 12^i. Itaque *SE* ad *EG* erit sicut circulus cuius semidiameter *GB* ad
circulum cuius semidiameter *BP*. Quamobrem duo coni quorum unius
quidem basis semidiameter *GB*, celsitudo autem *EG*, alterius autem
basis semidiameter *BP*, celsitudo autem *SE*, sunt per 12^{am} 12^i invicem
aequales (quandoquidem celsitudines sunt basibus mutuae). Per praece-

60 dentis autem corollarium conus cuius basis semidiameter *BG*, celsitudo
autem *EG*, aequalis est sectori sphaerico *BGDE*. Igitur sector sphaericus
BGDE aequalis est cono cuius basis quidem semidiameter *BP*, celsitudo
autem *SE*. Apponatur utrobique conus cuius basis semidiameter *BP* et
celsitudo *PE*; eritque segmentum sphaericum *BGD* aggregatum, scilicet ex

65 sectore *BGDE* et dicto cono, aequale conis quorum basium semidiameter
BP et celsitudines *PE*, *ES* simul sumptae. Sed tales coni simul sumpti
sunt per 22^{am} aequales cono cuius basis semidiameter *BP* et celsitudo
PS. Ergo et conus cuius basis semidiameter *BP*, celsitudo autem *PS*
aequalis est segmento sphaerico *BGD*, quod fuit reliquum ex demon-

70 strandis. Itaque vera est tota propositi sententia.

/ Corollarium.

Hinc manifestum est quod sphaericum segmentum ad conum eiusdem
basis ac verticis est sicut linea constans ex semidiametro sphaerae et
axe reliqui segmenti ad ipsum axem. Namque, ut paucis agam, conus

75 *BRD* ad conum *BAD* eiusdem basis est sicut *RP* altitudo ad *PA* altitudinem; fuit autem sphaericum segmentum *BAD* aequale cono *BRD*. Atque *RP* linea ad lineam *PA*, sicut aggregatum *EGP* ad axem *GP*. Igitur sphaericum segmentum *BAD* ad conum *BAD*, sicut *EGP* aggregatum ad axem *GP*, quod est propositum. Non aliter ostendam quod segmentum

80 sphaerae *BGD* ad conum *BGD* erit sicut aggregatum *EAP* ad axem *AP*, sicut infert corollarium.

Propositio XXXII.

SI CIRCULUS SPHAERAM SECET, SPHAERICORUM SEGMENTORUM UTRUMLIBET AEQUUM EST CONO CUIUS BASIS SEMIDIAMETER EST AEQUALIS IPSIUS SEGMENTI AXI, CEL-
5 SITUDO VERO AEQUALIS AGGREGATO EX SEMIDIAMETRO SPHAERAE ET EX AXE RELIQUI SEGMENTI.

Repetita descriptione praecedenti [cf. Fig. III.5C.31], aio quod segmentum sphaericum *ABD* aequum est cono cuius basis semidiameter aequalis est axi *AP*, celsitudo vero aggregato ex *EG* semidiametri (!)
10 et *GP* axe reliqui segmenti. Item aio quod segmentum sphaericum *BGD* aequale est cono cuius basis semidiameter aequalis est axi *GP*, celsitudo vero aequalis aggregato ex *AE* semidiametro et *AP* axe reliqui segmenti.

Primum sic ostendo. Quoniam ratio *GP* ad *PA* dupla est eius quae *BP*
15 ad *PA* per 8^{am} sexti, cuius et dupla est per 2^{am} 12^{i} ratio circuli cuius semidiameter *BP* ad circulum cuius semidiameter *PA*, ideo *GP* ad *PA* erit sicut circulus cuius semidiameter *BP* ad circulum cuius semidiameter *PA*. Fuit autem in praemissa sicut *GP* ad *PA*, sic *EA*, vel *EG*, ad *AR*; et coniunctim, sicut *EG*, *GP* simul ad *PR* totum, sic *PG* ad *PA* per 13^{am}
20 5^{i}; et ideo sicut circulus cuius semidiameter *BP* ad circulum cuius semidiameter *PA*. Quamobrem duo coni quorum unius quidem basis semidiameter est *AP* et celsitudo aggregatum ex *EG*, *GP*, alterius vero basis semidiameter *BP* et celsitudo *PR*, sunt aequales per 12^{am} 12^{i} quandoquidem celsitudines basibus reciprocae. Verum per praecedentem
25 conus cuius basis semidiameter *BP* et celsitudo *PR* fuit aequalis sphaerico segmento *ABD*. Ergo sphaericum segmentum *ABD* aequale est et cono cuius basis semidiameter *AP* et celsitudo aggregatum ex *EG*, *GP*, quod est unum ex demonstrandis. Accipe reliquum.

Quoniam ratio *AP* ad *PG* dupla est eius quae *BP* ad *PG* per 8^{am} sexti,
30 cuius et dupla est per 2^{am} 12^{i} ratio circuli cuius semidiameter *BP* ad circulum cuius semidiameter *PG*, ideo *AP* ad *PG* erit sicut circulus cuius semidiameter *BP* ad circulum cuius semidiameter *PG*. Fuit autem in praemissa sicut *AP* ad *PG*, sic *EG*, vel *EA*, ad *GS*. Quare per 13^{am} 5^{i} sicut *EA*, *AP* simul ad *PS* totum, sic *AP* ad *PG*, et ideo sicut circulus
35 cuius semidiameter *BP* ad circulum cuius semidiameter *PG*. Quamobrem
79 duo coni quorum unius quidem basis semidiameter est / *PG* et celsitudo aggregatum ex *AE*, *AP*, alterius autem basis semidiameter est *BP* et celsi

tudo *PS*, sunt ad invicem aequales per 12^{am} 12^i, quandoquidem reciproca sunt basibus fastigia. Verum ex praecedenti conus cuius basis semi-
40 diameter *BP* et celsitudo *PS* aequalis est sphaerico segmento *BGD*. Ergo sphaericum segmentum *BGD* aequale est cono cuius basis semidiameter est *PG* et celsitudo aggregatum ex *AE*, *AP*, quod supererat demonstrandum.

Aliter
PROPOSITAM SPHAERAM AD DATAM RATIONEM SECARE.
Sphaerae propositae diameter sit *AB* [Fig. III.5C.32]; data ratio *CD* ad
DE, oportet sphaeram *AB* secare ita ut segmentum ad segmentum sit
5 sicut *CD* ad *DE*.

Ponatur *AF* dimidium ipsius *AB*; et sicut est *CE* ad *ED* sic sit *FA* ad *AG*; quibus intersit media proportionalis *AH*, ut scilicet *FA*, *AH*, *AG* sint continuae proportionales; et circa axem *FB* describatur parabola *FHLK*. Itemque circum non coincidentes *FB*, *XB* per punctum *G* incedat hyper-
10 bola *GL* secans parabolen apud *L* punctum; et compleatur parallelogrammum *LMBX*, itemque parallelogrammum *AGNB*; eruntque per 12^{am} 2^i conicorum elementorum parallelogramma *LB*, *BG* praedicta inter se aequalia. Quare per 15^{am} sexti Euclidis erit sicut *AB* ad *BM*, sic *LM* ad *GA*. Itaque cum ratio quadrati *LM* ad quadratum *GA*, et ideo ratio quadrati
15 *AB* ad quadratum *BM*, componatur ex rationibus quadrati *LM* ad quadratum *HA* atque quadrati *HA* ad quadratum *AG*; et cum ratio *FM* ad *AG* componatur ex rationibus *MF* ad *FA* et *FA* ad *AG*, cumque per 39^{am} (! 20^{am}) primi con[ic]orum elementorum sit sicut quadratum *LM* ad quadratum *HA*, sic *MF* ad *FA*, et per 17^{am} sexti Euclidis sicut quadratum
20 *HA* ad quadratum *AG*, sic *FA* ad *AG*, propterea ex aequa proportione erit sicut quadratum *LM* ad quadratum *AG*; et ideo quadratum *AB* ad quadratum *BM*, sic *MF* ad *AG*; et per 2^{am} 12^i Euclidis sic etiam circulus cuius semidiameter *AB* ad circulum cuius semidiameter *BM*; itaque per 12^{am} [12^i] praedicti conus cuius axis *AG* basisque semidiameter *AB*
25 aequalis erit cono cuius axis *FM* basisque semidiameter *BM*, quandoquidem celsitudines basibus sunt reciprocae. Sed per 11^{am} 12^i Euclidis sicut conus cuius axis *FA* basisque semidiameter *AB* ad conum cuius axis *GA* basisque semidiameter *AB*, sic *FA* ad *AG*. Ergo sicut conus cuius axis *FA* basisque semidiameter *AB* ad conum cuius axis *FM*
30 basisque semidiameter *MB*, sic etiam *FA* ad *AG*, hoc est, *CE* ad *ED*. Verum per praecedentem 32^{am} conus cuius axis *FM* basisque semidiameter *BM* aequalis est sphaerico segmento cuius axis *BM*. Item per corollarium 25^{ae} huius conus cuius axis *FA* basisque semidiameter *AB* aequalis est sphaerae cuius diameter *AB*. Igitur sicut sphaera *AB* ad sphaericum
35 segmentum cuius axis *BM*, sic *CE* ad *ED*; et disiunctim, sicut *CD* ad *DE*, sic sphaericum segmentum cuius axis *AM* ad sphaericum segmentum cuius axis *BM*. Quare sphaera cuius diameter *AB* in / puncto *M* ipsius diametri secatur in duo segmenta quorum ratio est quae *CE* ad *ED*, quod fuit faciendum.

40   Haec propositio sumpta est ex Dionysiodoro (!) mathematico antiquissimo, ut tradit Eutotius Ascalonica in commentariis Archimedis.

Propositio XXXIII.

PYRAMIS ET CONUS SUB AEQUIS FASTIGIIS SUNT AD INVICEM SICUT BASES.

Sit pyramis AC super basim rectilineam A [Fig. III.5C.33]. Item conus
5   BGF super circulum BG eiusdem altitudinis.

Aio quod sicut est rectilineum A ad circulum BG, sic est pyramis AC ad conum BGF.

Sit enim, si possibile est, pyramis AC ad conum BGF sicut rectilineum A ad circulum maiorem minoremve ipso circulo BG. Et primo minorem,
10 qui sit DE concentricus ipsi BG; et ipsi BG inscribatur rectilineum polygonium per 13^{am} 12^l laterum minime tangentium circulum DE; et super rectilineum BG intelligatur pyramis eiusdem altitudinis cum cono BGF eiusdem verticis; eritque per 7^{am} (! 6^{am}) 12^l sicut rectilineum A ad rectilineum BG, sic pyramis AC ad pyramidem BGF. Sed per 8^{am} 5^l maior
15 est proportio rectilinei A ad circulum DE quam rectilinei A ad rectilineum BG. Igitur maiorem rationem habet per 12^{am} 5^l pyramis AC ad conum BGF quam pyramis AC ad pyramidem BGF. Ergo per 10^{am} 5^l pyramis BGF maior est cono BGF, pars toto, quod est impossibile. Non est ergo pyramis AC ad conum BGF, sicut rectilineum A ad circulum aliquem
20 minorem BG.

Sed nec maiorem, sit enim, brevitatis causa, circulus conusque propositus DEF eiusdem altitudinis cum pyramide AC.

Aio quod pyramis AC ad conum DEF non erit sicut rectilineum A ad circulum quempiam maiorem ipso DE circulo, ut puta ad circulum BG.
25 Ponatur enim BG circulus ipsi DE concentricus, et inscribatur ut prius circulo BG rectilineum laterum minime tangentium circulum DE, et super rectilineum BG intelligatur pyramis eiusdem verticis cum cono DEF. Itaque quoniam sicut rectilineum A ad circulum BG, sic pyramis AC ad conum DEF, et per 7^{am} (! 6^{am}) 12^l sicut rectilineum A ad recti-
30 lineum BG, sic pyramis AC ad pyramidem BGF, atque maior est ratio rectilinei A ad rectilineum BG quam eiusdem rectilinei A ad circulum BG per octavam 5^l: propterea per 12^{am} 5^l maiorem rationem habet pyramis AC ad pyramidem BGF quam eadem pyramis AC ad conum DEF. Ergo per 10^{am} 5^l conus DEF maior est pyramide BGF, pars toto, quod
35 est impossibile. Similiter non erit pyramis AC ad conum BGF sicut rectilineum A ad circulum quempiam maiorem ipso circulo BG; sed nec minorem, ut fuit ostensum. Superest ergo ut pyramis AC ad conum BGF sit sicut rectilineum A ad circulum ipsum BG, quod est propositum.

/ Corollarium I.

81
40   Manifestum est est (!) ergo quod prisma et cylindrus sub aequalibus fastigiis sunt ad invicem sicut bases. Namque tam prisma ad suam

pyramidem per 8^{am} 12^i quam cylindrus ad suum conum per 9^{am} eiusdem triplus est.

Corollarium II.

45   Itaque pyramis et conus quorum bases et celsitudines sunt aequales erunt ad invicem aequales. Hoc idem de prismate et cylindro pronunciandum.

Propositio XXXIV.

PYRAMIS ET CONUS SUPER AEQUALES ERECTI BASES SUNT AD INVICEM SICUT CELSITUDINES.

   Pyramis AB et conus GD stent super aequales bases B, D, existentibus
5 scilicet rectilineo B et circulo D aequalibus; sintque pyramidis et coni celsitudines AB, GD [Fig. III.5C.34].

   Aio quod pyramis AB ad conum GD est sicut celsitudo AB ad celsitudinem DG.

   Nam si celsitudines tales sint aequales, patet propositum per corol-
10 larium primum praecedentis. Si inaequales, tunc super circulum D constituatur conus eiusdem altitudinis cum pyramide AB, qui sit conus ED, eritque per praemissae corollarium secundum pyramis AB aequalis cono ED. Sed per 11^{am} 12^i conus ED ad conum GD est sicut celsitudo ED ad celsitudinem GD. Ergo et pyramis AB ad conum GD, sicut celsitudo
15 ED, et ideo sicut celsitudo AB (sunt enim aequales), ad celsitudinem GD, quod est propositum.

Corollarium.

   Manifestum est ergo quod prisma quoque et cylindrus super aequas bases sunt ad invicem sicut celsitudines, nam prisma pyramidis et
20 cylindrus coni triplus est.

/ Propositio XXXV.

PYRAMIDIS ET CONI RATIO COMPONITUR EX BASIUM ET CELSITUDINUM RATIONIBUS.

   Sit pyramis AB, cuius basis B, et celsitudo AB [Fig. III.5C.35].
5 Item conus GD, cuius basis D et celsitudo GD.

   Aio quod ratio pyramidis AB ad conum GD componitur ex ratione basis B ad basim D et ex ratione altitudinis AB ad altitudinem GD.

   Ponatur enim conus EZ super circulum Z aequalem circulo D, vel pyramis EZ super basim Z aequalem basi D, cuius altitudo EZ sit
10 aequalis altitudini AB; eritque per 33^{am} praemissam pyramis AB ad conum EZ, vel per 8^{am} 12^i ad pyramidem EZ, sicut basis B ad basim Z, et per 11^{am} 12^i. Si EZ sit conus (vel per praemissam si sit pyramis), erit conus EZ, vel pyramis EZ, ad conum GD, sicut celsitudo EZ ad celsitudinem GD. Verum ratio pyramidis AB ad conum GD componitur ex ratione

15 pyramidis *AB* ad conum sive pyramidem *EZ* et ex ratione coni sive pyramidis *EZ* ad conum *GD*. Ergo eadem ratio pyramidis *AB* ad conum *GD* componetur ex ratione basis *B* ad basim *Z* vel basim *D* (cum sint aequales) et ex ratione celsitudinis *EZ* vel celsitudinis *AB* (cum sint aequales) ad celsitudinem *GD*, et hoc erat demonstrandum.

Corollarium I.

20

Manifestum est ergo quod similiter prismatis et cylindri ratio ex basium et celsitudinum rationibus componitur, cum prisma pyramidis, et cylindrus coni, sit triplus.

Corollarium II.

25 Unde facillime sequitur ut pyramis et conus quorum bases sunt fastigiis reciprocae sint invicem aequales; contra, si pyramis et conus ad invicem extiterint aequales, eorum bases erunt fastigiis reciprocae; et hoc idem de prismate et cylindro sentiendum.

Propositio XXXVI.

SPHAERA AEQUALIS EST PYRAMIDI CUIUS BASIS AEQUALIS EST SPHAERICAE SUPERFICIEI, CELSITUDO AUTEM SPHAERAE SEMIDIAMETRO.

5 Esto sphaera *A*; pyramis autem *P* basim habeat aequalem superficiei sphaerae *A*, celsitudinem vero aequalem semidiametro sphaerae *A* [Fig. III.5C.36].

Aio quod aequalis est *P* pyramis sphaerae *A*.

Sit enim conus *M* basim habens aequalem superficiei sphaerae *A*, celsitu/dinem vero aequalem semidiametro sphaerae *A*. Itaque *P* pyramis

83

11 et *M* conus et bases et celsitudines invicem aequales habent. Quare per 2^{um} corollarium 33^{ae} praecedentis aequalis est *P* pyramis cono *M*. Sed conus *M* per 24^{am} (! 25^{am}) aequalis est sphaerae *A*. Ergo et *P* pyramis aequalis est sphaerae *A*, quod est propositum.

Corollarium I.

15

Unde manifestum est quod prisma cuius basis aequalis est superficiei sphaerae, celsitudo autem aequalis semidiametro sphaerae, est triplum ad sphaeram, quoniam per 8^{am} 12^{i} tale prisma triplum est suae pyramidis, quae aequalis est sphaerae.

Corollarium II.

20

Quare prisma cuius basis aequalis est superficiei sphaerae, celsitudo vero tertia pars semidiametri sphaerae, est aequale ipsi sphaerae.

Corollarium III.

Et quoniam superficies sphaerae per 10^{am} huius aequalis est rectangulo
25 quod fit ex diametro sphaerae in peripheriam sui maximi circuli, ideo

sphaera aequalis est ei solido parallelepipedo cuius basis est dictum rectangulum, celsitudo autem tertia pars semidiametri sphaerae.

Corollarium IV.

30 Manifestum est ergo quod ex ductu semidiametri sphaerae in superficiem eius producitur triplum soliditatis sphaerae, nam tale productum est soliditas prismatis habentis basim aequalem superficiei sphaerae et altitudinem aequalem semidiametro sphaerae.

Corollarium V.

35 Quamobrem ex ductu tertiae partis semidiametri in superficiem sphaerae producitur soliditas sphaerae.

Propositio XXXVII.

SECTOR SPHAERICUS AEQUALIS EST PYRAMIDI CUIUS BASIS AEQUALIS EST BASI SPHAERICAE SECTORIS, CELSITUDO VERO SEMIDIAMETRO SPHAERAE.

5 Sit sector sphaericus A [Fig. III.5C.37]; pyramis autem P habeat basim aequalem sphaericae basi sectoris A, celsitudinem autem semidiametro sphaerae cuius est sector A.

84 Aio quod pyramis P aequalis est / ipsi A sectori.

Sit enim conus M, cuius basis aequalis sit sphaericae basi sectoris A, 10 celsitudo vero semidiametro sphaerae. Eritque per 2^{um} corollarium 33^{ae} huius pyramis P aequalis cono M. Sed conus M per 30^{am} aequalis est sectori A; igitur et pyramis P aequalis est sectori A, quod erat propositum.

Corollarium I.

15 Unde manifestum est quod prisma cuius basis aequalis est sphaericae basi sectoris sphaerici, celsitudo autem aequalis semidiametro sphaerae, est triplum ipsius sectoris, quandoquidem per 8^{am} 12^{i} tale prisma triplum est suae pyramidis, quae aequalis est sectori. Quare prisma cuius basis aequalis est sphaericae basi sectoris sphaerici, celsitudo autem tertia pars semidiametri sphaerae, aequale est sphaerico sectori dicto.

20 Et quoniam basis sphaerica sectoris sphaerici per corollarium 16^{ae} huius aequalis est circulo cuius semidiameter est recta quae a vertice segmenti sphaerici ad peripheriam basis, hic autem circulus per 4^{am} de dimensione circuli aequalis est rectangulo contento sub sua semidiametro et dimidio peripheriae, ideo parallelepipedum solidum cuius basis est dictum rectangulum, celsitudo vero tertia pars semidiametri sphaerae aequale est
25 dicto sectori sphaerico.

Corollarium II.

Manifestum est ergo quod ex ductu semidiametri sphaerae in basim sphaericam sectoris sphaerici producitur triplum sectoris. Ex ducto vero
30 tertiae partis semidiametri in dictam basim producitur sector ipse.

Propositio XXXVIII.

SPHAERA AD CUBUM SUAE DIAMETRI RATIONEM HABET QUAM UNDECIM AD UNUM ET VIGINTI FERE.

Sit sphaera A, cubus R, cuius latus aequum sit diametro sphaerae
5   A [Fig. III.5C.38].

Aio quod sphaera A ad cubum R est fere sicut 11 ad 21, quod sic concludam.

Sit cylindrus S, cuius basis diameter diametro sphaerae A et celsitudo eidem diametro sit aequalis. Itaque prismatis R et cylindri S celsitudo
10   est una. Quare per corollarium 33^{ae} huius erit sicut basis prismatis R ad basim cylindri S, sic prisma R ad cylindrum S. Sed basis prismatis R est quadratum quod ex diametro basis cylindri S; estque per 8^{am} de dimensione circuli quadratum quod ex diametro circuli ad circulum sicut 14 ad 11 fere. Igitur prisma R ad cylindrum S est fere sicut 14 ad 11.
15   Per corollarium autem 26^{ae} huius vel per 28^{am} cylindrus S sesquialter est ad sphaeram A, et ideo sicut 21 ad 14. Ergo per 23^{am} 5^i ex aequa proportione erit sicut 21 ad 11, sic cubus R ad sphaeram A, quod est propositum.

/ Corollarium I.

20   Manifestum est ergo quod 21 sphaerae sunt aequales 11 cubis qui fiunt ex diametro sphaerae. Patet, nam per praemissam sphaera ad cubum suae diametri est sicut 11 ad 21. Ergo per 13^{am} quinti 11 sphaerae ad 11 cubos sunt sicut 11 ad 21. Ergo conversim 11 cubi ad 11 sphaeras sunt sicut 21 ad 11. Sed 21 sphaerae ad 11 sphaeras sicut 21 ad 11.
25   Igitur 21 sphaerae eam habent rationem ad 11 sphaeras quam 11 cubi ad easdem 11 sphaeras. Quare per 9^{am} quinti 21 sphaerae aequales sunt 11 cubis qui ex diametro sphaerae.

Corollarium II.

Unde manifestum est si cubus qui ex diametro cuiuspiam sphaerae
30   multiplicetur undecies, producti pars 21^a erit solidum sphaerae; contra si sphaera vicies semel multiplicetur, producti pars 11^a erit cubus qui ex diametro sphaerae. Sed haec supponunt rationem peripheriae ad diametrum triplam sesquiseptimam. Quod si peripheriae ad diametrum ratio supponatur tripla superpartiens 10/71—quoniam tunc per 8^{am} de
35   dimensione circuli quadratum quod ex diametro circuli ad ipsum circulum est sicut 284 ad 223 et per 28^{am} huius sphaera ad cylindrum (! cylindrus ad sphaeram) est sicut 426 ad 284 scilicet sequialtera—ideo ex 23^a quinti ex aequa perversim proportione erit sicut 426 ad 223, sic cubus R ad sphaeram A. Quamobrem tunc 426 sphaerae erunt aequales cubis 223 qui
40   fiunt ex diametro sphaerae, quod sequitur ex dicta argumentatione. Unde si cubus qui ex diametro sphaerae multiplicetur 223es, producti pars 426^a erit solidum sphaerae. Contra, si sphaera 426es multiplicetur

producti pars 223ᵃ erit cubus qui ex diametro sphaerae. Verum prima suppositio, quae facit peripheria[m] ad diametrum esse triplam sesquisep-
45 timam, facit et sphaeram aliquanto maiorem vero; altera autem quae peripheriam ad diametrum profert triplam superpartientem 10/71 facit sphaeram aliquanto minorem veritate. Non enim licuit in hoc punctum geometricum attingere—non ipsi quidem Archimedi quamvis geo-metrarum omnium acutissimo.

50

Finis

Messanae 10. Septembris octavae Indictionis 1534.

The Book of Archimedes On the Sphere and the Cylinder From the Tradition of Eutocius, Augmented, Restored in the Best Order, and Emended by Fran-cesco Maurolico the Mamertine, Most Zealous in the Mathematical Discipline

Proposition I.

THE SURFACE OF A PYRAMID ERECTED ON AN EQUIANGU-LAR AND EQUILATERAL BASE (THE SURFACE BEING A COL-LECTION OF TRIANGLES MEETING AT THE APEX OF THE PYRAMID) IS EQUAL TO THE RIGHT TRIANGLE ONE OF WHOSE SIDES ABOUT THE RIGHT ANGLE IS EQUAL TO THE PERPEN-DICULAR DRAWN FROM THE APEX TO A SIDE OF THE BASE AND THE OTHER TO THE PERIMETER OF THE BASE.[1]

For example, let a pyramid with apex Z be erected on the equilateral and equiangular pentagon $ABGD$ as a base, so that the straight line from point Z to the center of a circle circumscribing pentagon ABG be perpen-dicular to the pentagon [see Fig. III.5C.1]. Whence it happens that the triangles which meet at apex Z are mutually equilateral and equiangular. Moreover, let perpendicular ZH fall from point Z to one of the sides of the pentagon, say AB. And let right triangle TKL be posited, with its right angle at K and whose side TK is equal to ZH and side KL to the whole perimeter of pentagon ABG.

And so I say that the whole surface of the pyramid, which is a collection of triangles joined at apex Z, is equal to $\triangle TKL$.

For let KL be cut into as many segments as there are sides of base ABG, i.e., in this example, into five parts: KM, MN, NX, XO, OL, each equal to a single side of base ABG. And let TM, TN, TX, TO be joined. And so since the bases AB and KM of triangles AZB and TKM are equal, as are

[1] See the Commentary to Text C, Prop. I, lines 5–11.

the perpendiculars ZH and TK, therefore, by I.38 [of Euclid] these triangles are equal. But, by VI.1, $\triangle TKL / \triangle TKM = KL / KM$, and hence as 5 : 1; so also the surface of pyramid ABG—which surface is the collection of the five triangles meeting at point Z—is five times $\triangle AZB$. Therefore, the said surface of pyramid ABG is equal to $\triangle TKL$. Q.E.D.

Proposition II.

THE CURVED [i.e. LATERAL] SURFACE OF A [RIGHT] CONE IS EQUAL TO THE RIGHT TRIANGLE ONE OF WHOSE SIDES CONTAINING THE RIGHT ANGLE IS EQUAL TO THE CONICAL SLANT HEIGHT AND THE OTHER TO THE CIRCUMFERENCE OF THE BASE.[1]

Let there be a right triangle ABG with right angle at B [see Fig. III.5C.2]. When this triangle is rotated about AB as an axis until it returns to its [starting] place, a cone is described which has A as its apex and circle GDE as its base. And let there be a right triangle ZHT having a right angle at H and whose side ZH is equal to slant height AG and where side HT is equal to the circumference of circle GDE.

I say that the conical surface of cone $AGDE$ which line AG describes is equal to $\triangle ZHT$.

For if $\triangle ZHT$ is not equal to the conical surface of $AGDE$ it will be completely equal to the conical surface of a cone having a base less or greater than circle GD and an altitude the same as of cone AGD. Therefore, in the first place let $\triangle ZHT$ be equal to the conical surface of a cone having its apex at A and its base at circle KL less than circle EG, and a side AL. This cone is evidently the one which is described by $\triangle ABL$. And, by XII.13 [of Euclid][2] let a regular polygon be described within circle EG whose sides do not at all touch circle KL; and from apex A let slant heights descend to the angles of the polygon, which with the sides of the polygon will contain the pyramid having the same apex as cone AGD and included therein. And so let a perpendicular AM be drawn from apex A to one of the sides of polygon EG, e.g., to side GD. And, by the preceding [proposition], the surface of pyramid AGD, which is the collection of triangles meeting at apex A, is equal to the right triangle one of whose sides about the right angle is equal to perpendicular AM while the other is equal to the perimeter of polygon GDE. Therefore, a triangle of this sort is less than $\triangle ZHT$ whose sides about the right angle, ZH and HT, are greater, being equal to side AB and the circumference of circle GDE, which are [respectively] greater than perpendicular AM and the perimeter of polygon GDE. But $\triangle ZHT$ was equal to the surface of cone ALK. Therefore, the said surface of pyramid AGD is less than the surface of cone ALK which is impossible since the surface of the pyramid is greater than the included surface of cone ALK. On thus: since $\triangle ZHT$ is greater than the triangle to which the surface of the pyramid is equal, while the surface of the pyramid

Prop. II
[1] See Com., Prop. II, lines 2–5.
[2] *Ibid.*, line 19.

is greater than the included surface of cone ALK; therefore $\triangle ZHT >$ surf. cone ALK; and so it is not equal to it as was supposed.

Now let $\triangle ZHT$ be equal, if possible, to the conical surface of a cone having a greater base than circle GD and having the same altitude as cone AGD; and, for the sake of brevity, let the supposed cone be ALK and side ZH of $\triangle ZHT$ be equal to slant height AL and side HT equal to the circumference of circle KL.

I say now that it is not possible for $\triangle ZHT$ to equal the lateral surface of some cone having as its base a circle greater than circle KL and its apex at H.

For if it is possible, let $\triangle ZHT$ be equal to the surface of a cone whose base circle $EG >$ circle KL; and, as before, let there be described within circle EG a regular polygon not touching circle KL, and on the polygon [let there be erected] pyramid GD with apex A, from which we let drop perpendicular AM to side DG of the polygon. And, by the preceding [proposition], the surface of pyramid AGD, being a collection of triangles at the apex, is equal to a right triangle one of whose sides about the right angle is equal to perpendicular AM while the remaining side is equal to the perimeter of polygon GDE. And so this triangle is greater than $\triangle ZHT$ since it has sides about the right angle which are greater. But $\triangle ZHT$ was equal to the surface of cone $AGDE$. Therefore, the said surface of the pyramid is greater than the surface of the cone $AGDE$, that is, the "included" is greater than the "including," which is impossible. Hence, in the same way, if side ZH of $\triangle ZHT$ is equal to slant height AG and side HT to the circumference of circle EG, $\triangle ZHT$ would not be equal to the surface of some cone whose base is greater than circle EG and has its apex in A. And it was demonstrated that the same triangle is not equal to the surface of any cone whose base is less than circle EG and has its apex at A. It remains, therefore, that the same $\triangle ZHT$ is equal to the conical surface of cone $AGDE$. Q.E.D.

Corollary I.

Therefore, it is evident that a conical surface is produced from the product of a conical side (i.e. slant height) and one-half the circumference of the base, just as the area of a [right] triangle (by I.41 [of Euclid]) arises from the product of one of the sides about the right angle in the right triangle and one-half the other [side about the right angle].[3]

Corollary II.

Again, the conical surface is produced from the product of the radius of the conical base and the semicircumference of the circle whose radius is the conical side (i.e. slant height). This follows out of the third corollary of the sixth [proposition].[4]

[3] *Ibid.*, lines 66–69.
[4] *Ibid.*, lines 71–73.

Corollary III.

Then the circle whose radius is the mean proportional between the slant height and the radius of the conical base is equal to the conical surface, which follows from the last corollary of the sixth [proposition].[5]

Proposition III.

THE CONICAL SURFACE IS TO THE BASE AS THE CONICAL SIDE [I.E. SLANT HEIGHT] IS TO THE RADIUS OF THE BASE.[1]

Let there be cone ADG described by the rotation around AB as an axis of $\triangle ABG$ having a right angle at B; the apex of the cone is A, its base [circle] DG and its [base] radius BG [see Fig. III.5C.3].

I say that the lateral surface of cone ABG is related to circle DG as slant height AG is to radius BG.

For let there be $\triangle ZHT$ having a right angle at Z and with side ZH equal to slant height AG while ZT is equal to the circumference of circle DG. Also, from ZH (which is greater than BG since AG is greater than it) let there be cut ZN equal to BG, and let NT be joined. By [Proposition] IV of the *Booklet on the Measurement of the Circle*, $\triangle ZNT$ = circle DG, while $\triangle ZHT$ (by the preceding [proposition]) is equal to the lateral surface of cone DAG. But, by VI.1 [of Euclid] and conjunct proportionality, $ZH / ZN = \triangle ZTH / \triangle ZTN$. Therefore, lat. surf. ADG / circle $DG = ZH / ZN = AG / BG$. Q.E.D.

Or as follows: By the corollary of [Proposition] IV of *On the Measurement of the Circle*, the area of circle DG is equal to the product of radius BG and semicircumference DG. By corollary [I] of the preceding [proposition], the conical surface of ADG is equal to the product of slant height AG and semicircumference DG. Therefore, by VI.1 again, AG / BG = lat. surf. ADG / circle DG. Q.E.D.

Proposition IV.

THE LATERAL SURFACE OF A [RIGHT] CYLINDER IS EQUAL TO THE RECTANGLE CONTAINED BY THE SIDES, ONE OF WHICH IS EQUAL TO THE AXIS OF THE CYLINDER, WHILE THE OTHER IS EQUAL TO THE CIRCUMFERENCE OF THE BASE.[1]

Let there be a rectangle $ABGD$. When it has been rotated once about side AB as the fixed [axis], a cylinder is described which has AB as its axis and circle GE as its base [Fig. III.5C.4]. And let there be rectangle ZHT whose side ZH is equal to axis AB while its side HT is equal to the circumference of circle EG.

[5] *Ibid.*, lines 75–77.
Prop. III
[1] See Com., Prop. III, lines 2–3.
Prop. IV
[1] See Com., Prop. IV, lines 2–5.

I say that the cylindrical surface which side *GD* describes is equal to rectangle *ZT*.

For if rectangle *ZHT* is not equal to the cylindrical surface which line *DG* describes, it will be equal to a cylindrical surface of some cylinder having a base greater or less than circle *EG* and described about the same axis. Therefore, in the first place, let rectangle *ZHT* be equal to the lateral surface of a cylinder whose base is a circle *KL* less than circle *EG* and whose axis is the same *AB*. This cylinder is evidently the one described by the rotation of rectangle *ABLM* about axis *AB* and whose lateral surface is described by line *LM*. And, by XII.13 [of Euclid],[2] let regular polygon *EG* be inscribed in circle *EG* and not at all touching circle *KL*. On this polygon let a prism be erected with the same altitude as the cylinder, the sides of the cylinder having been drawn perpendicular to the angles of the polygon. And the rectangular [surfaces] of the prism, whose bases are the sides of the polygon, together will be equal (by I.36 and VI.1 [of Euclid]) to the rectangle which arises from the product of axis *AB* and the perimeter of polygon *EG*. But this rectangle is less than rectangle *ZT*, which arises from the product of axis *AB* and the circumference of circle *EG*, which latter is greater than the perimeter of the polygon. But the rectangle *ZT* was equal to the cylindrical surface which line *LM* describes. Therefore, the [surface] rectangles of the prism whose base is polygon *EG* are [together] less than the cylindrical surface which line *LM* describes; the including surface, I say, is less than the included, which is impossible.

Or as follows. Since rectangle *ZT* is greater than the aforesaid [surface] rectangles of the prism while the rectangles of the prism are [together] greater than the cylindrical surface which line *LM* describes, since the "including" is greater than the "included," therefore, rectangle *ZT* is greater than the cylindrical surface which line *LM* describes. Therefore, it is not equal to it, as was proposed.

Now let rectangle *ZHT* be equal, if possible, to the cylindrical surface of a cylinder having a base greater than circle *EG* and an axis *AB*. And, for the sake of brevity, let the supposed cylinder be the one whose axis is *AB* and whose base circle is *KL*, the one evidently described by the rotation of rectangle *ABLM* once about axis *AB*, and side *ZH* of rectangle *ZT* is equal to axis *AB* while side *HT* is equal to the circumference of circle *KL*.

I say now that it is not possible for rectangle *ZT* to be equal to the lateral surface of some cylinder described about axis *AB* and having a base greater than circle *KL*.

For if it is possible, let rectangle *ZT* be equal to the lateral surface of the cylinder whose base circle *EG* is greater than circle *KL*, which is evidently the one described by the rotation of rectangle *ABGD* about axis *AB* and whose lateral surface is described by line *GD*. And, as before, let a regular polygon be inscribed in circle *EG* but not touching circle *KL*.

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[2] See Com., Prop. II, line 19.

And on this polygon let there be erected a prism included by the cylindrical surface which line *GD* describes. And the [surface] rectangles of the prism whose bases are sides of the polygon are together equal (by I.36 or VI.1 [of Euclid]) to the rectangle which arises from the product of axis *AB* and the perimeter of polygon *EG*. And so this rectangle is greater than rectangle *ZT* since one of its sides is equal to one side of rectangle *ZT* while the other is greater than its corresponding side of *ZT* (for the perimeter of polygon *EG* is greater than the circumference of *KL*). But rectangle *ZT* was equal to the cylindrical surface which line *GD* describes. Therefore, the said [surface] rectangles of the prism together are greater than the cylindrical surface which line *GD* describes. And so the included surface of the prism is greater than the including cylindrical surface, which is impossible. Therefore, rectangle *ZT* is not equal to some lateral surface of a cylinder about axis *AB* whose base is greater than circle *KL*.

Similarly, if side *ZH* of rectangle *ZT* is equal to axis *AB* and side *HT* to the circumference of circle *EG*, rectangle *ZT* will not be equal to the lateral surface of any cylinder about axis *AB* whose base is greater than circle *EG*. And it was demonstrated that neither is the same rectangle equal to the lateral surface of any cylinder about axis *AB* whose base is less than circle *EG*. Therefore, it remains that the same rectangle *ZT* is equal to the lateral surface whose axis is *AB* and whose base is circle *EG*, which is evidently the surface described by line *GD*. Q.E.D.

Corollary I.

It is evident, therefore, that a cylindrical surface is produced from the product of the cylindrical axis and the circumference of the base.[3]

Corollary II.

Again, the cylindrical surface is produced from the product of the diameter of the cylindrical base and the circumference of the circle whose diameter is [equal to] the cylindrical axis. This follows from the third corollary of the sixth [proposition].[4]

[Corollary III.]

Finally, the circle whose radius is the mean proportional between the cylindrical side or the axis and the diameter of the cylindrical base is equal to the cylindrical surface. This follows from the last corollary of the sixth [proposition].[5]

Proposition V.

THE CYLINDRICAL SURFACE IS RELATED TO A CONICAL SURFACE HAVING THE SAME AXIS AND BASE AS THE AXIS IS

[3] See Com., Prop. IV, lines 72–73.
[4] *Ibid.*, lines 75–77.
[5] *Ibid.*, lines 78–81.

TO ONE-HALF THE CONICAL SLANT HEIGHT. ALSO, THE CYLINDRICAL SURFACE IS RELATED TO THE BASE AS THE AXIS IS TO HALF THE RADIUS.[1]

With the rectangle *ABGD* rotated once about axis *AB* let a cylinder be described and with △*ABG* rotated about the same axis let a cone be described. And circle *EG*, which line *BG* describes, will be the base of the cylinder as well as the cone [Fig. III.5C.5].

I say that the cylindrical surface which line *DG* describes is related to the conical surface which line *AG* describes as axis *AB* is to one-half line *AG*, while it [i.e. the same cylindrical surface] is related to circle *EG* as axis *AB* is to one-half line *BG*.

For by the preceding [proposition], the cylindrical surface which line *GD* describes is equal to the product of *AB* and the circumference of circle *EG*. By the second [proposition] of this [work] the conical surface which line *AG* describes is equal to the product of *AG* and one-half the circumference *EG*; hence [it also equals] the product of one-half *AG* and the whole circumference *EG*. But, by the corollary to the fourth [proposition] of *On the Measurement of the Circle*, circle *EG* is equal to the product of *BG* and one-half the circumference *EG*. Therefore, [it is also equal] to the product of one-half *BG* and the whole circumference *EG*. Therefore, since the altitude of these three rectangles is the same, namely, a line equal to the circumference of circle *EG*, then by VI.1 [of Euclid] they will be related to each other as the bases. And so the cylindrical surface which line *DG* describes will be related to the conical [surface] which *AG* describes as *DG* is to one-half *AG*; but [it will also be related] to circle *EG* as line *AB* is to one-half of *BG*. Q.E.D.

Proposition VI.

THE CIRCUMFERENCES OF CIRCLES ARE PROPORTIONAL TO [THEIR] DIAMETERS.[1]

[A demonstration of this proposition is had in the eighth (proposition) of the Preamble of Maurolycus.][2]

For let there be two circles *AB* and *GD* whose diameters are *AB* and *GD* [see Fig. III.5C.6].

I say that diameter *AB* / diameter *GD* = circumference *AB* / circumference *GD*.

For let diameter *AB* / diameter *GD* = circumference *AB* / circumference *EZ*. And circumference *EZ* will be equal to circumference *GD*, for other-

Prop. V
[1] See Com., Prop. V, lines 2–5.
Prop. VI
[1] See Com., Prop. VI, lines 2–3.
[2] See above, Text A, Prop. VIII. This is, of course, an editorial comment made by Borelli or Cyllenius.

wise it will be greater or less [than it]. If less, let it be concentric to this GD, and by XII.13 [of Euclid] let regular polygon GD be inscribed [in circle GD] but not touching circumference EZ; and similarly, let another [regular polygon] similar to it [be inscribed] in circumference AB. On account of the similitude of the polygons, diameter AB / diameter GD = perimeter of polygon AB / perimeter of polygon GD. Hence, perimeter of polygon AB / perimeter of polygon GD = circumference AB / circum-ference EZ. But the perimeter of polygon GD is greater than the circum-ference EZ, the "containing" certainly being greater than the "contained." Therefore, by V.14 [of Euclid], the perimeter of polygon AB will be greater than the circumference of AB, i.e. the "included" will be greater than the "including," which is impossible. Now if circumference EZ is [supposed to be] greater than circumference GD, then inversely, circumference EZ / circumference AB = diameter GD / diameter AB = circumference GD / circumference TH, where circumference TH < circumference AB, by V.14 [of Euclid]. And so diameter GD / diameter AB = circumference GD / circumference TH, with circumference TH less than the circum-ference of diameter AB, which leads to the first impossibility. Wherefore, circumference EZ will not be greater than circumference GD. And it was demonstrated that neither was it less. Hence it will be completely equal to it. But circumference AB / circumference EZ = diameter AB / diameter GD. Therefore, circumference AB / circumference GD = diameter AB / diameter GD. Q.E.D.

[Corollary I.]

Hence, it is evident that just as the diameter of a proposed circle is to the aggregate of the diameters of any circles, so is the circumference of the proposed circle to the aggregate of the circumferences of all of these circles, for this follows from this proposition and V.13 [of Euclid].[3]

[Corollary II.]

Hence, if the diameter of the proposed circle be equal to the aggregate of the diameters of any number of circles, so also the circumference of the proposed circle will be equal to the aggregate of the circumference of all those circles.[4]

[Corollary III.]

Further, it follows that the product of the diameter of the first circle and the circumference of the second circle is equal to the product of the diameter of the second circle and the circumference of the first. The same thing ought to be said regarding radii and circumferences.[5] For this fol-lows from the present proposition and VI.15 [of Euclid].[6]

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[3] See Com., Prop. VI, lines 31–37 and 34.
[4] Ibid., lines 35–37.
[5] Ibid., lines 38–40.
[6] Ibid., line 41.

[Corollary IV.]

Finally, if the diameters of three circles are continually proportional, then, since their circumferences are proportional in the same ratio, and accordingly, the radius of the first is to the radius of the second as the semicircumference of the second is to the semicircumference of the third, therefore, by VI.15 [of Euclid], it will result that the product of the radius of the first circle and the semicircumference of the third is equal to the product of the radius of the second and the semicircumference of the second and in the same way is equal to the second circle, the area of a circle being in fact produced from the product of the radius and the semi-circumference.[7]

Another Demonstration of the Same Sixth [Proposition].[8]

Let there be two circles AB, GD whose diameters are AB, GD.

I say that circumference AB / circumference GD = diameter AB / diameter GD.

For if it is not so, then diameter AB will be to diameter GD as circumference AB is to some circumference that is either less or more than circumference GD. First, let diameter AB / diameter GD = circumference AB / circumference EZ, where EZ is less than circumference GD and concentric to it. And by XII.13 [of Euclid] let a regular polygon GD be inscribed in circle GD but not touching EZ. And let [another] similar polygon be inscribed in circle AB. Because of the similarity of the polygons, perimeter of polygon AB / perimeter of polygon GD = diameter AB / diameter GD; and, therefore, perim. polyg. AB / perim. polyg. GD = circumference AB / circumference EZ. Alternately, circum. AB / perim. polyg. AB = circum. EZ / perim. polyg. GD. But circumference of circle AB > perimeter of polygon AB; therefore, circumference of circle EZ > perimeter of polygon GD, i.e., the "included" is greater than the "including," which is impossible.

Then let diameter AB / diameter GD = circumference AB / circumference HT, where HT > circumference GD. And, inversely, diameter GD / diameter AB = circumference HT / circumference AB; and thus let this ratio be as circumference GD to some circumference EZ. Then, alternately, circumference HT / circumference GD = circumference AB / circumference EZ. But circumference HT > circumference GD [by supposition]. Therefore circumference AB > circumference EZ. Hence, diameter GD / diameter AB = circumference GD / circumference EZ, where EZ > circumference AB. Hence the same impossibility follows as before. And so this ratio of diameter AB to diameter GD is not as the ratio of circumference AB to some circumference that is either greater than or less than circumference GD. The ratio therefore will be as circumference AB to circumference GD, which was to be demonstrated [here],

[7] *Ibid.*, lines 42–49.
[8] *Ibid.*, line 50.

although the same thing is demonstrated abstractly in the fifth [proposition] of the *Booklet on the Measurement of the Circle*.

Corollary.

It is evident, therefore, that the diameter of a given circle is related to the aggregate of the diameters of any number of circles as the circumference of the given circle is to the aggregate of the circumferences of all those circles. For this is obvious from the present [proposition] and V.13 [of Euclid]. Accordingly, if the diameter is equal to the aggregate of the diameters, the circumference is equal to the aggregate of the circumferences.[9]

Corollary.

Also, [it is evident] that the product of the diameter of a circle and the circumference of another circle is equal to the product of the diameter of this second circle and the circumference of the first. This is obvious from the present [proposition] and VI.15 [of Euclid].[10]

Proposition VII.

THE LATERAL SURFACE OF A TRUNCATED CONE IS EQUAL TO THE PRODUCT OF THE SLANT HEIGHT OF THE CONE AND ONE-HALF [THE SUM OF] THE CIRCUMFERENCES OF THE BASES.[1]

Let there be a [right] triangle ABG having its right angle at B with [line] DE parallel to BG [see Fig. III.5C.7]. When $\triangle ABG$ has been rotated once about axis AB, the cone AGZ with base circle GZ has been described by $\triangle ABG$, while cone AHE with base circle HE has been described by $\triangle ADE$. Now the difference of these cones, which evidently is described by trapezium $DBGE$, is called a truncated cone, and it has two unequal base circles, namely, GZ and EH.

And so I say that the lateral surface of the truncated cone, that is, the surface described by line EG, is equal to the product of line EG and ½ (circum. circle GZ + circum. circle EH).

For let there be set out $\triangle TKL$ having its right angle at K and its side TK equal to slant height AG, while side KL is equal to circumference GZ. Then, by the second [proposition] of this [work] the lateral surface of cone AGZ will be equal to $\triangle TKL$. Let KM be placed equal to GE and

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[9] This "corollarium" is equivalent to Corollaries I and II to the first demonstration of the proposition.

[10] This is equivalent to Corollary III to the first demonstration. See Com., Prop. VI, lines 38–40. Note that Corollary IV to the first demonstration is not given here. Prop. VI.15 = Gr. VI.16. See Com., Prop. VI, line 41.

Prop. VII
[1] See Com., Prop. VII, lines 2–3.

let MN be drawn parallel to KL; and, by the similitude of the triangles, $[KL / MN] = TK / TM = GA / AE$. Therefore, radius BG / radius DE $= KL / MN$. But, by the preceding [proposition], radius BG / radius DE $=$ circum. GZ / circum. EH. Therefore, $KL / MN =$ circum. GZ / circum. EH; and, alternately, circum. $EH / MN =$ circum. GZ / KL. But circumference $GZ =$ line KL. Therefore, circum. $HE =$ line MN. And since $TM = AE$, therefore by the second [proposition] of this [work] the [lateral] surface of cone AHE is equal to $\triangle TMN$. But the lateral surface of cone AZG was equal to $\triangle TKL$. Therefore, the lateral surface of truncated cone EZ is equal to the trapezium $KMNL$.

And so let KN and ML be drawn. By I.7 [of Euclid], $\triangle KML = \triangle KNL$. But $KM \cdot KL = 2 \triangle KML$. Therefore, $KM \cdot KL = 2 \triangle KNL$. But $KM \cdot MN = 2 \triangle KMN$. Hence, $KM \cdot (KL + MN) = (2 \cdot$ whole trapezium $KMNL)$. And so $KM \cdot \frac{1}{2} (KL + MN) =$ trapezium $KMNL$. Now KM was equal to EG, and KL to the circumference ZG, and MN to the circumference EH. Therefore, $EG \cdot \frac{1}{2} ($circum. $GZ +$ circum. $EH) =$ trapezium $KMNL$, and, therefore, it is equal to the lateral surface of truncated cone EZ, to which the trapezium was equal. Q.E.D.

Corollaries.

[I] And so the product of the sum of the radii of the bases of a truncated cone and a circumference whose diameter is the conical slant height is equal to the conical surface. This follows from the second and third corollaries of the second [proposition].

[II] Further, the circle whose radius is the mean proportional between the slant height of a truncated cone and the sum of the radii of its bases is equal to the conical surface. This follows from the last corollary of the sixth [proposition].[2]

Proposition VIII.

IF THE HALF OF A REGULAR POLYGON TERMINATED AT THE ENDS OF THE DIAMETER OF THE CIRCLE IN WHICH IT IS DESCRIBED IS ROTATED COMPLETELY WITH THE DIAMETER REMAINING FIXED, THE CONICAL SURFACES OF THE SOLID DESCRIBED WILL TOGETHER BE EQUAL TO THE PRODUCT OF A SIDE OF THE POLYGON AND [THE SUM OF] ALL THE CIRCUMFERENCES OF THE CIRCLES DESCRIBED BY THE ANGLES OF THE POLYGON.[1]

In circle AB having diameter AB let there be described a regular polygon, i.e., decagon $AGDEZBHTKL$ [see Fig. III.5C.8], and let [lines] GL, DK, ET, ZH be drawn to cut diamter AB at right angles in points M, N, X, and

[2] *Ibid.*, lines 39–44.
Prop. VIII
[1] See Com., Prop. VIII, lines 2–8.

O. Then let either half of the polygon (e.g. the half ADB) be rotated completely with diameter AB remaining fixed. In its rotation this half of the polygon will describe a certain solid of rotation ADB composed of diverse bodies: namely of (1) cones which triangles AGM and BZO describe, (2) the truncated cones which trapezia $MGDN$ and $OZEX$ describe and (3) the cylinder which rectangle $NDEX$ describes. These will all be described, I say, in the rotation of the semipolygon. If the number of sides of this semipolygon is an even number, then there will be no cylinder intervening in the composition of the solid described, for there will be no middle side parallel to the diameter as there is here with side ED parallel to AB and describing a cylinder in its rotation.

And so I say that the conical surfaces of the solid of rotation AB, i.e., the whole surface of the solid, is equal to the product of side AG and the [sum of the] circumferences of the circles described by lines MG, ND, XE and OZ.

For, by the second [proposition] of this [work], the conical surface which line AG describes is equal to the product of AG and one-half the circumference described by angle G, and by the same [proposition] the conical surface which line BZ describes is equal to the product of BZ (or AG) and one-half the circumference described by angle Z.[2] Also, by the preceding [proposition], the lateral surface of the truncated cone which line GD describes is equal to the product of side GD (or AG) and one-half the [sum of the] circumferences described by angles G and D; and, by the same [proposition], the surface of the truncated cone which line EZ describes is equal to the product of side EZ (or AG) and one-half the [sum of the] circumferences described by angles Z and E. Further, the cylindrical surface which line DE describes, by [Proposition] IV of this [work], is equal to the product of side DE and the circumference described by angle D or angle E, since each describes the circumference of the cylindrical base. And hence the said cylindrical surface is equal to the product of side DE (or AG) and one-half the [sum of the] circumferences described by angles E and D. And so [in sum] all of the complete circumferences described by angles G, D, E, Z are multiplied by a side. Hence, by II.1 of the *Elements* [the sum of] all of the conical surfaces described by the semipolygon ADB, i.e., the total surface of the solid described by the semipolygon, is equal to the product of side AG and [the sum of] all of the complete circumferences described by angles G, D, E, and Z.

I shall demonstrate this same proposition in regard to the solid described by a semipolygon whose number of sides will be an even number. But since in this case there is no cylinder, it will not be necessary to cite the fourth [proposition] of this [work] but only the second on behalf of the cones which always are formed at the ends of the stationary diameter and the preceding [proposition] on behalf of the intermediary truncated cones. Therefore, what is proposed, is true.

[2] *Ibid.*, lines 9–50.

Proposition IX.

IF THE HALF OF A REGULAR POLYGON INSCRIBED IN A CIRCLE AND TERMINATED AT THE ENDS OF A DIAMETER THAT REMAINS STATIONARY IS ROTATED UNTIL IT RETURNS TO THE POSITION FROM WHICH IT STARTED, THE WHOLE SURFACE OF THE SOLID DESCRIBED WILL BE EQUAL TO THE PRODUCT OF THE CIRCUMFERENCE CONTAINING THE POLYGON AND THE LINE WHICH TOGETHER WITH THE DIAMETER OF THE CIRCLE AND A SIDE OF THE POLYGON FORMS A RIGHT TRIANGLE IN THE CIRCLE.[1]

In circle AB having diameter AB, let there be described a regular polygon of any number of sides, e.g., a dodecagon whose angles are at points $A, G, D, E, Z, H, B, T, K, L, M, N$ [see Fig. III.5C.9]; and let lines GN, DM, EL, ZK and HT be drawn so as to cut the diameter AB orthogonally at points X, O, P, R and S. Also let lines DN, EM, ZL, and HK be drawn to cut the said diameter at points Y, F, C and Q. Also let line BG be drawn, forming with side GA a right angle, by III.29 [of Euclid].[2] And with semipolygon AEB rotated through a complete turn on axis AB, let the solid of rotation be described, just as in the preceding [proposition].

I say that the whole surface of this sort of solid, i.e., the aggregate of the conical surfaces described by the sides of the semipolygon, is equal to the product of line BG and circumference AB of the circle.

For by the preceding [proposition], the product of AG and [the sum of] all the circumferences described by angles G, D, E, Z and H is equal to the whole surface of the solid described by the semipolygon, which demonstration, if it be pleasing, repeat as follows:

The conical surface described by side

$$\begin{aligned}
AG &= AG \cdot \tfrac{1}{2} \text{ circum. of diam. } GN \\
GD &= AG \cdot \tfrac{1}{2} \text{ (circum. of diam. } GN + \text{circum. of diam. } DM) \\
DE &= AG \cdot \tfrac{1}{2} \text{ (circum. of diam. } DM + \text{circum. of diam. } EL) \\
EZ &= AG \cdot \tfrac{1}{2} \text{ (circum. of diam. } EL + \text{circum. of diam. } ZK) \\
ZH &= AG \cdot \tfrac{1}{2} \text{ (circum. of diam. } ZK + \text{circum. of diam. } HT) \\
HB &= AG \cdot \tfrac{1}{2} \text{ circum. of diam. } HT
\end{aligned}$$

And this is by Propositions VII and II. Therefore, by II.1 of the *Elements*, the whole surface of the solid described by the semipolygon is equal to the product of the side AG and all of the circumferences whose diameters are GN, DM, EL, ZK and HT. But triangles GAX, $NQX, DQO, MCO, ECP, LFP, ZFR, KRY, HYS$ and TBS are similar to $\triangle BGA$, since evidently they are right triangles and the angles intercepting equal arcs are equal by III.26 [of Euclid];[3] and hence the remaining angles are equal.

Prop. IX
[1] See Com., Prop. IX, lines 2–9.
[2] *Ibid.*, line 16.
[3] *Ibid.*, line 37.

And so by VI.4 [of Euclid]:

$$\frac{BG}{GA} = \frac{GX}{XA} = \frac{XN}{XQ} = \frac{DO}{QO} = \frac{OM}{OC} = \frac{EP}{CP} = \frac{PL}{PF} = \frac{ZR}{FR} = \frac{RK}{RY} = \frac{HS}{YS} = \frac{ST}{SB}.$$

Hence by V.13 [of the *Elements*]:

$$\frac{BG}{GA} = \frac{(GN + DM + EL + ZK + HT)}{AB}.$$

And, therefore, by [the first] corollary of Proposition VI of this [work] BG is to GA as the circumference [of the circle] whose diameter is the sum of $GN + DM + EL + ZK + HT$ is to the circumference whose diameter is AB. But the circumference whose diameter is the sum of $GN + DM + EL + ZK + HT$ by the [second] corollary of Proposition VI of this [work] is equal to the sum of the circumferences whose diameters are GN, DM, EL, ZK and HT. Therefore, BG is to GA as the sum of the circumferences whose diameters are GN, DM, EL, ZK and HT is to the circumference whose diameter is AB. Therefore, by VI.15 of the *Elements*, $(BG \cdot \text{circum. of diam } AB) = (GA \cdot [\text{circum. of diam. } GN + \text{circum. of diam. } DM + \text{circum. of diam. } EL + \text{circum. of diam. } ZK + \text{circum. of diam. } HT])$. But, by the preceding [proposition], $(GA \cdot [\text{circum. of diam. } GN + \text{circum. of diam. } DM + \text{circum. of diam. } EL + \text{circum. of diam. } ZK + \text{circum. of diam. } HT])$ is equal to the surface of the solid described by semipolygon AEB. Therefore, the product of BG and the circumference whose diameter is AB is equal to the surface of the solid described by semipolygon AEB. Q.E.D.

Corollary I.

Whence the product of diameter AB and the circumference of the circle whose diameter is BG will be equal to the surface of the same solid of rotation. This indeed follows from this proposition and the third corollary of the sixth [proposition]. Also the circle whose radius is the mean proportional between lines AB and BG is equal to the surface of the above-mentioned solid of rotation. This follows from the last corollary of the sixth [proposition].[4]

Corollary II.

Hence it is obvious that the surface of the solid described by semipolygon AEB is equal to the cylindrical surface whose axis is line BG and base is circle AB. For by [Proposition] IV of this [work], the cylindrical surface is equal to the product of the axis and the circumference of the base.

[4] *Ibid.*, lines 64–69.

Corollary III.

Also the product of the perpendicular from the center of the circle to the side of the polygon and the circumference of the circle containing the polygon is one-half the whole surface of the solid described by the semi-polygon rotated once on the diameter.

For the perpendicular drawn from center P to side AG is one-half of BG just as AP is one-half AB. And so the product of the said perpendicular and circumference AB is one-half the product of BG and circumference AB and therefore is one-half the surface of the aforesaid solid described by the semipolygon.

Proposition X.

THE SURFACE OF A SPHERE IS EQUAL TO THE RECTANGLE ARISING FROM THE PRODUCT OF THE DIAMETER OF THE SPHERE AND THE CIRCUMFERENCE OF ITS GREATEST CIRCLE.[1]

Let there be circle $ABGD$ with center E and diameter AG; and, with either of the semicircles (e.g. semicircle ABG) rotated once while the diameter AG stands still, let sphere ABG be described [see Fig. III.5C.10].

I say that the surface of sphere ABG is equal to the product of diameter AG and the circumference of circle ABG.

For, if the product of AG and the circumference of circle ABG is not equal to the surface of sphere ABG, it will be equal to the surface of some sphere greater or less than sphere ABG. Hence, first let it be equal to the surface of sphere ZHT, less than sphere ABG and concentric with it, and with diameter ZT. This sphere semicircle ZHT describes when rotated about diameter ZT. And, by XII.13 [of Euclid][2], let a regular polygon be inscribed in circle ABG but not at all touching circle ZHT, which polygon let be $ABGD$, one of whose sides is AK. And with GK drawn and with half of the polygon rotated about diameter AG, let a solid of rotation [composed] of conical surfaces be described which does not at all touch sphere ZHT. And, by the preceding [proposition], the product of GK and the circumference of circle ABG will be equal to the surface of the whole solid ABG described by the semipolygon. But the product of diameter AG and the circumference of circle ABG is greater than the product of GK and the same circumference. Therefore, the product of diameter AG and the circumference of circle ABG (and hence the surface of sphere ZHT) will be greater than the surface of the solid described by semipolygon ABG, that is, the "included" will be greater than the "including," which is impossible.

Prop. X

[1] See Com., Prop. X, lines 2–4.
[2] *Ibid.*, line 15.

Or as follows: since the product of diameter AG and the circumference of circle ABG is greater than the surface of the solid described by semipolygon ABG, and this is greater than the surface of the included sphere ZHT, therefore the product of diameter AG and the circumference of circle ABG is greater than the surface of sphere ZHT; therefore, it is not equal to it as was supposed.

Then let the product of AG and the circumference of circle ABG be equal to the surface of a sphere greater than sphere ABG; but for the sake of brevity let the proposed sphere be ZHT.

I say that the product of diameter ZT and the circumference of circle ZHT will not be equal to the surface of a sphere greater than sphere ZHT.

For, if it is possible, let it be equal to the surface of sphere ABG concentric to ZHT, the sphere which semicircle ABG describes about diameter AG; and, as before, let a regular polygon be inscribed in circle ABG and not touching circle ZHT. Let one of its sides be AK, and let GK be drawn, and EL perpendicular to AK. And, with the semipolygon ABG rotated about diameter AG, let a solid of rotation [composed] of conical surfaces be described, which does not at all touch sphere ZHT. Now $GK = 2\,EL$ just as $AG = 2\,AE$. But $ZT = 2\,ZE$ and $ZE < EL$. Therefore $GK > ZT$. Hence, $(GK \cdot$ circum. circ. $ABG) > ($diam. $ZT \cdot$ circum. circ. $ZHT)$. But, by the preceding [proposition], the product of GK and the circumference of circle ABG is equal to the surface of the solid described by semipolygon ABG. Now, by hypothesis, (diam. $ZT \cdot$ circum. circ. ZHT) = surface sphere ABG. Therefore, the surface of the solid described by semipolygon ABG is greater than the surface of sphere ABG, i.e. the "included" is greater than the "including," which is impossible. Therefore, the product of diameter ZT and circumference ZHT is not equal to the surface of some sphere greater than sphere ZHT. Similarly, I may demonstrate that the product of diameter AG and the circumference of circle ABG is not equal to the surface of some sphere greater than sphere ABG. But neither [was it equal to the surface of a sphere] less [than ABG], as was demonstrated. Therefore, it remains that the product of diameter AG and the circumference ABG is equal to the surface of sphere ABG. Q.E.D.

Corollary.

It is evident, therefore, that the surface of a sphere is equal to a circle whose radius is equal to the diameter of the sphere.[3] For, by [Proposition] IV of *On the Measurement of the Circle*, the area of such a circle is equal to the product of its radius, which is the diameter of the sphere, and one half of its circumference, which by [Proposition] VI of this [work] is equal to the circumference of the greatest circle in the sphere.

[3] *Ibid.*, lines 60–61.

Proposition XI.

THE SURFACE OF A SPHERE IS QUADRUPLE ITS GREATEST CIRCLE AND IS EQUAL TO THE LATERAL SURFACE OF THE CYLINDER WHOSE AXIS AS WELL AS ITS BASE DIAMETER IS EQUAL TO THE DIAMETER OF THE SPHERE.[1]

Let there be sphere *ABG*, which semicircle *ABC* describes when it is rotated and diameter *AG* stands still; and let its center be *D* [see Fig. III.5C.11].

I say that the surface of sphere *ABG* = 4 circle *ABG*.

For, by the preceding [proposition], surface of sphere *ABG* = (diam. *AG* · circum. circ. *ABG*). Now, by Proposition IV of the *Booklet on the Measurement of the Circle*, the area of the circle is equal to the product of radius *DG* and one-half the circumference of the whole circle *ABG* and, by VI.18 [of Euclid],[2] the product of diameter *AG* and the circumference of circle *ABG* is quadruple the product of radius *DG* and the circumference of semicircle *ABG* (since each of the former factors is double each of the latter). Therefore, the surface of sphere *ABG* is quadruple *ABG*, the greatest circle in the sphere, which is the first of the things we proposed. I prove the second as follows:

Let there be a cylinder whose axis as well as base diameter is equal to diameter *AG*.

I say that the surface of sphere *ABG* is equal to the lateral surface of a cylinder of this sort.

For the base of such a cylinder will be circle *ABG*; hence, by Proposition IV of this [work], the lateral surface of this cylinder will be equal to the product of diameter *AG* and the circumference of circle *ABG*. But this [latter product] is equal to the surface of the sphere. Therefore, the spherical surface will be equal to the lateral surface of such a cylinder, which had remained to be demonstrated.

Corollary.

It is evident, therefore, that the total surface of the cylinder whose axis as well as base diameter is equal to the diameter of the sphere is to the surface of the sphere as 3 is to 2.[3] For the two bases of the cylinder are [equal to] one-half of the spherical surface; and, since the spherical surface is quadruple to one of those [base circles] and the lateral surface of the cylinder is equal the spherical surface, therefore the total surface of the cylinder, which consists of the lateral surface and the bases, contains the spherical surface one and one-half times.

Prop. XI

[1] See Com., Prop. XI, lines 2–5.

[2] *Ibid.*, line 12.

[3] *Ibid.*, lines 28–30.

Proposition XII.

IF THERE ARE TWO REGULAR AND SIMILAR FIGURES [I.E. POLYGONS] AND ONE IS INSCRIBED IN AND THE OTHER CIRCUMSCRIBED ABOUT A CIRCLE, AND IF, WITH THE DIAMETER STANDING STILL, THE SEMICIRCLE AS WELL AS THE HALVES OF THE FIGURES [I.E. SEMIPOLYGONS] ARE ROTATED UNTIL THEY RETURN TO THEIR [INITIAL] POSITION, THE SURFACE OF THE SPHERE DESCRIBED BY THE SEMICIRCLE IS THE MEAN PROPORTIONAL BETWEEN THE SURFACES OF THE SOLIDS DESCRIBED BY THE HALVES OF THE FIGURES.[1]

Let a regular polygon be described in circle ABG with diameter AG and center D [see Fig. III.5C.12]. Let one of its sides be AB. Let arc AB be bisected in point E, and let radius DE be drawn so as to bisect chord AB in point Z. [Further] let HET be drawn orthogonally, [i.e.] at right angles, to this DE. And, therefore, by III.15 [of Euclid][2] it is tangent to the circle at point E and it meets radii DA and DB extended at points H and T. [Then] TH will be the side of regular polygon circumscribing circle ABG and similar to inscribed polygon ABG. And so let [the exterior] circumscribed figure be completed, and let it be THK. And let semicircle ABG as well as semipolygons ABG and THK be rotated once on axis TK which stands still.

And so I say that the surface of the sphere which semicircle ABG describes is the mean proportional between the surfaces of the solids which semipolygons ABG and THK describe. This I shall demonstrate as follows.

Let perpendicular EL drop from point E to DB and [thus] triangles DEL and DBZ will be mutually equilateral; therefore, $DZ = DL$. But by the third corollary of [Proposition] IX of this [work], from the product of EZ (or hence DL) and the circumference ABG arises one-half of the surface of the solid described by semipolygon ABG. But, by [Proposition] X of this [work], from the product of radius DB and the circumference of semicircle ABG arises one-half the surface of the sphere described by semicircle ABG. Further, by the third corollary of [Proposition] IX [of this work], from the product of DE and the circumference of circle THK circumscribing the polygon [THK] (and, hence from the product of DH and the circumference of circle ABG, for these two products are equal by the second corollary of [Proposition] VI of this [work]) arises one-half the surface of the solid described by semipolygon THK. Therefore, these three products [namely (1) $DL \cdot$ circum. ABG, (2) $DB \cdot$ circum. ABG, and (3) $DH \cdot$ circum. ABG], which are [respectively] the halves of the said surfaces, are, by VI.1 [of Euclid], mutually as the lines DL, DE, and DH, the bases of the products, for the [common] altitude [of the three products]

is equal to the circumference of circle *ABG*. But *DL*, *DE* and *DH* are continual proportionals because of the similitude of triangles *DEL* and *DHE*. Therefore, the three products, which are the halves of the said surfaces, are continual proportionals. Therefore, their doubles, i.e., the whole surfaces, are continual proportionals. And so the surface of the solid described by semipolygon *ABG*, the surface of the sphere described by semicircle *ABG* and the surface of the solid described by semipolygon *THK* are continuous proportionals in the proportion evidently of these lines *DL*, *DE* and *DH*. Q.E.D.

Scholium.

In the case of two circles one of which is circumscribed about and the other inscribed in a regular polygon, and, with the diameter remaining fixed, if the semipolygon as well as the two semicircles is rotated, the surface of the solid described by the semipolygon is the mean proportional between the surfaces of the spheres described by the semicircles.

For to the data of the preceding [demonstration] let there be added circle *THK* and then let a sphere be described by the rotation of semicircle *THK*.

I say that the surface of solid *THK* is the mean proportional between the surfaces of spheres *ABG* and *THK*.

For, by [Proposition] X, from the product of straight line *HD* and the circumference of circle *THK* arises half the surface of sphere *THK*, and, as in the preceding [proposition], it has been demonstrated that from the product of line *HD* and the circumference of circle *ABG* arises half of the surface of solid *THK*. Hence, by VI.1 [of Euclid], the surface of sphere *THK* will be to the surface of solid *THK* as the circumference of circle *THK* is to the circumference of circle *ABG*, and, therefore, by [Proposition] VI [of this work], as the radius *DH* is to the radius *DE*. But, in the preceding proof, *DH* / *DE* = surf. solid *THK* / surf. sphere *ABG*. Therefore, surf. solid *THK* / surf. sphere *ABG* = surf. sphere *THK* / surf. solid *THK*. And so the surface of solid *THK* is the mean proportional between the surfaces of spheres *THK* and *ABG*. Q.E.D.

Corollary.

It is evident, therefore, that if a regular polygon is inscribed in a circle and then a circle is inscribed in that polygon and further [another] polygon with corresponding angles in the [second] circle, and this process [of alternate inscription] is repeated as many times as you please, and if the semicircles as well as the semipolygons are rotated while the diameter remains fixed, the surfaces of the described spheres and solids of rotation are continuously proportional following their order of inscription.

To this point [we have treated] of the surface of the solid of rotation and the sphere; now we shall speak of the surfaces of segments cut from the said solids.

Proposition XIII.

THE CONICAL SURFACES OF THE SEGMENT INCLUDED BETWEEN THE APEX OF THE SOLID DESCRIBED BY A REGULAR SEMIPOLYGON AND ONE OF THE CIRCLES DESCRIBED BY THE ANGLES ARE TOGETHER EQUAL TO THE PRODUCT OF (1) A SIDE OF THE CIRCUMSCRIBED [POLYGON] AND (2) THE [SUM OF THE] CIRCUMFERENCES DESCRIBED BY THE ANGLES MINUS HALF OF THE CIRCUMFERENCE OF THE CIRCLE CUTTING THE SEGMENT FROM THE SOLID.[1]

For within circle AB with diameter AB let there be described regular polygon AGB [see Fig. III.5C.13]. By rotating half of this polygon once about fixed diameter AB let a solid of rotation be described. From this solid let there be taken the segment included between apex A and one of the circles described by the angles, for example, the circle described by angle E. This segment then is AEZ.

I say that the conical surfaces of segment AEZ, which evidently are those described by sides AG, GD and DE, are together equal to $AG \cdot$ (circum. descr. by angle G + circum. descr. by angle D + circum descr. by angle $E - \frac{1}{2}$ circum. descr. by angle E). For let GT, DH and EZ be drawn so as to cut diameter AB at K, L, and N. Moreover, they will intersect it at right angles. And so, by [Proposition] II of this [work], the conical surface which line AG describes is equal to the product of AG and one-half the circumference which point G describes. Also, by [Proposition] VII, the conical surface of the truncated cone which line GD describes is equal to the product of GD and one-half the [sum of the] circumferences described by points G and D. Also, by [Proposition] VII, the conical surface which DE describes (or, by [Proposition] IV, if $LDEM$ is a parallelogram, the cylindrical surface described) is equal to the product of DE (or AG) and half the [sum of the] circumferences described by D and E. Here then these add up to the whole circumferences described by points G and D plus one-half of that described by the final point E. Hence, by II.1 of the *Elements*, all the conical surfaces described by lines AG, GD and DE, that is, the surface of segment AEZ, are equal to the product of side AG and of all the circumferences described by angles G, D, E less one-half of the circumference described by the lowest angle E (which last circumference is the circumference of the circle cutting segment AEZ from the whole solid described by the semipolygon). Q.E.D.

This same [conclusion] is demonstrated even if the chord AG is not the side of a regular polygon inscribed in the circle so long as AG, GD and DE are equal.

Proposition XIV.

THE CONICAL SURFACES OF THE SEGMENT INCLUDED BETWEEN THE APEX OF THE SOLID DESCRIBED BY A REGU-

Prop. XIII
[1] See Com., Prop. XIII, lines 2–8.

LAR SEMIPOLYGON AND ONE OF THE CIRCLES DESCRIBED
BY THE ANGLES ARE TOGETHER EQUAL TO THE PRODUCT OF
(1) THE CIRCUMFERENCE OF A CIRCLE WHOSE DIAMETER
IS THE AXIS OF THIS SEGMENT AND (2) THE LINE WHICH
TOGETHER WITH THE DIAMETER OF THE CIRCLE CONTAIN-
ING THE POLYGON AND THE SIDE OF THE POLYGON FORMS A
RIGHT TRIANGLE IN THE CIRCLE.[1]

What will be necessary for the description? I assume the whole descrip-
tion of [Proposition] IX, but from the solid which semipolygon AEB
describes I take a segment whose apex is A and whose base is some one
of the circles described by the angles of the polygon, e.g., the circle
whose diameter is DM. Let this segment be designated DAM [see
Fig. III.5C.14].

And so I say that the conical surfaces of segment DAM, which evidently
are those described by lines AG and GD, are together equal to the product
of line BG and the circumference of a circle whose diameter is AO (and
AO I call the axis of segment DAM).

The demonstration of this is almost the same as the demonstration of
[Proposition] IX. For, as there, triangles GAX, NQX and DQO are similar
to $\triangle BGA$, that is they are equiangular [with it]. Hence, by VI.4 [of Euclid],
$BG / GA = GX / AX = XN / XQ = DO / QO$. And, therefore, by V.13
[of Euclid][2] $BG / GA = (GN + DO) / AO$. And so by a corollary of
[Proposition] VI of this [work], $BG / GA =$ circum. of diam. $(GN + DO) /$
circum. of diam. AO. But, by a corollary of [Proposition] VI, the cir-
cumference whose diameter is the sum of GN and DO is equal to the sum
of the circumferences whose diameters are GN and DO. Therefore, BG
is to GA as the sum of the circumferences of diameters GN and DO
is to the circumference of diameter AO. But, by a corollary of [Proposition]
VI, the circumference of diameter DO is equal to one-half the circum-
ference of diameter DM, since the diam. $DO = \frac{1}{2}$ diam. DM. Therefore,
$BG / GA = ($circum. of diam. $GN + \frac{1}{2}$ circum. of diam. $DM) /$ circum.
of diam. AO. Hence by VI.15 of the *Elements*,[3] $BG \cdot ($circum. of diam.
$AO) = GA \cdot ($circum. of diam. $GN + \frac{1}{2}$ circum. of diam. $DM)$. But by the
preceding [proposition] $GA \cdot ($circum. of diam. $GN + \frac{1}{2}$ circum. of diam.
$DM) =$ the conical surfaces of solid segment DAM. Therefore, $BG \cdot ($cir-
cum. of diam. $AO) =$ the conical surfaces of solid segment DAM. Q.E.D.

I shall show in the same way that line $BG \cdot ($circum. of diam. $AP) =$ the
conical surfaces of solid segment EAL, nor shall I show differently that
$BG \cdot ($circum. of diam. $AR) =$ the conical surfaces of solid segment ZAK,
since the axis of segment EAL is this AP, while the axis of segment
ZAK is this line AR. Therefore that which is proposed is true.

This same thing will be demonstrated even if chord AG is not the side

Prop. XIV
[1] See again Com., Prop. XIII, lines 2–8.
[2] See Com., Prop. XIV, line 22.
[3] *Ibid.*, lines 33–34.

of a regular polygon inscribed in circle AB so long as the sides of the segment are equal.

Corollaries.

[I] Therefore, (line $AO \cdot$ circum. of diam. BG) = the conical surfaces of DAM. This follows from the third corollary of [Proposition] VI. [II] Further, a circle whose radius is the mean proportional between lines AO and BG is equal to those same conical surfaces of segment DAM.[4] This follows from the fourth corollary of [Proposition] VI.

Proposition XV.

IF A SPHERE IS CUT BY A PLANE, THE SPHERICAL SURFACE OF EACH SEGMENT IS EQUAL TO THE RECTANGLE WHICH ARISES FROM THE PRODUCT OF THE DIAMETER OF THE SPHERE AND THE CIRCUMFERENCE OF THAT CIRCLE WHOSE DIAMETER IS THE AXIS OF THE SEGMENT.[1]

Let there be circle $ABGD$, whose diameter is AG and center is E [see Fig. III.5C.15]. Further, let diameter AG be cut at right angles by line BD in any point Z. And with either semicircle (e.g. ABG) rotated once about fixed axis AG, let sphere $ABGD$ be described. Hence in such a rotation line ZB describes a circle which will cut the sphere into two segments: BAD, BGD.

And so I say that the spherical surface of either segment (e.g. of segment BAD) is equal to the product of diameter AG and the circumference of a circle whose diameter is AZ designated as the axis of segment BAD; the demonstration of this is similar to the demonstration of [Proposition] X.

For if the product of diameter AG and the circumference of a circle whose diameter is AZ is not equal to the surface of spherical segment ABD, let it be equal to the spherical surface of some segment greater or less than the surface of segment BAD. And first [let it be equal to a spherical surface] less than [that of BAD], namely to the spherical surface of segment THL cut off from sphere $HTKL$ by means of the plane of the said circle, which sphere semicircle HTK describes about diameter HK and which sphere is concentric to sphere ABG. And let arcs AB and AD be divided repeatedly until, by XII.13 [of Euclid],[2] the chords of the arcs do not touch circumference THL. And let one of the chords be AM, and let GM be joined. And the product of diameter AG and the circumference of the circle of diameter AZ will be greater than the product of GM (which is less than the diameter [AG]) and the circumference

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[4] *Ibid.*, lines 53–55.

Prop. XV

[1] See Com., Prop. XV, lines 2–5.

[2] See Com., Prop. X, line 15.

of the circle of diameter AZ. But the product of diameter AG and the circumference of the circle of diameter AZ is by hypothesis equal to the surface of spherical segment THL; and the product of GM and the circumference of the circle of diameter AZ is, by the preceding [proposition], equal to the conical surfaces described by the chords of the arcs into which the greater arc AB is cut. Therefore, the spherical surface of segment THL is greater than the conical surfaces described by the chords of the parts of arc AB. Therefore, the "included" surface is greater than the "including," which is impossible.

Or [proceed] as follows: Since the product of diameter AG and the circumference of the circle of diameter AZ is greater than the product of GM and that circumference and therefore is greater than the conical surfaces described by the chords of the parts of arc AB, and [since] these surfaces are greater than the included spherical surface of segment THL, therefore the product of diameter AG and the circumference of the circle of diameter AZ is greater than the spherical surface of segment THL. Therefore, it is not equal to it, as was supposed.

Then let the product of diameter AG and the circumference of the circle of diameter AZ be equal to a spherical surface greater than the spherical surface of segment BAD. But, for the sake of brevity, let the supposed segment be THL [in the same figure as before].

I say that the product of diameter HK and the circumference of the circle of diameter HZ is not equal to the spherical surface of some segment greater than the spherical surface of segment THL.

For, if it is possible, let it be equal to the spherical surface of segment BAD which is greater than the surface [of segment] THL cut off from sphere $ABGD$ by means of the plane of the circle cutting sphere $[A]B[G]D$ [as well as sphere $HTKL$, which it originally cut]. This sphere $ABGD$ is the one described by semicircle ABG on diameter AG [and is concentric with sphere $HTKL$]. And, by XII.13 [of Euclid][3] let the arcs AB and AD be divided [repeatedly] until the chords of the parts do not touch arc THL. And let one of the chords be AM, let GM be joined, and let EN be drawn perpendicular to AM. And since $AG = 2AE$ and hence $GM = 2EN$ (with $HK = 2HE$ and $HE < EN$), therefore $HK < GM$. Hence the product of GM and the circumference of the circle of diameter HZ is greater than the product of HK and the circumference of the circle of diameter HZ. But the product of GM and the circumference of the circle of diameter AZ is, by the preceding [proposition], equal to the conical surfaces described by the chords of the arcs into which arc AB is divided. Now the product of HK and the circumference of the circle of diameter HZ is by hypothesis equal to the spherical surface of segment BAD. Therefore, the conical surfaces described by the chords of the parts of arc AB are greater than the spherical surface of segment BAD. And so the included surface is greater than the including, which is impossible.

[3] *Ibid.*

Therefore, it is not [true] that the product of *HK* and the circumference of the circle of diameter *HZ* is equal to the spherical surface of some segment greater than the spherical surface of segment *THL*. In the same way, I shall demonstrate that the product of *AG* and the circumference of the circle of diameter *AZ* is not equal to the spherical surface of some segment greater than the spherical surface of segment *BAD*. And it was demonstrated [that the said product is] not [equal] to [a spherical surface] less [than that of segment *BAD*]. Therefore, [speaking] generally, the product of *AG* and the circumference of the circle of diameter *AZ* is equal to the spherical surface of segment *BAD*. Q.E.D.

Now we have supposed the spherical segment *BAD* to be less than a hemisphere. But it can be shown in the same way that the product of *AG* and the circumference of the circle of diameter *GZ* is equal to the spherical surface of segment *BGD*. Or if you please, let line *BE* be drawn in place of line *BZ* and *ED* in place of line *ZD*, these lines cutting the circumference of circle *HTK* in points *X* and *O*. And the demonstration is effected through the spherical segments described by the similar arcs *BG* and *XK* and through the conical surfaces described by the chords of the parts of arc *BG* not touching the circumference *XK*. But in the second part of the demonstration *XO* is drawn to cut *EZ* at [point] *P* and *GZ* will be the axis of segment *BGD* greater than the axis *KP* of segment *XKO*, which is necessary for the second part of the demonstration. Whence, concluding in the same way as before, the product of *AG* and the circumference of the circle of diameter *GZ* is more than every segmental spherical surface described by an arc less than *BG* and similar to it, and it is less than every segmental spherical surface described by an arc greater than *BG* and similar to it and, therefore, it is equal to the segmental spherical surface described by arc *BG*.

But just as you will have demonstrated this for one of the spherical segments, so accordingly can it be easily demonstrated for the remaining [segment] as follows.

Since it has been demonstrated that the product of *AG* and the circumference of the circle of diameter *AZ* is equal to the spherical surface of segment *BAD*, from this I shall demonstrate that the product of *AG* and the circumference of the circle of diameter *GZ* will be equal to the spherical surface of segment *BGD*. For, by [Proposition] X of this [work], the whole surface of sphere *ABG*, which is the aggregate of the spherical surfaces of the two segments *BAD*, *BGD*, is equal to the product of *AG* and the circumference of circle *ABG*. But, by the first corollary of [Proposition] VI, the circumference of circle *ABG* is equal to the sum of the circumferences of the circles whose diameters are *AZ* and *ZG*. Therefore, the aggregate of the spherical surfaces of the segments *BAD* and *BDG* is equal to the product of *AG* and the circumferences of the circles whose diameters are *AZ* and *ZG*. Thence if the spherical surface of segment *BAD* is subtracted, or if its demonstrated equal is subtracted, namely the product of *AG* and the circumference of the circle of diameter

AZ, then the remaining spherical surface of segment *BDG* is equal to the product of *AG* and the circumference of the circle of diameter *ZG*, and this was proposed.

Corollary I.

It is evident, therefore, that the spherical surface of a spherical segment is equal to the product of the axis of the segment and the circumference of the greatest circle in the sphere and, therefore, to the lateral surface of a cylinder whose base is the greatest circle of the sphere and whose altitude is the axis of the segment. This is obvious from the third corollary of [Proposition] VI.

Corollary II.

Also, the circle whose radius is the mean proportional between the axis of a spherical segment and the diameter of the sphere is equal to the spherical surface of this segment, which is clear from the fourth corollary of [Proposition] VI.[4] From this [present] corollary, the following sixteenth proposition will be easily demonstrated.

Proposition XVI.

IF A SPHERE IS CUT BY A PLANE, THE SURFACE OF EACH SEGMENT IS EQUAL TO THE PRODUCT OF (1) THE LINE DRAWN FROM THE APEX OF THIS SEGMENT TO THE CIRCUMFERENCE OF THE CUTTING CIRCLE AND (2) THE CIRCUMFERENCE OF THE CIRCLE OF WHICH THE SAID LINE IS THE DIAMETER. AND, THEREFORE, THIS SPHERICAL SURFACE OF THE SEGMENT IS EQUAL TO THE CIRCLE OF WHICH THE SAID LINE IS THE RADIUS.[1]

Let diameter *AG* of circle *ABGD* cut line *BD* at right angles in any point *Z*; and, with semicircle *ABG* rotated once about diameter *AG*, let sphere *ABGD* be described [consult previous Figure III.5C.15]. In this rotation line *ZB* will describe the circle which is the common boundary of the two spherical segments *BAD* and *BGD*. And so let straight lines *AB* and *BG* be drawn from the apexes of the spherical segments to the circumference of the cutting circle whose diameter is *BD*.

I say that the spherical surface of segment *BAD* is equal to the product of *AB* and the circumference of the circle of which the diameter is *AB*. For, by VI.8 [of Euclid], *AG* / *BA* = *BA* / *AZ*. Therefore, by [Proposition] VI of this work, *GA* / *AB* = circum. of diam. *AB* / circum. of diam. *AZ*. Hence, by VI. 15 [of Euclid],[2] (*AB* · circum. of diam. *AB*) = (*AG* · cir-

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⁴ See Com., Prop. XV, lines 111–14.
Prop. XVI
¹ See Com., Prop. XVI, lines 2–8.
² See Com., Prop. XIV, lines 33–34.

cum. of diam. *AZ*). But by the preceding [proposition], the product of *AG* and the circumference of diameter *AZ* is equal to the spherical surface of segment *BAD*. Therefore, the spherical surface of segment *BAD* is equal to the product of *AB* and the circumference of diameter *AB*. And this is the first of the things proposed.

And since the circumference of diameter *AB* is one-half the circumference of radius *AB* by [Proposition] VI, hence from the product of *AB* and half of the circumference of radius *AB* also arises the spherical surface of the segment *BAD*. But from the product of *AB* and one-half the circumference of radius *AB* arises a circle whose radius is *AB*, by [Proposition] IV of *On the Measurement of the Circle*. And so the circle of radius *AB* is equal to the spherical surface of segment *BAD*, which is the second of the things proposed.

By methods that are completely the same, I shall show that the spherical surface of segment *BGD* is equal to the product of *BG* and the circumference of the circle whose diameter is *BG*, or to the circle whose radius is *BG*.

But if this has been demonstrated for one of the segments, e.g. *BAD*, the same can be demonstrated for the remaining [segment] as follows. Since ∠*ABG* is a right angle, by III.30 [of Euclid],[3] therefore, by [Proposition] IX of *On the Measurement of the Circle*, two circles of radii *AB* and *BG* are equal to the circle of radius *AG*. But, by [Proposition] X of this [work], this [last] circle is equal to the [whole] spherical surface and hence to the spherical surfaces of segments *BAD* and *BDG*. Therefore, the circles of radii *AB* and *BG* are equal to the spherical surfaces of segments *BAD* and *BDG*. But the circle of radius *AB* was equal to the spherical surface of segment *BAD*. Therefore, the circle of radius *BG* remains equal to the spherical surface of segment *BDG*. Q.E.D.

Corollary.

It is evident, therefore, that the spherical surface of a spherical segment is equal to the circle whose radius is the line from the apex of the segment to the circumference of the circle cutting the sphere. For, by [Proposition] VI, the circumference of the circle of diameter *AB* is equal to one-half the circumference of radius *AB*. Therefore, the product of *AB* and the circumference of diameter *AB* is equal to the product of *AB* and one-half the circumference of radius *AB*. But the product of *AB* and the circumference of diameter *AB* is equal to the spherical surface of segment *BAD*. Now the product of *AB* and one-half the circumference of radius *AB* is, by [Proposition IV of *On the Measurement of the Circle*, equal to the circle of radius *AB*. Therefore, the spherical surface of segment *BAD* is equal to the circle of radius *AB*, which is the thing proposed. I shall demonstrate no differently that the spherical surface of segment *BGD* is equal to the circle of radius *BG*.

³ See Com., Prop. XVI, lines 37–38.

Proposition XVII.

IF A SPHERE IS CUT BY A PLANE, THE SPHERICAL SURFACES OF THE SEGMENTS ARE MUTUALLY AS THE AXES OF THE SEGMENTS.[1]

Repeating the preceding description [i.e. of Fig. III.5C.15 but see Fig. III.5C.15A], I say that the spherical surface of segment BAD is to the spherical surface of segment BGD as axis AZ is to axis ZG.

For, by VI. 8 [of Euclid], $AZ \mid ZG = (AZ \mid ZB) \cdot (AZ \mid ZB)$: and $AZ \mid ZB = AB \mid BG$ (because of the similarity of triangles AZB and ABG). Therefore, $AZ \mid ZG = (AB \mid BG) \cdot (AB \mid BG)$. And, by VI.18 [of Euclid] $AB^2 \mid BG^2 = (AB \mid BG) \cdot (AB \mid BG)$. Therefore, $AB^2 \mid BG^2 = AZ \mid ZG$. But by XII.2 [of Euclid], circle of radius AB / circle of radius $BG = (2AB)^2 \mid (2BG)^2 = AB^2 \mid BG^2$. Therefore, circle of radius AB / circle of radius $BG = AZ \mid ZG$. But, by the corollary of the preceding [proposition], the circle of radius AB is equal to the spherical surface of segment BAD. [And the circle of radius BG is equal to the spherical surface of segment BGD.] Therefore, the spherical surface of segment BAD is to the spherical surface of segment BDG as axis AZ to axis ZG. Q.E.D.

Corollary.

It is evident, therefore, that the surface of a sphere is to the spherical surface of its segment as the diameter of the sphere is to the axis of the segment; this is obvious from conjunct proportionality.

Proposition XVIII.

IF A SPHERE IS CUT BY A PLANE, THE SPHERICAL SURFACE OF EITHER SEGMENT IS TO THE CUTTING CIRCLE AS THE DIAMETER OF THE SPHERE IS TO THE AXIS OF THE REMAINING SEGMENT.[1]

I shall still stay with the same description [i.e. of Fig. III.5C.15, but see Fig. III.5C.15A above], while asserting that the spherical surface of segment BAD is to the cutting circle of diameter BD as diameter AG is to axis GZ. Also, [I say] that the spherical surface of segment BGD is to the circle of diameter BD as diameter AG is to axis AZ.

For, by VI.8 [of Euclid], $AG \mid GZ = (AG \mid GB) \cdot (AG \mid GB)$; but $AG \mid GB = AB \mid BZ$ (because of the similarity of triangles AGB and ABZ). Therefore, $AG \mid GZ = (AB \mid BZ) \cdot (AB \mid BZ)$. But, by XII.2 [of Euclid], circle of radius AB / circle of radius $BZ = (AB \mid BZ) \cdot (AB \mid BZ)$. Therefore, circle of radius AB / circle of radius $BZ = AG \mid ZG$. But, by

Prop. XVII
[1] See Com., Prop. XVII, lines 2–3.
Prop. XVIII
[1] See Com., Prop. XVIII, lines 2–4.

[Proposition] XVI, the circle of radius *AB* is equal to the spherical surface of segment *BAD* while the circle of radius *BZ* is equal to circle *BD* cutting the sphere. Therefore, the spherical surface of segment *BAD* is to the circle of radius *BZ* (which separates the segments) as diameter *AG* is to *GZ*, the axis of the remaining segment.

I shall demonstrate in no way differently that the spherical surface of segment *BGD* is to the circle of radius *BZ* as diameter *AG* is to *AZ*, the axis of the remaining segment. Q.E.D.

Proposition XIX.

IF A SPHERE IS CUT BY TWO PARALLEL PLANES, THE SPHERICAL SURFACE OF THE INTERCEPTED SEGMENT, I.E. THE SPHERICAL ZONE, IS EQUAL TO THE PRODUCT OF THE DIAMETER OF THE SPHERE AND THE CIRCUMFERENCE OF THE CIRCLE WHOSE DIAMETER IS THE AXIS OF THE SEGMENT.[1]

To the preceding description, I add the straight line *RS* so as to cut diameter *AG* in point *Y* [see Fig. III.5C.16]. And so by the rotation of the semicircle [*ABG*] *RY* will describe a circle parallel to the circle of diameter *BD*.

And so I say that the spherical surface of the spherical segment *BS* intercepted between the two parallel circles of diameters *BD* and *RS*, i.e. the zone described by arc *BR*, is equal to the product of diameter *AG* and the circumference of the circle whose diameter is axis *ZY*.

For, by [Proposition] XV above, the product of *AG* and the circumference of the circle of diameter *AY* is equal to the spherical surface of segment *RAS*, while the circumference of the circle whose diameter is *AY* is, by the first corollary of [Proposition] VI, equal to the circumferences of the circles whose diameters are *AZ* and *ZY*. Therefore, the product of *AG* and the circumferences of the circles of diameters *AZ* and *ZY* is equal to the spherical surface of segment *RAS*. But, by [Proposition] XV, the product of *AG* and the circumference of the circle of diameter *AZ* is equal to the spherical surface of segment *BAD*. And so from the product of *AG* and the circumferences of the circles of diameters *AZ* and *ZY* let us subtract the product of *AG* and the circumference of the circle of diameter *AZ*. Also from the spherical surface of segment *RAS* let us subtract the spherical surface of segment *BAD* (evidently two equals are being subtracted from two equals). And on the one hand, the product of *AG* and the circumference of the circle of diameter *ZY* remains and, on the other, the spherical surface of the intercepted segment *BS*. Consequently these are equals. Q.E.D.

You may show the same thing by the corollary of [Proposition] XV.

Prop. XIX
  [1] See Com., Prop. XIX, lines 2–6.

For, by that corollary, from the product of line AY and the circumference of circle ABG arises the spherical surface of segment RAS and from the product of AZ and the same circumference arises the spherical surface of segment BAD. Whence it follows from II.1 of the *Elements* that from the product of ZY and the said circumference arises the spherical surface of the intercepted segment BS. Q.E.D.

Corollary I.

It is evident, therefore, that the spherical zone of the spherical segment included between parallel circles is equal to the product of the axis of the segment and the circumference of the greatest circle in the sphere; and, therefore, [the zone is equal] to the lateral surface of the cylinder whose base is the greatest circle of the sphere and whose altitude is the axis of the segment.

Corollary II.

Also it is evident that when a sphere has been cut by any number of parallel planes, the spherical surfaces intercepted by the planes are to each other as the axes of these spherical segments. This is obvious from the corollary of the preceding [proposition] and from VI.1 of the *Elements*. This corollary could also be demonstrated very easily by the corollary of Proposition XVII, and by equal and disjunct proportionality.

Proposition XX.

IF A SPHERE AND A CYLINDER HAVE A COMMON DIAMETER AND A COMMON AXIS, AND IF THEY ARE CUT BY ANY NUMBER OF PARALLEL PLANES PERPENDICULAR TO THE AXIS, EACH OF THE SPHERICAL SURFACES WILL BE EQUAL TO EACH OF THE CYLINDRICAL SURFACES INTERPOSED BETWEEN THE SAME CUTTING PLANES.[1]

Let square $EZHT$ be circumscribed about circle $ABGD$, and let the points of contact be A, B, G, D, and let AG be joined [see Fig. III.5C.17]. Also, between sides ZH and ET, let any lines you wish KL and MN be drawn parallel to EZ and HT and cutting axis AG at points X and O, circumference ABG at P, R, S and Y. And let it [the whole figure] be rotated about the unmoved diameter AG until it returns to its place; [then] the parallelogram will describe a cylinder, and the semicircle a sphere, and AG will be the axis of the sphere as well as of the cylinder. Also, EZ and HT, the diameters of the bases of the cylinders, [are] equal to the diameter of the sphere; and the cylinder will circumscribe the sphere. For the cylindrical bases touch the spherical surface in points

Prop. XX
[1] See Com., Prop. XX, lines 2–7.

A, G; and also the cylindrical surface touches the spherical [surface] and the contact [of the two surfaces] will be the circumference of the circle described by the point of contact B. Also, these lines KX, MO when rotated will describe parallel circles cutting the cylinder, and the points P, S will describe the circumferences of circles which will be the common sections of the said circles cutting the cylinder and the spherical surface.

And so I say that the spherical surface lying between the circles of diameters ZE and KL, i.e. the surface described by arc AP, is equal to the cylindrical surface interposed between the same circles, i.e., to the surface described by line ZK. Also [I say] that the spherical surface which is between the circles of diameters KL and MN, i.e., the surface described by arc PS, is equal to the cylindrical surface which is between these same circles, i.e. to the surface described by line KM. Also [I say] that the spherical surface which is between the circles of diameters MN, TH, i.e., the surface described by arc SBG, is equal to the cylindrical surface which is included between these same circles, i.e. to the surface described by line MH.

The first and third of these [assertions] is obvious by the corollary to Proposition XV above, since cylinder ZL has an altitude equal to AX, the axis of spherical segment PAR, and a base which is the greatest circle of the sphere, and also since cylinder MT has an altitude OG, which is the axis of spherical segment SGY, and a base which is the greatest circle in the sphere. The second [assertion] moreover is obvious by the first corollary of the preceding [proposition], since cylinder KN has altitude XO, which is the axis of spherical segment $SPRY$, and a base which is the greatest circle of the sphere. I can demonstrate the same thing for the spherical and cylindrical surfaces which any two planes parallel to the bases of the cylinder intercept and on which planes the axis of the cylinder stands perpendicularly. Q.E.D.

I shall demonstrate the same thing in another way. Since, by [Proposition] IV of this [work], from the product of AG and the circumference of circle ABG, the base of cylinder ZT, arises the lateral surface of cylinder ZT, and also from the product of AX and the circumference of the same circle arises the lateral surface of cylinder ZL, therefore by VI.1 [of Euclid] and conjunct proportionality, the lateral surface of cylinder ZT will be to the lateral surface of cylinder ZL as AG to AX. But by the corollary of [Proposition] XVII above, AG / AX = surf. sphere ABG / spher. surf. seg. PAR. Therefore, surf. sphere ABG / spher. surf. seg. PAR = lat. surf. cylind. ZT / lat. surf. cylind. ZL. And, by alternation, surf. sphere ABG / lat. surf. cylind. ZT = spher. surf. seg. PAR / lat. surf. cylind. ZL. But, by [Proposition] XI of this [work], the surface of sphere ABG is equal to the lateral surface of cylinder ZT. Therefore, the spherical surface of segment PAR is equal to the lateral surface of cylinder ZL. Similarly, I shall show that the spherical surface of segment SAY is equal to the lateral surface of surface of cylinder ZL.

Consequently, the spherical surface of segment PY, which is the spherical zone described by arc PS, is equal to the lateral surface of cylinder KN. And also the surface of the spherical segment SGY is equal to the lateral surface of cylinder MT. I can show this same thing for any planes parallel to the cylindrical base and cutting both the cylinder and the sphere circumscribed by the cylinder, and this is what our proposition signifies.

So much for the surfaces of solids of rotation and of spherical segments. The order [of proof] demands that now we ought to discuss first the volume of [such] solids of rotation and then the volume of a sphere.

Proposition XXI.

THE CONE WHOSE BASE IS EQUAL TO THE SUM OF THE BASES OF ANY NUMBER OF CONES AND WHOSE ALTITUDE IS THE SAME [AS THEIRS] IS EQUAL TO THE SUM OF THOSE [CONES].[1]

For example, the base of cone A is equal to the sum of the bases of some number of cones, say of the three [cones] B, G, and D [see Fig. III.5C.18]. Let all of the cones have the same altitude.

I say that cone A = cone B + cone G + cone D.

For, since their altitudes are equal, by XII.11 [of Euclid] cone B / cone A = base of cone B / base of cone A. Also, cone G / cone A = base of cone G / base of cone A. Therefore, by V.24 [of Euclid] (cone B + cone G) / cone A = (base of cone B + base of cone G) / base of cone A. Further, cone D / cone A = base of cone D / base of cone A. Therefore, again by V.24 (cone B + cone G + cone D) / cone A = (base of cone B + base of cone G + base of cone D) / base of cone A. But by hypothesis, (base of cone B + base of cone G + base of cone D) = base of cone A. Therefore, cone B + cone G + cone D = cone A. Q.E.D. We shall demonstrate the same thing for any number by cones.

You can conclude the same thing for cylinders whose altitudes are equal.

Proposition XXII.

THE CONE WHOSE ALTITUDE IS EQUAL TO THE SUM OF THE ALTITUDES OF ANY NUMBER OF CONES ON EQUAL BASES IS EQUAL TO THE SUM OF THOSE [CONES].

For example, let the altitude of cone A be equal to [the sum of] the altitudes of some number of cones, say of the three [cones] B, G and D [see Fig. III.5C.19]. And let the bases of all the cones be equal.

I say that cone A = cone B + cone G + cone D.

For, since their bases are equal, by XII.11 [of Euclid] cone B / cone A = altitude of cone B / altitude of cone A. Also, cone G / cone A = altitude

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Prop. XXI

[1] See Com., Prop. XXI, lines 2–4.

of cone G / altitude of cone A. Therefore, by V.24 [of Euclid], (cone B + cone G) / cone A = (altitude of cone B + altitude of cone G) / altitude of cone A. Further, cone D / cone A = altitude of cone D / altitude of cone A. Therefore, again by V.24 [of Euclid], (cone B + cone G + cone D) / cone A = (alt. cone B + alt. cone G + alt. cone D) / alt. cone A. But (alt. cone B + alt. cone G + alt. cone D) = alt. cone A. Therefore, cone B + cone G + cone D = cone A. Q.E.D. We shall demonstrate the same thing for any number of cones, by repeating [Prop.] V.24 as many times as is necessary. Q.E.D.

Proposition XXIII.

THE SOLID WHICH IS DESCRIBED BY A TRIANGLE ROTATED ABOUT ONE OF ITS SIDES AS AN AXIS UNTIL IT RETURNS TO ITS [INITIAL] POSITION IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE CONICAL SURFACE DESCRIBED BY THE ROTATION OF ANOTHER OF THE SIDES OF THE TRIANGLE AND WHOSE ALTITUDE IS EQUAL TO A PERPENDICULAR FALLING ON THE SAME SIDE FROM THE OPPOSITE ANGLE. HOWEVER, IF A [STRAIGHT] LINE IS LED IN ANY WAY FROM ONE OF THE TERMINI OF THE FIXED SIDE TO THE OPPOSITE SIDE, THE SOLID DESCRIBED BY THIS TRIANGLE [THUS] CUT OFF IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE LATERAL SURFACE DESCRIBED BY THE SEGMENT OF THAT SIDE CUT OFF AND WHOSE ALTITUDE IS EQUAL TO THE PERPENDICULAR FROM THE SAID TERMINUS OF THE FIXED SIDE TO THE SAME SIDE. FURTHER, IF THE SIDE OF THE TRIANGLE IS PARALLEL TO THE AXIS ABOUT WHICH THE TRIANGLE IS ROTATED AND ONLY [ONE] ANGLE OF THE TRIANGLE IS TERMINATED IN THE AXIS, THE SOLID DESCRIBED BY THE TRIANGLE WILL BE EQUAL TO THE CONE WHOSE BASIS IS EQUAL TO THE CYLINDRICAL SURFACE DESCRIBED BY THE SIDE PARALLEL TO THE AXIS AND WHOSE ALTITUDE IS EQUAL TO THE PERPENDICULAR DRAWN FROM THE AXIS ITSELF TO THE SAME SIDE.[1]

I shall demonstrate this three-part proposition very briefly. For let there be triangle ABG which when rotated once about one side (say, about side AB) as an axis describes solid AGB [see Fig. III.5C.20(a)]. Now from the angle opposite AG, that is B, let perpendicular BD fall on opposite side AG. And let there be a cone H, whose base is equal to the conical surface which side AG describes and whose altitude is equal to the perpendicular BD.

I say that cone H is equal to solid AGB.

For let triangle ABG first be posed as having a right angle which is

Prop. XXIII
[1] See Com., Prop. XXIII, lines 2–21.

at *B* and then, by the definition of a cone, solid *ABG* described by the triangle is a cone whose axis is *AB* and whose base radius is *BG*. Hence, by the third [proposition] of this [work] the conical surface which line *AG* describes (and hence the base of cone *H* equal to it) is to the base of cone *ABG* as line *AG* is to radius *BG* and therefore (because of the similarity of triangles *AGB* and *ABD*) as axis *AB* is to perpendicular *BD*, which is the altitude of cone *H*. And so the bases of cones *AGB* and *H* are mutually proportional to the altitudes. Therefore, cones *AGB* and *H* are equal by XII.12 [of Euclid].[2] Q.E.D.

Now let the angles of △*ABG* at *A* and *B* be acute [see Fig. III.5C.20(b)] and let perpendicular line *GE* be drawn from point *G* to the axis *AB*, which perpendicular in the rotation of △*ABG* will describe the circle of radius *GE*. And the solid *ABG* will be composed of two cones *AGE* and *BGE* having as a common base the circle of radius *GE*. And so let there be cone *K* having its base equal to the said circle and its altitude equal to *AB*. And, by the preceding [proposition] cone *K* = cone *AGE* + cone *BGE* = solid *AGB*. And so, since by the third [proposition] the conical surface which line *AG* describes (and hence the base of cone *H* equal to it) is to the base of cone *AGE* (and hence to the base of cone *K* equal to it) as *AG* is to *GE*, and therefore (because of the similarity of triangles *AGE* and *ABD*) as altitude *AB* of cone *K* is to altitude *BD* of cone *H*, therefore, by XII.12 [of Euclid],[3] cone *H* = cone *K* = solid *AGB*. Q.E.D.

Then in the third case [see Fig. III.5C.20(c)], let the angle *B* (of the angles of △*ABG*) be obtuse. Then let perpendicular *GE* be drawn from point *G* to axis *AB* extended, so that in the rotation of the triangle line *GE* describes a circle which will be the base of the cone *AGE* described by △*AGE* as well as that of cone *BGE* described by △*BGE*. And so let there be cone *K* having a base equal to the said circle and an altitude equal to straight line *AB*. And, therefore, by what has been premised, cone *K* + cone *BGE* = cone *AGE*. Then from both sides [of the equation] let the common cone *BGE* be subtracted, and it will result that cone *K* is equal to the solid described by the rotation of △*ABG*. And so, since by the third [proposition] the conical surface described by line *AG* (and hence the base of cone *H* equal to it) is to the base of cone *AGE* (and hence to the base of cone *K* equal to it) as *AG* is to *GE*, and therefore (because of the similarity of triangles *AGE* and *ABD*) as altitude *AB* of cone *K* is to altitude *BD* of cone *H*, [therefore] by XII.12 [of Euclid][4] cone *H* = cone *K* = solid *AGB*. Q.E.D. To this point the first part of the proposition has been demonstrated.

The second part is proved as follows [see Fig. III.5C.21]. In this △*ABG* let line *BZ* be drawn from terminus *B* (one of the termini of

2 *Ibid:*, line 37.
3 *Ibid.*
4 *Ibid.*

line AB) to the opposite side; and let there be cone N having its base equal to the conical surface described by line GZ in revolution and its altitude equal to perpendicular BD.

I say that cone N is equal to the solid described by $\triangle BGZ$ in rotation.

For let there be cone H having its base equal to the conical surface described by line AG [in revolution] and its altitude [equal to] BD. Also [let there be] cone M having its base equal to the conical surface described by line AZ and its altitude [equal to] BD. And, as was demonstrated, cone H will be equal to the solid described by $\triangle ABG$; but cone M is equal to the solid described by $\triangle ABZ$. Also, since the conical surface described by line AG is equal to the sum of the conical surfaces described by lines AZ and ZG, therefore the base of cone H, which was equal to the conical [surface], will be equal to the sum of the bases of cones M and N, whose surfaces are equal to the conical surfaces; and, since the altitudes of cones H, M, and N are equal, hence, by the proposition premised before, cone H = cone M + cone N. And so let the equal magnitudes of cone M and solid AZB be subtracted from the equal magnitudes of cone H and solid AGB, and there will remain the equal magnitudes of cone N and the solid described by $\triangle BGZ$. Q.E.D.

The third part of the proposition remains [see Fig. III.5C.22]. As before let $\triangle BGZ$ be rotated about axis AB, and let GZ be the side parallel to AB, and let cone H have its base equal to the cylindrical surface described by line GZ and its altitude equal to the perpendicular BD drawn from angle B to side GZ.

I say that cone H is equal to the solid described by the rotation of $\triangle BZG$.

For let rectangle $AZGX$ be completed. In rotation, this rectangle will describe a cylinder whose axis is AX and whose base radii are AZ and GX. Hence BD will describe a circle, which will be the terminus of cylinders AD and DX. Also, as a matter of fact, the triangles ABZ and BGX will describe cones whose bases are those of the cylinder and whose [common] apex is B. And so let there be cone K having its base equal to the base of cylinder AG and its altitude double AX. By XII.11 [of Euclid][5] cone K will be double the cone having the same base and an axis equal to AX. But, by the [proposition] premised, this latter cone is equal to the sum of cones AZB and BGX. Therefore, cone $K = 2 \cdot$(cone AZB + cone BGX). But the solid described by $\triangle BGZ$ is double the sum of these same cones AZB and BGX since by XII.9 [of Euclid][6] the whole cylinder AG is triple these same cones. And so cone K is equal to the solid described by $\triangle BGZ$.

By the fifth [proposition] of this [work] the cylindrical surface described by line GZ (and hence the base of cone H equal to it) is to the base of cylinder AG (and hence to the base of cone K) as axis AX is to ½ radius

[5] See Com., Prop. XXXIV, line 13.
[6] See Com., Prop. XXIII, line 104.

GX, and, therefore, as the altitude of cone *K* (which is double axis *AX*) is to radius *GX* (which is double its own half) and hence to altitude *BD* of cone *H*. Therefore, by XII.12 [of Euclid],[7] cone *H* = cone *K*, and hence cone *H* = the solid described by △*BGZ*. Q.E.D.

This is when the perpendicular *DB* falls between points *G* and *Z*. But if the perpendicular from point *B* to *GZ* is one of these [two lines], *BZ*, *BG*, the demonstration will still be the same, but within the cylinder *AG* [only] one cone will be described, not two. Now if the perpendicular *DB* falls outside of points *G* and *Z* [see Fig. III.5C.23], I shall demonstrate this part of the proposition as follows. With *ZD* extended at will to point *O*, let *BO* be connected. And let there be cone *M* having its base equal to the cylindrical surface described by line *OG* and its altitude equal to *BD*. Also, let there be cone *N* having its base equal to the cylindrical surface described by line *OZ* and its altitude to *BD*. And, as was demonstrated before, cone *M* will be equal to the solid described by △*OBG*, while cone *N* is equal to the solid described by △*OBZ*. And since the cylindrical surface described by line *OG* is equal to the sum of the cylindrical surfaces described by lines *OZ* and *ZG*, therefore the base of cone *M* is equal to the sum of the bases of cones *N*, *H*. Since the altitude of these latter cones is the same, by the previously premised [proposition], cone *M* will be equal to the sum of cones *N* and *H*. And so let cone *N* be subtracted from cone *M* and the solid described by △*OBZ* from the solid described by △*OBG*, i.e., equals from equals, and the result is that cone *H* is equal to the solid described by △*BZG*. Q.E.D.

It is concluded, therefore, that the solid described by a triangle applied laterally or angularly to an axis existing in the same plane and rotated in a complete revolution about the same axis is equal to the cone whose base is equal to the conical or cylindrical surface described by the side of the triangle opposite the angle applied to the axis and whose altitude is equal to the perpendicular which proceeds from the said angle to the said side wherever it falls. And this is the whole proposition which serves for the succeeding demonstrations.

Proposition XXIV.

THE SOLID DESCRIBED BY HALF OF A REGULAR POLYGON INSCRIBED IN A CIRCLE WHEN IT HAS ROTATED IN A COMPLETE ROTATION ABOUT A FIXED DIAMETER IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE WHOLE SURFACE OF THE SOLID AND WHOSE AXIS IS THE PERPENDICULAR WHICH PROCEEDS FROM THE CENTER OF THE CIRCLE TO ANY SIDE OF THE POLYGON.[1]

[7] *Ibid.*, line 37.
Prop. XXIV
[1] See Com., Prop. XXIV, lines 2–7.

In circle AB with diameter AB, let there be inscribed a regular polygon, i.e., the decagon AB [see Fig. III.5C.24].[2] Let the solid AB with conical surfaces be described by the single rotation about axis AB of the half of the decagon $AGDEZB$. Let perpendicular HT be drawn from the center of sphere H to one of the sides of the polygon, say to side AG; and let there be cone K, whose base is equal to the whole surface of solid AB and whose altitude is equal to perpendicular HT.

I say that cone K is equal to solid AB.

For let angles G, D, E and Z be connected to the center H; and let the following cones with altitude HT be constructed: cone L having a base equal to the conical surface described by line AG; cone M [with a base equal] to that [surface] described by line BZ; cone N [with a base equal] to that [surface] described by line GD; cone X [with a base equal] to that [surface] described by line ZE; cone O [with a base equal] to that [surface] described by line DE, which [surface] is cylindrical. Whence the bases of cones L, M, N, X and O taken together will be equal to the whole surface of solid AB and therefore will be equal to the base of cone K. Therefore, by [Proposition] XXI of this [work] the sum of all these cones L, M, N, X and O will be equal to cone K. Now by the first part of the preceding [proposition] the solid which is described by $\triangle HAG$ in rotation is equal to cone L and that described by $\triangle HBZ$ is equal to cone M; by the second part of the preceding [proposition] that described by $\triangle HGB$ is equal to cone N and that described by $\triangle HZE$ is equal to cone X. Finally, by the third part of the preceding [proposition] that [solid described] by $\triangle HDE$ is equal to cone O. And so solid AB, which is described by the whole semipolygon ADB, is equal to the sum of cones L, M, N, X, O; but this sum was equal to cone K. Therefore, solid AB will be equal to cone K. And this was to be demonstrated.

Corollary.

It is clear, therefore, that solid AB is equal to the cylinder whose base is equal to the surface of solid AB and whose axis is equal to one-third part of perpendicular HT, for by XII.11 and XII.9 [of Euclid] such a cylinder is equal to cone K and therefore to solid AB etc.

Proposition XXV.

A SPHERE IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE SPHERICAL SURFACE AND WHOSE ALTITUDE IS EQUAL TO THE RADIUS OF THE SPHERE.[1]

Let there be a circle $ABGD$ with center E and diameter AG [see Fig. III.5C.25]. And let sphere ABG be described by the complete rotation

[2] *Ibid.*, line 9.
Prop. XXV
[1] See Com., Prop. XXV, lines 2–4.

of semicircle *ABG* upon immobile diameter *AG*. And let there be a cone *M* whose base is equal to the surface of sphere *ABG* and whose altitude is equal to radius *AE*.

I say that cone *M* is equal to sphere *ABG*.

For if cone *M* is not equal to sphere *ABG*, it will be equal to some other sphere that is greater or less than sphere *ABG*. Hence first, if it is possible, let cone *M* be equal to sphere *ZHT*, which is less than sphere *ABG* but concentric with it and which has a diameter *ZT* and which semicircle *ZHT* describes in rotation about diameter *ZT*. And by XII.13 [of Euclid][2] let there be inscribed in circle *ABG* a regular polygon that does not touch circle *ZT* at all. And let this polygon be *ABGD*, one of whose sides is *AK* to which perpendicular *EL* is drawn from the center. And by the complete rotation of half the polygon about diameter *AG* let there be described a solid of conical surfaces that does not touch sphere *ZHT* at all. Then by the preceding [proposition] solid *ABG* will be equal to the cone whose base is equal to the surface of solid *ABG* and whose altitude is equal to perpendicular *EL*. And so cone *M* is greater than a cone of this sort, inasmuch as it is greater in both base and altitude (since clearly the surface of sphere *ABG* is greater than the surface of the included solid *ABG* and line *AE* is greater than perpendicular *EL*). And therefore sphere *ZHT* [posited as] equal to cone *M* will be greater than solid *ABG*, i.e. the part greater than the whole, which is impossible.

Or [prove it] thus. Since cone *M* was greater than the said cone equal to solid *ABG* and therefore is greater than solid *ABG*, and since solid *ABG* is greater than sphere *ZHT*, accordingly cone *M* will be greater than sphere *ZHT* and hence is not equal to it as was supposed.

Then, if it is possible, let cone *M* be equal to a sphere greater than sphere *ABG*. But for the sake of brevity, let it be supposed that this sphere is *ZHT* [instead of *ABG*] and let cone *M* have a base equal to the surface of sphere *ZHT* and an altitude equal to radius *EZ*.

I say that cone *M* will not be equal to some sphere that is greater than sphere *ZHT*.

For, if it is possible, let it be equal to sphere *ABG*, which is greater than sphere *ZHT* but concentric with it. Within sphere *ABG* let there be described as before a solid *ABG* whose surface does not touch sphere *ZHT* etc. And by the preceding [proposition] solid *ABG* will be equal to the cone whose base is equal to the surface of solid *ABG* and whose altitude is equal to perpendicular *EL*. Therefore such a cone is greater than cone *M* since it exceeds it in both base and altitude. Therefore solid *ABG* is greater than cone *M*. But cone *M* was by hypothesis equal to sphere *ABG*. Therefore solid *ABG* is greater than sphere *ABG*, i.e. the part than the whole, which is impossible. Therefore cone *M* is not equal to some sphere greater than sphere *ZHT*. Similarly, if cone *M* has a base equal to the surface of sphere *ABG* and an altitude equal to radius *EA*,

[2] *Ibid.*, line 15.

it will not be equal to any sphere greater than sphere ABG. And it was demonstrated that [it will] not [be equal] to [any] lesser [sphere]. Therefore it remains that cone M is completely equal to sphere ABG. Q.E.D.

It is evident, therefore, that a sphere is equal to the cone whose base radius is equal to the diameter of the sphere and whose axis is equal to the radius of the sphere. This is obvious, for by the corollary to [Proposition] X a circle whose radius is equal to the diameter of the sphere is equal to the surface of the sphere.

Proposition XXVI.

A SPHERE IS DOUBLE THE CONE WHOSE BASE IS EQUAL TO THE GREATEST CIRCLE IN THE SPHERE AND WHOSE ALTITUDE IS EQUAL TO THE DIAMETER OF THE SPHERE.

Let A be a sphere whose greatest circle is BCD and let cone N have a base equal to circle BCD and an altitude equal to the diameter BD of the sphere [see Fig. III.5C.26].

I say that sphere A is double cone N.

For let cone M be posited which has a base equal to the surface of sphere A and an altitude AD equal to the radius of sphere D. By the preceding [proposition] cone M will be equal to sphere A. Also let cone X be posited which has a base that is double circle BCD and an altitude equal to the diameter BD of the sphere. And so the base of cone M, which is equal to the surface of the sphere A, is by [Proposition] XI quadruple circle BCD and the base of cone X is double circle BCD. Therefore the base of cone M is double the base of cone X, while the altitude of cone X is double the altitude of cone M. Therefore the bases of cones M and X are reciprocally proportional to their altitudes, and hence by XII.12 [of Euclid][1] the cones are equal. But cone M is equal to sphere A; therefore cone X is also equal to sphere A. But cone X is double cone N, for it is of the same altitude and has a double base (since the base of cone X is double circle BCD and the base of cone N is equal to the same circle). Therefore, sphere A is double cone N. Q.E.D.

Corollary.

It is evident, therefore, that the cylinder whose base is equal to the greatest circle in a sphere and whose altitude is equal to the diameter of the sphere is $3/2$ the sphere.[2] For by XII.9 [of Euclid][3] the cylinder is triple its cone; and the sphere is double that cone as was demonstrated. Hence the cylinder will become $3/2$ the sphere.

Prop. XXVI
   [1] See Com., Prop. XXIII, line 37.
   [2] See Com., Prop. XXVI, lines 23–25.
   [3] *Ibid.*, line 25.

Proposition XXVII.

IF THE AXIS OF A CYLINDER IS EQUAL TO THE DIAMETER OF ITS BASE AND IF THE BASE OF A CONE IS EQUAL TO THE WHOLE SURFACE OF THE CYLINDER WHILE ITS ALTITUDE IS EQUAL TO THE RADIUS OF THE BASE OF THE CYLINDER, THE CYLINDER WILL BE EQUAL TO THE CONE.[1]

Let side AB of rectangle $ABGD$ be double side BG [see Fig. III.5C.27], and let cylinder AG be described by the complete rotation of the rectangle around side AB. And so, since BG, which is the radius of the base, is one-half the axis AB, the whole diameter will be equal to the axis. Also let E be a cone whose base is equal to the whole surface of cylinder AG, i.e. to the sum of the lateral surface described by line GD and the bases described by lines AD and BG, and whose altitude is equal to radius AD.

I say that cylinder AG and cone E are equal.

For let axis AB be bisected at Z, and let DZ and ZG be drawn. Thus it happens that in the rotation of rectangle AG these triangles ZDA and ZGB describe cones whose bases are the bases of the cylinder and whose apexes are at point Z. Now let a perpendicular fall from point Z to DG, which perpendicular we let be ZH. And let there be a cone T whose base is equal to the cylindrical surface described by line GD and whose altitude is equal to perpendicular ZH and hence to radius AD, which is the altitude of cone E. And by the third part of [Proposition] XXIII above, cone T is equal to the solid described by $\triangle DZG$. Accordingly, cone T plus the cones described by triangles ZDA and ZGB will be equal to the whole cylinder AG; and also the base of cone T plus the bases of the cylinder will be equal to the whole surface of the cylinder. Therefore the base of cone T plus the bases of cones ADZ and BGZ (which are also the bases of the cylinder) will be equal to the base of cone E, which was equal to the whole surface of the cylinder. And the altitude of each of the four cones is the same since lines AD, AZ, ZB, ZH, which are the altitudes of the cones, are equal. Therefore, by [Proposition] XXI above cone E is equal to the sum of cones T, ADZ and BGZ. But this sum was equal to cylinder AG. Therefore cylinder AG will be equal to cone E. Q.E.D.

Proposition XXVIII.

THE CYLINDER WHOSE AXIS AND BASE ARE EACH EQUAL TO THE DIAMETER OF A SPHERE IS THREE-HALVES THE SPHERE.[1]

Prop. XXVII
[1] See Com., Prop. XXVII, lines 2–5.
Prop. XXVIII
[1] See again Com., Prop. XXVII, lines 2–5.

Although this proposition has been demonstrated as a corollary to [Proposition] XXVI above, still I would like to demonstrate it in another way here.

Let A be a sphere and Z a cylinder [see Fig. III.5C.28], and let the axis of cylinder Z and its diameter be equal to diameter CB of sphere A.

I say that cylinder Z is ³⁄₂ sphere A.

For let M be a cone whose base is equal to the surface of sphere A and whose altitude is equal to the radius AB of the sphere. By [Proposition] XXV above cone M will be equal to sphere A. Also let E be a cone whose base is equal to the whole surface of cylinder Z and whose altitude is equal to the radius of the base of the cylinder. By the preceding [proposition] cone E will be equal to cylinder Z (since the radius of sphere A and the radius of the base of the cylinder—which are the altitudes of cones M, E—are equal by hypothesis) and the whole surface of cylinder Z (which surface is the base of cone E), is, by the corollary of [Proposition] XI, ³⁄₂ the surface of sphere A (which surface is the base of cone M), and by XII.11 [of Euclid] cones of the same altitude are mutually as their bases. Now cone E will be ³⁄₂ cone M; but cone M was equal to sphere A and cone E to cylinder Z. Therefore cylinder Z is also ³⁄₂ sphere A. Q.E.D.

Corollary.

Hence it is again obvious that a sphere is double the cone whose axis and base diameter each are equal to the diameter of the sphere. For, since by the preceding [proposition] a cylinder of this sort is ³⁄₂ the sphere, while by XII.9 [of Euclid][2] it is triple the cone, the sphere will be double the cone.

And so the sphere is related to the cylinder and the remaining solids of rotation in such manner. Now we shall pass on to the volumes of segments.[3]

Proposition XXIX.

IF ANY ARC OF A CIRCLE IS CUT INTO EQUAL SEGMENTS AND CHORDS ARE SUBTENDED TO THESE ARCS, AND IF THE EXTREMES OF THE ARC ARE CONNECTED TO THE CENTER OF THE CIRCLE, AND IF THE RECTILINEAR FIGURE INCLUDED BY THE CHORDS AND RADII IS ROTATED COMPLETELY ABOUT ONE OF THE RADII AS IT REMAINS STATIONARY THE SOLID DESCRIBED IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE CONICAL SURFACES DESCRIBED BY THE CHORDS AND WHOSE ALTITUDE IS EQUAL TO THE

[2] See Com., Prop. XXVI, line 25.
[3] See Com., Prop. XXVIII, lines 28–29.

PERPENDICULAR DRAWN FROM THE CENTER OF THE CIRCLE TO ANY OF THE CHORDS.[1]

In circle *AD* with center *H* let arc *AD* be cut into any number of equal segments [see Fig. III.5C.29], e.g., into the two segments *AG* and *GD*. Let chords *AG* and *GD* subtend those segments. And with radii *AH* and *DH* drawn, let the rectilinear figure *AD* be rotated about radius *AH* as it remains stationary, so that the solid *AD* is described. And let *K* be a cone whose base is equal to the conical surfaces described by chords *AG* and *GD* and whose altitude is equal to *HT*, the perpendicular drawn to side *AG*.

I say that cone *K* is equal to solid *AD*.

For let *GH* be joined and let two cones be constructed each with altitude *HT*, namely cone *L* having a base equal to the conical surface described by line *AG* and cone *N* having a base equal to the conical surface described by line *GD*. Whence the bases of cones *L* and *N* together will be equal to the base of cone *K*, which base was equal to the said conical surfaces. Therefore by [Proposition] XXI of this [work] cones *L* and *N* together will be equal to cone *K*. But by [Proposition] XXIII the solid described by △*HAG* is equal to cone *L* and that described by △*GHD* is equal to cone *N*. Therefore the solid *AD* described by the whole rectilinear figure *AD* is equal to the sum of cones *L* and *N*. But this sum was equal to cone *K*. Therefore solid *AD* will be equal to cone *K*. Q.E.D.

We shall demonstrate the same thing however many equal chords there are [but with these provisos noted] that we must construct as many cones as there are triangles, with each cone equal to each solid described by the triangles. Further, we must continually apply [Propositions] XXI and XXIII of this [work]. And [note that] the demonstration of [Proposition] XXIV of this [work] is similar.

Proposition XXX.

IF A CONICAL SURFACE HAVING ITS APEX IN THE CENTER OF THE SPHERE SEPARATES THE SPHERE INTO TWO SECTORS, EACH OF THE SECTORS IS EQUAL TO THE CONE WHOSE BASE IS EQUAL TO THE SPHERICAL BASE OF ITS SECTOR AND WHOSE ALTITUDE IS EQUAL TO THE RADIUS OF THE SPHERE.[1]

From circle *ABGD* with center *E* and diameter *AG* let arc *AB* be taken [see Fig. III.5C.30]. Let radius *EB* be drawn. And a sphere will be described by the rotation of semicircle *ABG* around its fixed diameter. And in that same rotation the sector *ABE* of the circle will describe the solid sector *ABED* of the sphere. Let *M* be posited as a cone whose

Prop. XXIX
  [1] See Com., Prop. XXIX, lines 2–11.
Prop. XXX
  [1] See Com., Prop. XXX, lines 2–6.

base is equal to the base of the spherical sector *ABD*, i.e. [the surface] which arc *AB* describes, and whose altitude is equal to radius *AE*.

I say that cone *M* is equal to spherical sector *ABD*.

For otherwise it will be equal to some sector that is more or less [than *ABD*]. Hence first let cone *M* be equal to spherical sector *ZHT* less than sector *ABD* but concentric with it, sector *ZHT* being described by arc *ZH* which lies between lines *AE* and *EB*. And let arc *AB* be continuously bisected until by XII.13 [of Euclid][2] the chords of the arcs do not touch arc *ZHT*, and let one of the chords be *AK*. Let *EL* be perpendicular to *AK*. Let solid *ABD* be described by the rotation of the rectilinear figure *AKBE* around axis *AE*. By the preceding [proposition] this solid will be equal to the cone whose base is equal to the conical surfaces described by chords *AK* and *KB* and whose altitude is equal to perpendicular *EL*. And so cone *M* is greater than this cone since it is greater in both base and altitude. But cone *M* was equal to the spherical sector *ZT*; therefore the spherical sector *ZT* is greater than solid *ABD*, i.e. the part than the whole, which is impossible.

Or [prove it] thus. Cone *M* is greater than solid *ABD*, inasmuch as it is greater than the cone to which the solid is equal; therefore it is greater than spherical sector *ZT* and hence it is not equal to it as was supposed.

Then let cone *M* be equal to a spherical sector greater than sector *ABD*. But for the sake of brevity let the assumed sector be spherical sector *ZHTE* and let cone *M* have a base equal to the spherical base of sector *ZT*, i.e. to [the surface] which arc *ZH* describes, and an altitude equal to radius *EH*.

I say that cone *M* will not be equal to some spherical sector greater than sector *ZT*.

For, if it is possible, let it be equal to sector *BD* concentric with sector *ZT* but greater than it, [sector *BD* being that] which circular sector *ABE* describes. And as before let there be described in arc *AB* chords that do not touch circle *ZH*; and let solid *BD* be formed by the rotation of the rectilinear figure *ABE*. By the preceding [proposition] this solid will be equal to the cone whose base is equal to the conical surfaces described by chords *AK* and *KB* and whose altitude is equal to perpendicular *EL*. Therefore this cone will be greater than cone *M*, which it exceeds in both base and altitude. But by hypothesis cone *M* was equal to sector *BD*. Therefore solid *BD* is also greater than sector *BD*, i.e. the part than the whole, which is impossible. Therefore, cone *M* is not equal to some spherical sector greater than sector *ZT*. Similarly, if cone *M* is supposed as having a base equal to the spherical base of sector *BD* and an altitude equal to radius *AE*, cone *M* will not be equal to some sector greater than sector *BD*. But neither [is it equal to a sector] less [than *BD*] as

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[2] See Com., Prop. XXV, line 15.

was demonstrated. Therefore it remains that cone M is completely equal to spherical sector $ABED$. Q.E.D.

And since sphere ABG was cut into the two sectors which the conical surface described by line EB delimits, we shall be able to demonstrate the same thing for spherical sector BGD described by circular sector EBG. Hence let N be a cone whose base is equal to the spherical base of sector BGD, i.e. [the surface] which arc BG describes [and whose altitude is equal to radius AE]. And we shall demonstrate in the same way [as before] that cone N will be equal to spherical sector BGD.

Or [prove it] thus. Let there be a cone X having a base equal to the whole surface of sphere ABG and an altitude equal to radius AE. By [Proposition] XXV cone X will be equal to sphere ABG; and the base of cone X is equal to the bases of cones M and N taken together. Accordingly, by [Proposition] XXI cone X is equal to the sum of cones M and N. Therefore cones M and N taken together will be equal to sphere ABG. And so let the spherical sector ABD be subtracted from sphere ABG and cone M from the sum of cones M and N, i.e., equals from equals. Therefore spherical sector BGD and cone N will remain as equals. Q.E.D.

Corollary.

It is evident, therefore, that a spherical sector is equal to the cone whose base radius is equal to the line which proceeds from the apex of the spherical segment to the circumference of its base and whose altitude is equal to the radius of the sphere. For, by the corollary of [Proposition] XVI the circle whose radius is the line which proceeds from the apex of the spherical segment to the circumference of the base is equal to the spherical base of the sector. Therefore, the cone whose base is equal to the circle with a radius equal to the line which proceeds from the apex of the spherical segment to the circumference of its base and whose altitude is equal to the radius of the sphere will immediately be equal to the cone whose base is equal to the base of the spherical sector and whose altitude is equal to the radius of the sphere. But this latter cone was equal to the spherical sector. Therefore the former cone, whose base is the circle having as a radius the line which proceeds from the apex of the spherical segment to the circumference of its base and whose altitude is equal to the radius of the sphere, will also be equal to the spherical sector, just as the corollary adduces.

Proposition XXXI.

IF A CIRCLE CUTS A SPHERE, EACH OF THE SPHERICAL SEGMENTS IS EQUAL TO THE CONE WHOSE BASE IS THE CUTTING CIRCLE AND WHOSE ALTITUDE IS THE STRAIGHT LINE WHICH IS RELATED TO THE AXIS OF THE SEGMENT AS THE SUM OF THE RADIUS OF THE SPHERE AND THE

AXIS OF THE OTHER SEGMENT IS RELATED TO THE SAME AXIS.[1]

Let $ABGD$ be a sphere which circle ABG describes in a complete rotation about fixed axis AG [see Fig. III.5C.31]. Let the sphere be cut by a circle [i.e., plane] perpendicular to diameter AG and whose diameter is BD, which cuts AG at P. Let the segments be ABD and BGD with axes respectively AP and PG, and let $(EA + PG) / PG = RP / AP$.

I say that the cone whose base radius is BP and whose altitude is RP is equal to the spherical segment ABD.

Also let $(EG + PA) / PA = SP / GP$.

I further say that the cone whose base radius is BP and whose altitude is SP is equal to the spherical segment BGD.

The first [statement] I demonstrate as follows. Since $(EA + PG) / PG = RP / AP$, by disjunct proportionality $EA / PG = RA / AP$, and inversely $PG / EA = AP / RA$; and by alternation $PG / AP = EA / RA$; then inversely $AP / PG = RA / AE$. And by conjunct proportionality $AG / GP = RE / EA$. But $AG / GP = (AG / GB)^2$, by VI.8 [of Euclid]. Now $AG / GB = AB / BP$, because of the similarity of triangles AGB and ABP. Therefore, $AG / GP = (AB / BP)^2$. But $AG / GP = RE / EA$. Therefore, $RE / EA = (AB / BP)^2$. But circle of radius AB / circle of radius $BP = (AB / BP)^2$ (inasmuch as the circles are mutually as the squares of their diameters, by XII.2 [of Euclid]). And so $RE / EA =$ circle of radius AB / circle of radius BP. Accordingly, the two cones of which one has a base radius AB and an altitude EA and the other a base radius BP and an altitude RE are equal to each other by XII.12 [of Euclid][2] since their altitudes are inversely proportional to their bases. But by [Proposition] XXII of this [work] the cone whose base radius is BP and whose altitude is RE is equal to two cones each of whose base radii is BP and whose altitudes are respectively RP and PE. And by the corollary of the preceding [proposition] the cone whose base radius is AB and whose altitude is AE is equal to the spherical sector $ABED$. Therefore the two cones each of whose base radii is BP and whose altitudes are respectively RP and PE when taken together are equal to the spherical sector $ABED$, which is itself the aggregate of the spherical segment ABD and the cone whose base radius is BP and whose altitude is PE. Therefore, if the common cone whose base radius is BP and whose altitude is PE, is subtracted, the cone whose base radius is BP and whose altitude is RP will remain equal to the spherical segment ABD, which was the first thing to be demonstrated. I [now] prove the rest.

Since $(EG + PA) / PA = SP / GP$, by disjunct proportionality $EG / PA = SG / PG$, and inversely $PA / EG = GP / SG$. By alternation $AP / PG = EG / GS$, and inversely $GP / PA = SG / GE$. By conjunct proportionality

Prop. XXXI
   [1] See Com., Prop. XXXI, lines 2–7.
   [2] See Com., Prop. XXIII, line 37.

$GA / AP = SE / EG$. But by VI.8 [of Euclid] $GA / AP = (GA / AB)^2$; and $GA / AB = GB / BP$ because of the similarity of triangles GAB and GBP. Therefore, $GA / AP = (GB / BP)^2$. Now $GA / AP = SE / EG$; therefore, $SE / EG = (GB / BP)^2$. But circle of radius GB / circle of radius $BP = (GB / BP)^2$, by XII.2 [of Euclid]. And so SE / EG = circle of radius GB / circle of radius BP. Accordingly, the two cones of which one has base radius GB and altitude EG and the other base radius BP and altitude SE are equal to each other by XII.12 [of Euclid][3] (inasmuch as the altitudes are inversely proportional to the bases). By the corollary to the preceding [proposition] the cone whose base radius is BG and whose altitude is EG is equal to the spherical sector $BGDE$. Therefore the spherical sector $BGDE$ is equal to the cone whose base radius is BP and whose altitude is SE. If the cone whose base radius is BP and whose altitude is PE is added to both [the aforesaid magnitudes], [then] the spherical segment BGD (i.e. the aggregate of sector $BGDE$ and the said cone) is equal to the sum of the cones each of whose base radii is BP and whose altitudes are respectively PE and ES. But by [Proposition] XXII such cones taken together are equal to the cone whose base radius is BP and whose altitude is PS. Therefore the cone whose base radius is BP and whose altitude is PS is also equal to the spherical segment BGD, which is the rest that had to be demonstrated. And so the whole proposition is true.

Corollary.

Thence it is evident that a spherical segment is to a cone of the same base and same apex as the line consisting of the radius of the sphere and the axis of the other segment is to that same axis. For, and I shall proceed succinctly, cone BRD / cone BAD = altitude RP / altitude PA (since the cones are on the same base). But the spherical segment BAD was equal to cone BRD, and $RP / PA = (EG + GP) / GP$. Therefore, a spherical segment BAD / cone $BAD = (EG + GP) / GP$. Q.E.D. I shall demonstrate in a nowise different way that spherical segment BGD / cone $BGD = (EA + AP) / AP$, as the corollary adduces.

Proposition XXXII.

IF A CIRCLE CUTS A SPHERE, EACH OF THE SPHERICAL SEG-MENTS IS EQUAL TO THE CONE WHOSE BASE RADIUS IS EQUAL TO THE AXIS OF THE SEGMENT AND WHOSE ALTITUDE IS EQUAL TO THE SUM OF THE RADIUS OF THE SPHERE AND THE AXIS OF THE OTHER SEGMENT.[1]

Repeating the preceding description [i.e. of Fig. III.5C.31], I say that spherical segment ABD is equal to the cone whose base radius is equal

[3] *Ibid.*
Prop. XXXII
[1] See Com., Prop. XXXII, lines 2–6.

to axis AP and whose altitude is equal to the sum of the radius EG and the axis GP of the other segment. I also say that spherical segment BGD is equal to the cone whose base radius is equal to axis GP and whose altitude is equal to the sum of radius AE and the axis AP of the other segment.

I shall prove the first [statement] as follows. Since $GP / PA = (BP / PA)^2$ by VI.8 [Euclid], and by XII.2 [of Euclid] circle of radius BP / circle of radius $PA = (BP / PA)^2$, therefore $GP / PA =$ circle of radius BP / circle of radius PA. But it was assumed before that $GP / PA = EA / AR$ or EG / AR. And by conjunct proportionality $(EG + GP) / PR = PG / PA$, by V.13 [of Euclid]. Therefore circle of radius BP / circle of radius $PA = (EG + GP) / PR$. Accordingly, the two cones of which one has a base radius AP and an altitude $(EG + GP)$ and the other a base radius of BP and an altitude PR are equal by XII.12 [of Euclid][2] since their altitudes are inversely proportional to their bases. But by the preceding [proposition] the cone whose base radius is BP and whose altitude is PR was equal to spherical segment ABD. Therefore spherical segment ABD is also equal to the cone whose base radius is AP and whose altitude is $(EG + GP)$, which is one of the things to be demonstrated. [Now] take up the other.

Since $AP / PG = (BP / PG)^2$ by VI.8 [of Euclid], and by XII.2 [of Euclid][3] circle of radius BP / circle of radius $PG = (BP / PG)^2$, therefore $AP / PG =$ circle of radius BP / circle of radius PG. But it was premised before that $AP / PG = EG / GS$ or EA / GS. Therefore, by V.13 [of Euclid], $(EA + AP) / PS = AP / PG$. Therefore $(EA + AP) / PS =$ circle of radius BP / circle of radius PG. Accordingly, the two cones of which one has a base radius PG and an altitude $(AE + AP)$ and the other a base radius of BP and an altitude PS are equal to one another by XII.12 [of Euclid],[4] inasmuch as their altitudes are inversely proportional to their bases. But from the preceding [proposition] the cone whose base radius is BP and whose altitude is PS is equal to the spherical segment BGD. Therefore the spherical segment BGD is [also] equal to the cone whose base radius is PG and whose altitude is $(AE + AP)$, which is what remained to be demonstrated.

In another way

TO CUT A PROPOSED SPHERE IN A GIVEN RATIO.[1]

Let the diameter of the proposed sphere be AB and the given ratio CD / DE [see Fig. III.5C.32]. It is necessary to cut the sphere AB so that one segment is to the [other] segment as CD is to DE.

Let AF be posited as half of AB, and $CE / ED = FA / AG$. Between FA

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[2] See Prop. XXIII, line 37.

[3] *Ibid.*

[4] *Ibid.*

Prop. XXXII-Aliter

[1] See Com., Prop. XXXII-Aliter, lines 2–41.

and AG insert a mean proportional AH, so that FA, AH and AG are continual proportionals; and about axis FB describe parabola $FHLK$. Also though point G let hyperbola GL be constructed with FB and XB as asymptotes; let it intersect the parabola at point L, and let the rectangle $LMBX$ be completed and also the rectangle $AGNB$.. By II.12 of the *Conical Elements* [of Apollonius][2] the rectangles LB and BG will be equal. Therefore, by VI.15 [of Euclid][3] $AB / BM = LM / GA$. And so, since $LM^2 / GA^2 = AB^2 / BM^2 = (LM^2 / HA^2) \cdot (HA^2 / AG^2)$, and since $FM / AG = (MF / FA) \cdot (FA / AG)$, and since by I.20 of the *Conical Elements*[4] $LM^2 / HA^2 = MF / FA$, and by VI.17 of Euclid $HA^2 / AG^2 = FA / AG$, therefore by equality of proportion $LM^2 / AG^2 = AB^2 / BM^2 = MF / AG$. And by XII.2 of Euclid circle of radius AB / circle of radius $BM = MF / AG$. And so by XII.12[5] the cone whose axis is AG and whose base radius is AR will be equal to the cone whose axis is FM and whose base radius is BM, inasmuch as their altitudes are inversely proportional to their bases. But by XII.11 of Euclid the cone with axis FA and base radius AB is to the cone with axis GA and base radius AB as FA to AG. Therefore the cone with axis FA and base radius AB is to the cone with axis FM and base radius MB as FA is to AG, i.e. as CE is to ED. But by the preceding [Proposition] XXXII, the cone with axis FM and base radius BM is equal to the spherical segment whose axis is BM. Also by the corollary to [Proposition] XXV of this [work] the cone whose axis is FA and whose base radius is AB is equal to the sphere with diameter AB. Therefore sphere AB / spher. segm. with axis $BM = CE / ED$; and by disjunct proportionality spher. segm. with axis AM / spher. segm. with axis $BM = CD / DE$. Therefore the sphere with diameter AB may be cut at point M in this diameter into two segments whose ratio is CE / ED, which was to be done.

This proposition has been taken from Dionysodorus, a mathematician of very ancient times, as Eutocius of Ascalon passed it on in his commentaries on Archimedes.

Proposition XXXIII.

A PYRAMID AND A CONE WITH EQUAL ALTITUDES ARE TO EACH OTHER AS THEIR BASES.[1]

Let there be a pyramid AC on rectilinear base A [see Fig. III.5C.33] and cone BGF on circle BG and both of the same altitude.

I say that pyramid AC is to cone BGF as rectilinear figure A is to circle BG.

For, if it is possible, let pyramid AC be to cone BGF as rectilinear figure A is to a circle greater or less than circle BG. And in the first place, to

[2] See Com., to Text A, Prop. XXXI, line 16.
[3] See Com., to Text C, Prop. XIV, lines 33–34.
[4] See Com., Prop. XXXII-Aliter, lines 17–18.
[5] See Com., Prop. XXIII, line 37.

Prop. XXXIII

[1] See Com., Prop. XXXIII, lines 2–3.

a circle less than but concentric with *BG* and let [that circle] be *DE*. Then by XII.13 [of Euclid][2] let there be inscribed in *BG* a rectilinear polygon with sides that do not at all touch circle *DE*. And on rectilinear polygon *BG* let there be understood to be a pyramid of the same altitude as cone *BGF* and with the same apex. And by XII.6 [of Euclid] pyramid *AC* is to pyramid *BGF* as rectilinear figure *A* is to rectilinear polygon *BG*. But by V.8 [of Euclid] rect. fig. *A* / circle *DE* > rect. fig. *A* / rect. polyg. *BG*. Therefore by V.12 [of Euclid][3] pyramid *AC* / cone *BGF* > pyramid *AC* / pyramid *BGF*. Therefore, by V.10 [of Euclid], pyramid *BGF* > cone *BGF*, i.e., the part than the whole, which is impossible. Therefore the ratio of pyramid *AC* to cone *BGF* is not as the ratio of rectilinear figure *A* to some circle less than *BG*.

But neither [is it to some circle] greater [than *BG*]. For, in the cause of brevity, let there be a circle and [on it] the proposed cone *DEF* of the same altitude as pyramid *AC*.

I say that pyramid *AC* will not be to cone *DEF* as rectilinear figure *A* is to any circle greater than circle *DE*, e.g. to circle *BG*.

For let circle *BG* be posited as concentric with *DE*, and, as before, let there be inscribed in circle *BG* a rectilinear polygon with sides that do not at all touch circle *DE*; and on rectilinear polygon *BG* let there be understood to be a pyramid having the same apex as cone *DEF*. And so, since pyramid *AC* / cone *DEF* = rect. fig. *A* / circle *BG*, and by XII.6 [of Euclid] pyramid *AC* / pyramid *BGF* = rect. fig. *A* / rect. polyg. *BG*, and by V.8 [of Euclid] rect. fig. *A* / rect. polyg. *BG* > rect. fig. *A* / circle *BG*, hence by V.12 [of Euclid][4] pyramid *AC* / pyramid *BGF* > pyramid *AC* / cone *DEF*. Therefore by V.10 [of Euclid] cone *DEF* > pyramid *BGF*, i.e., the part than the whole, which is impossible. Similarly, the ratio of pyramid *AC* to cone *BGF* is not equal to the ratio of rectilinear figure *A* to any circle greater than circle *BG*. But neither [was it to any] lesser [circle], as was proved. It remains therefore that pyramid *AC* / cone *BGF* = rect. fig. *A* / circle *BG*. Q.E.D.

Corollary I.

It is evident, therefore, that a prism and a cylinder with equal altitudes are to each other as their bases. For by XII.8 [of Euclid][5] a prism is triple its pyramid just as by XII.9[6] a cylinder is triple its cone.

Corollary II.

And so a pyramid and a cone whose bases and altitudes are equal will be equal to each other. The same thing is to be declared for a prism and a cylinder.

[2] See Com., Prop. XXV, line 15.
[3] See Com., Prop. XXXIII, line 16.
[4] *Ibid.*
[5] *Ibid.*, line 42.
[6] See Com., Prop. XXVI, line 25.

Proposition XXXIV.

A PYRAMID AND A CONE ERECTED ON EQUAL BASES ARE TO EACH OTHER AS THEIR ALTITUDES.

Let pyramid *AB* and cone *GD* stand on equal bases *B* and *D*, *B* being a rectilinear figure and *D* a circle [see Fig. III.5C.34]; and let the altitudes of the pyramid and cone respectively be *AB* and *GD*.

I say that pyramid *AB* is to cone *GD* as altitude *AB* is to altitude *DG*.

For if such altitudes are equal, the proposition is obvious by the first corollary to the preceding [proposition]. If unequal, then let there be constructed on circle *D* a cone of the same altitude as pyramid *AB*, namely cone *ED*, and by the second corollary to the preceding [proposition] pyramid *AB* is equal to cone *ED*. But by XII.11 [of Euclid][1] cone *ED* / cone *GD* = altitude *ED* / altitude *GD*. Therefore, pyramid *AB* is to cone *GD* as altitude *ED* (and hence as altitude *AB*, for they are equal) is to altitude *GD*. Q.E.D.

Corollary.

It is evident, therefore, that a prism and a cylinder on equal bases are to each other as their altitudes, for the prism is triple its pyramid and the cylinder its cone.

Proposition XXXV.

THE RATIO OF A PYRAMID TO A CONE IS COMPOSED OF THE RATIOS OF THE BASES AND THE ALTITUDES.

Let *AB* be a pyramid with base *B* and altitude *AB*; also let *GD* be a cone with base *D* and altitude *GD* [see Fig. III.5C.35].

I say that pyramid *AB* / cone *GD* = (base *B* / base *D*) · (altitude *AB* / altitude *GD*).

For let cone *EZ* be placed on circle *Z* equal to circle *D*, or a pyramid *EZ* on base *Z* equal to base *D*, and let the altitude of *EZ* be equal to the altitude of *AB*. By [Proposition] XXXIII above, pyramid *AB* is to cone *EZ* (or by XII.8[1] to pyramid *EZ*) as base *B* is to base *Z*. And also by XII.11[2] if *EZ* is a cone (or by the previous [proposition] if it is a pyramid), cone *EZ* (or pyramid *EZ*) is to cone *GD* as altitude *EZ* is to altitude *GD*. But the ratio of pyramid *AB* to cone *GD* is composed of the ratio of pyramid *AB* to cone (or pyramid) *EZ* and of the ratio of cone (or pyramid) *EZ* to cone *GD*. Therefore the same ratio of pyramid *AB* to cone *GD* is composed of the ratio of base *B* to base *Z* (or to base *D*, since they are equal) and of the ratio of altitude *EZ* (or altitude *AB*, since they are equal) to altitude *GD*. Q.E.D.

Prop. XXXIV
[1] See Com., Prop. XXXIV, line 13.
Prop. XXXV
[1] See Com., Prop. XXXV, line 11.
[2] See Com., Prop. XXXIV, line 13.

Corollary I.

It is evident, therefore, that similarly the ratio of a prism to a cylinder is composed of the ratios of the bases and the altitudes since the prism is triple its pyramid and the cylinder its cone.

Corollary II.

Whence it easily follows that a pyramid and a cone whose bases are inversely proportional to their altitudes are equal to one another. Contrariwise, if a pyramid and a cone are equal, their bases will be inversely proportional to their altitudes. The same thing is to be grasped for a prism and a cylinder.

Proposition XXXVI.

A SPHERE IS EQUAL TO THE PYRAMID WHOSE BASE IS EQUAL TO THE SURFACE OF THE SPHERE AND WHOSE ALTITUDE IS EQUAL TO THE RADIUS OF THE SPHERE.[1]

Let A be a sphere [see Fig. III.5C.36]. Let P be a pyramid having a base equal to the surface of the sphere and an altitude equal to the radius of a sphere A.

I say that pyramid P is equal to sphere A.

For let M be a cone having a base equal to the surface of sphere A and an altitude equal to the radius of sphere A. And so pyramid P and cone M have mutually equal bases and altitudes. Therefore by the second corollary to [Proposition] XXXIII above, pyramid P is equal to cone M. But by [Proposition] XXV cone M is equal to sphere A. Therefore pyramid P is also equal to sphere A. Q.E.D.

Corollary I.

Whence it is evident that the prism whose base is equal to the surface of a sphere and whose altitude is equal to the radius of the sphere is triple the sphere, since by XII.8 [of Euclid][2] such a prism is triple its pyramid and the pyramid is equal to the sphere.

Corollary II.

Therefore a prism whose base is equal to the surface of a sphere but whose altitude is one-third part of the radius of the sphere is equal to the sphere.

Corollary III.

And since by [Proposition] X of this [work] the surface of a sphere is equal to the rectangle which arises from the product of the diameter of the

Prop. XXXVI
[1] See Com., Prop. XXXVI, lines 2–4.
[2] See Com., Prop. XXXIII, line 42.

sphere and the circumference of its greatest circle, therefore the sphere is equal to a solid [rectangular] parallelepiped whose base is the said rectangle and whose altitude is one-third part of the radius of the sphere.

Corollary IV.

It is evident, therefore, that from the product of the radius of a sphere and its surface a volume triple that of the sphere is produced, for such a product is equal to the volume of a prism having a base equal to the surface of the sphere and an altitude equal to the radius of the sphere.

Corollary V.

Therefore, from the product of one-third part of the radius and the surface of a sphere the volume of the sphere is produced.

Proposition XXXVII.

A SPHERICAL SECTOR IS EQUAL TO THE PYRAMID WHOSE BASE IS EQUAL TO THE SPHERICAL BASE OF THE SECTOR AND WHOSE ALTITUDE IS EQUAL TO THE RADIUS OF THE SPHERE.[1]

Let A be a spherical sector, and let P be a pyramid having a base equal to the spherical base of sector A and an altitude equal to the radius of the sphere of which A is the sector [see Fig. III.5C.37].

I say that pyramid P is equal to sector A.

For let M be a cone whose base is equal to the spherical base of sector A and whose altitude is equal to the radius of the sphere. By the second corollary to [Proposition] XXXIII of this [work] pyramid P is equal to cone M. But by [Proposition] XXX cone M is equal to sector A. Therefore pyramid P is also equal to sector A. Q.E.D.

Corollary I.

Whence it is evident that the prism whose base is equal to the spherical base of a spherical sector and whose altitude is equal to the radius of the sphere [from which the sector is taken] is triple this sector, since by XII.8 [of Euclid] such a prism is triple its pyramid and the pyramid is equal to the sector. Therefore the prism whose base is equal to the spherical base of a spherical sector and whose altitude is one-third part of the radius of the sphere is equal to the said spherical sector.

And since by the corollary to [Proposition] XVI of this [work] the spherical base of a spherical sector is equal to a circle whose radius is the straight line which proceeds from the apex of the spherical segment to the circumference of its base and since by [Proposition] IV of *On the Measurement of the Circle* this circle is equal to the rectangle contained

Prop. XXXVII
[1] See again Com., Prop. XXXVI, lines 2–4.

by its radius and half its circumference, therefore the solid parallelepiped whose base is the said rectangle and whose altitude is one-third part of the radius of the sphere is equal to the said spherical sector.

Corollary II.

It is evident, therefore, that from the product of the radius of a sphere and the spherical base of a spherical sector [of that sphere] a volume is produced which is triple the sector, while from the product of one-third part of the radius and the said base the volume of the sector itself is produced.

Proposition XXXVIII.

A SPHERE HAS TO THE CUBE OF ITS DIAMETER ALMOST THE RATIO THAT ELEVEN HAS TO TWENTY-ONE.[1]

Let A be a sphere and R the cube whose edge is equal to the diameter of sphere A [see Fig. III.5C.38].

I say that sphere A is to cube R almost as 11 is to 21, and I prove it as follows.

Let S be a cylinder whose base diameter is equal to the diameter of sphere A and whose altitude is equal to the same diameter. And so the altitude of prism R and cylinder S is the same. Therefore by the [first] corollary to [Proposition] XXXIII of this [work] prism R / cylinder S = base of prism R / base of cylinder S. But the base of prism R is the square of the diameter of the base of cylinder S. And by [Proposition] VIII of *On the Measurement of the Circle* the square of the diameter of a circle is to the circle almost as 14 to 11. Therefore prism R is to cylinder S almost as 14 to 11. But by the corollary to [Proposition] XXVI of this [work] or by [Proposition] XXVIII cylinder S = $\frac{3}{2}$ sphere A; therefore [they are related] as 21 to 14. Hence by V.23 [of Euclid] and equal proportionality cube R / sphere A = 21 / 11. Q.E.D.

Corollary I.

It is evident, therefore, that 21 spheres are equal to 11 cubes of the diameter of a sphere. For by the [proposition] above a sphere is to the cube of its diameter as 11 is to 21. Therefore by V.13 [of Euclid] 11 spheres / 11 cubes = 11/21. Inversely, 11 cubes / 11 spheres = 21/11. But 21 spheres / 11 spheres = 21/11. Therefore 21 spheres have the same ratio to 11 spheres as 11 cubes have to these same 11 spheres. Hence by V.9 [of Euclid] 21 spheres are equal to 11 cubes of the diameter of a sphere.

Prop. XXXVIII
[1] See Com., Prop. XXXVIII, lines 2–3.

Corollary II.

Whence it is evident that, if a cube of the diameter of any sphere is multiplied 11 times, $\frac{1}{21}$ of the product will be the volume of the sphere. Contrariwise, if a sphere is multiplied 21 times, $\frac{1}{11}$ of the product will be the cube of the diameter of the sphere. But these considerations assume that the ratio of the circumference to the diameter is $3\frac{1}{7}$. But if the ratio of the circumference to the diameter is assumed to be $3\frac{10}{71}$, then by [Proposition] VIII of *On the Measurement of the Circle* the square of the diameter of a circle is to that circle as 284 is to 223, and by [Proposition] XXVIII of this [work] a cylinder is to its sphere as 426 is to 284, i.e. as 3 is to 2. Therefore by V.23 [of Euclid] and equal proportionality cube R / sphere $A = 426/223$. Therefore 426 spheres will be equal to 223 cubes of the diameter of the sphere, which follows from the said argument. Whence if the cube of the diameter of a sphere is multiplied 223 times, $\frac{1}{426}$ of the product will be the volume of the sphere. Contrariwise, if the sphere is multiplied 426 times, $\frac{1}{223}$ of the product will be the cube of the diameter of the sphere. But the first supposition which makes the [ratio of] circumference to diameter to be $3\frac{1}{7}$ is actually a little too great, while the other supposition which makes the [ratio of] circumference to diameter to be $3\frac{10}{71}$ is in truth too little. For it was not permitted in this matter to attain geometrical exactness, not even to Archimedes himself, though he was the most acute of all geometers.

The End
At Messina, 10 September of the eighth indiction, 1534.

Commentary to Text C

Proposition I

5–12   Although this proposition is not given as a separate proposition in the *Liber de curvis superficiebus*, the substance of its enunciation and proof is found there in Proposition I (Vol. 1, pp. 454–56, lines 48–57). In Tinemue's work the regular polygon was an octagon; in Maurolico's a pentagon, thus reflecting the procedure of Maurolico's *De circuli dimensione*, Proposition III (see Text B above).

Proposition II

2–5   "Coni . . . basis." This was drawn by Maurolico from Proposition I of the *Liber de curvis superficiebus* (Vol. 1, p. 450, lines 1–5). Maurolico has altered the terminology. For *rotunda pyramis* he used *conus*, for *ypothenusa* the more modern *conicum latus*. But in this proposition he retained *curva superficies*, though he occasionally changed it in other places. Maurolico's proof of this proposition is a free adaptation of the medieval proof and is adequately described by the summary I have given

in Volume 1, pp. 507–08. It is the first of the proofs in which Maurolico used the so-called "easier way," which I have discussed in Section III of this chapter. A.-M. Legendre offers a similar proof using the "easier way" in his *Elements of Geometry and Trigometry*, tr. of E. D. Brewster (Edinburgh, 1824), Bk. VIII, Prop. VII, pp. 202–03:

Proposition VII. Theorem.

The convex surface of a cone is equal to the circumference of its base multiplied by half its side. [See Fig. III.5C.39.]

Let AO be a radius of the given cone's base, S its vertex, and SA its side: the surface will be circ. $AO \cdot \frac{1}{2}SA$. For, if possible, let circ. $AO \cdot SO$ be the surface of a cone having S for its vertex, and for its base a circle whose radius OB is greater than AO.

About the smaller circle describe a regular polygon $MNPT$, the sides of which shall not meet the circle whose radius is OB; and let $SMNPT$ be the regular pyramid, having this polygon for its base and the point S for its vertex. The triangle SMN, one of those which compose the convex surface of the pyramid, has for measure its base MN multiplied by half its altitude SA, or half the side of the given cone; and since this altitude is the same in all the other triangles SNP, SPQ, etc., the convex surface of the pyramid must be equal to the perimeter $MNPTM$ multiplied by $\frac{1}{2}SA$. But the contour $MNPTM$ is greater than circ. AO; hence the convex surface of the pyramid is greater than circ. $AO \cdot \frac{1}{2}SA$, and consequently greater than the convex surface of the cone having the same vertex S and the circle whose radius is OB for its base. On the contrary, however, the surface of this cone is greater than that of the pyramid; because, if two such pyramids are adjusted to each other base to base, and two such cones base to base, the surface of the double cone will envelope on all sides that of the double pyramid, and therefore (Lemma 2.) be greater than it; hence the surface of the cone is greater than that of the pyramid. The reverse of this resulted from our hypothesis; hence that hypothesis was false; hence, in the first place, the circumference of the cone's base multiplied by half the side cannot measure the surface of a greater cone.

We are next to shew that it cannot measure the surface of a smaller cone. Let BO be the radius of the given cone's base; and, if possible, let circ. $BO \cdot \frac{1}{2}SB$ be the surface of a cone having S for its vertex, and AO less than OB for the radius of its base.

The same construction being made as above, the surface of the pyramid $SMNPT$ will still be equal to the perimeter $MNPT \cdot \frac{1}{2}SA$. Now this perimeter $MNPT$ is less than circ. OB; likewise SA is less than SB: hence, for a double reason, the convex surface of the pyramid is less than circ. $OB \cdot \frac{1}{2}SB$, which, by hypothesis, is the surface of the cone having SA for the radius of its base; hence the surface of the pyramid must be less than that of the inscribed cone. On the contrary, however, it is greater; for, adjusting two such pyramids to each other base to base, and two such cones base to base, the surface of the double

pyramid will envelope that of the double cone, and (Lemma 2.) will be greater than it. Hence, in the second place, the circumference of the given cone's base multiplied by half the side cannot be the measure of the surface of a smaller cone.

Hence finally, the convex surface of a cone is equal to the circumference of its base multiplied by half its side.

19 "per 13^{am} 12^{i}" Campanus XII.13 = Greek and Zamberti XII.16. See the *Elementa* (Basel, 1546), pp. 413–14.

66–69 "Manifestum . . . trigoni." The thrust of this corollary is obvious. If by the proposition the lateral surface is equal to the right triangle with sides about the right angle equal respectively to the slant height and the circumference of the base, then by Proposition I.41 of the *Elements* it is equal to the product of one of the sides and half the other. In general, throughout the treatise, Maurolico introduced corollaries that convert the rectangular figures to which the curved figures are equal to products, that is to algebraic forms with which the early modern mathematicians began to be more comfortable.

71–73 "Rursum . . . primae (*!* sextae)." This is easily proved when we realize that the reference to the "third corollary of the first" is an error for the "third corollary of the sixth" proposition. It might seem strange for the author to have cited a later proposition until it is realized that the proof of Proposition VI is independent of the first five propositions. The cited corollary holds that if there are two unequal circles, then $r \cdot c' = r' \cdot c$ when r and r' are their radii, c and c' their circumferences. Now this equation is applicable to the corollary here if we consider the first circle to be the base circle of the cone with radius r and circumference c and the second circle to be one with radius r' equal to the slant height of the cone and circumference c'. Thus by the third corollary of the sixth proposition $r \cdot c'/2 = r' \cdot c/2$. But the surface of the cone $S = r' \cdot c/2$ by Corollary I of this Proposition II. Hence $S = r \cdot c'/2$. Q.E.D.

75–77 "Demum . . . primae (*!* sextae)." This is equivalent to Proposition I.14 of Archimedes' *On the Sphere and the Cylinder* (see Vol. 2, 25vV). As I have noted in Section III of this chapter, it could have been seen by Maurolico in Johannes de Muris' *De arte mensurandi*, Chap. X, if Maurolico had not yet seen Archimedes' work in 1534. Maurolico's proof (completely different from Archimedes') is obvious once we realize that the citation "ex postremo corollario primae" is actually to the last (i.e. fourth) corollary of the "sixth" proposition. This is the same error which I have described in the preceding comment. The cited corollary holds that if we have three circles (with radii r, r' and r'') whose diameters (and hence radii) are in continual

proportion, then $r \cdot c''/2 = r' \cdot c'/2 = $ Area of 2nd circle. In the corollary to Proposition II here being proved, the first circle is that which has a radius r equal to the slant height, the second circle is that which has a radius r' equal to the mean proportional between r and r'' and the third circle is that whose radius r'' is equal to the radius of the base circle of the cone. Now by Corollary I to Proposition II, the surface of the cone $S = r \cdot c''/2$, and by Corollary IV of Proposition VI $r \cdot c''/2 = r' \cdot c'/2 = $ Area of 2nd circle. Hence $S = $ Area of 2nd circle. Q.E.D. This corollary is the first of several added by Maurolico to convert the rectangular measures of curved surfaces to circular measures. It provides authority for the assumption of a cone with base circle equal to the surface of another cone needed in later propositions (e.g. see Proposition XXIII).

Proposition III

2–3 "Conica . . . basis." This proposition is identical with the corollary to Proposition I of the *Liber de curvis superficiebus* (Vol. 1, pp. 458–60, lines 105–17), and with Proposition I.15 of Archimedes' *On the Sphere and the Cylinder* (see Vol. 2, 26rK). That Maurolico took it from the former rather than the latter seems to be assured by the fact that, like the author of the former, Maurolico used Proposition IV of *De dimensione circuli* (=Proposition I of *On the Measurement of the Circle*) while Archimedes used Proposition XII.2 of Euclid.

Proposition IV

2–5 "Cylindri . . . aequale." Maurolico has taken this enunciation from Proposition II of the *Liber de curvis superficiebus* (Vol. 1, p. 460, lines 1–4), changing *columpna rotunda* to *cylindrus*, *tetragonum* to *rectangulum* and *circumferentia* to *peripheria*. In his proof Maurolico follows the instructions of the author of the *Liber de curvis superficiebus* by constructing a proof similar to that given in Proposition II (i.e. Proposition I of the medieval work). He thus employs the "easier way." See also Legendre's use of the "easier way" in his *Elements of Geometry*, Bk. VIII, Prop. IV (*ed. cit.*, pp. 198–99):

PROPOSITION IV. THEOREM.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.

Let CA be the radius of the given cylinder's base, H its altitude; the circumference whose radius is CA, being represented by circ. CA, we are to shew that circ. $CA \cdot H$ will be the convex surface of the cylinder. [See Fig. III.5C.40.] For, if this proposition is not true, then circ. $CA \cdot H$ must be the surface of a greater cylinder, or of a less one. Suppose it first to be the surface of a less cylinder; of the cylinder, for example, the radius of whose base is CD, and whose altitude is H.

About the circle whose radius is *CD*, circumscribe a regular polygon *GHIP*, the sides of which shall not meet the circle whose radius is *CA*; conceive a right prism having *H* for its altitude, and the polygon *GHIP* for its base. The convex surface of this prism will be equal (2.VIII.) to the contour of the polygon *GHIP* multiplied by the altitude *H*: this contour is less than the circumference whose radius is *CA*; hence the convex surface of the prism is less than circ. $CA \cdot H$. But, by hypothesis, circ. $CA \cdot H$ is the convex surface of the cylinder whose base has *CD* for its radius; which cylinder is inscribed in the prism: hence the convex surface of the prism must be less than that of the inscribed cylinder. On the other hand (3.VIII.) it is greater; hence our hypothesis was false: *hence, in the first place, the circumference of a cylinder's base multiplied by its altitude cannot be the measure of a smaller cylinder.*

We are next to shew that this product cannot be the measure of a greater cylinder. For, retaining the present figure, let *CD* be the radius of the given cylinder's base; and, if possible, let circ. $CD \cdot H$ be the convex surface of a cylinder, which with the same altitude has for its base a greater circle, the circle, for instance, whose radius is *CA*. The same construction being performed as above, the convex surface of the prism will again be equal to the contour of the polygon *GHIP* multiplied by the altitude *H*. But this contour is greater than circ. *CD*; hence the surface of the prism must be greater than circ. $CD \cdot H$, which, by hypothesis, is the surface of [the] cylinder having the same altitude, and *CA* for the radius of its base. Hence the surface of the prism must be greater than that of the prism (*!* cylinder). But even though this prism were inscribed in the cylinder, its surface (3.VIII.) would be less than the cylinder's; still farther is it less when the prism does not reach so far as to touch the cylinder. Hence our last hypothesis also was false; hence, in the second place, *the circumference of a cylinder's base multiplied by the altitude cannot measure the surface of a greater cylinder.*

Hence, finally, the convex surface of a cylinder is equal to the circumference of its base, multiplied by the altitude.

72–73 "Manifestum . . . basis." Since the cylindrical surface is equal to a rectangle with sides equal respectively to the axis of the cylinder and the circumference of the base and such a rectangle equals the product of these sides, then the cylindrical surface is equal to the same product.

75–77 "Rursum . . . primae (*!* sextae)." The reasoning is precisely like that of Corollary II to Proposition II. Cf. my Commentary, Prop. II, lines 71–73.

78–81 "Demum . . . primae (*!* sextae)." This corollary is equivalent to Proposition I.13 of Archimedes' *On the Sphere and the Cylinder* (see Vol. 2, 25vD-E). But again I note that Maurolico's source need not have been the actual text of Archimedes. He could have found this proposition in Johannes de Muris' *De arte mensurandi*, Chap. X, as I have suggested in Section III of this

chapter. Its proof here (again completely different from Archimedes') is similar to that of Corollary III to Proposition II. Cf. my Commentary to that corollary, lines 75–77. As before, this corollary provides justification for the assumption in later propositions (e.g. Propositions XXIII and XXVII) of the existence of a cone having as its base circle the surface of some cylinder.

Proposition V

2–5 "Cylindrica . . . semidiametri." This two-part proposition was taken by Maurolico from the two-part corollary to Proposition II of the *Liber de curvis superficiebus* (Vol. 1, p. 462, lines 23–27), with some terminological changes. Maurolico's proof makes specific the general instructions for proof in the medieval work.

Proposition VI

2–3 "Circulorum . . . proportionales." The enunciation and proof (in fact the two proofs, since they are virtually identical) were drawn by Maurolico from Proposition III of the *Liber de curvis superficiebus* (see Vol. 1, pp. 462–66), once more with terminological changes. The proofs use the "easier way" and their character is described in my summary of the proof of Proposition III of the medieval work (see Vol. 1, pp. 510–11). We should note that in these proofs we have an example of the case where the second half of the proof is so constructed that it leads to the very same impossibility refuted in the first half. Such was not the case in Maurolico's proofs of Propositions II and IV above, where the second half of each proof is a *reductio* argument similar to but independent of the first half. The fact that Maurolico gives two almost identical proofs for the propositions is not unique. A similar practice is also reflected in Maurolico's treatment of the various solutions of the problem of finding two mean proportionals, as I shall indicate in Section III of the next chapter. Of the two proofs given by Maurolico, the first is probably the one preferred by him, for it is always the set of corollaries to the first proof that he cites in other propositions. I suspect that the editor was at a loss as to which proof Maurolico wanted in the published text and so he printed them both. Incidentally, Maurolico had previously proved this proposition in Proposition V of his *De circuli dimensione* (see above Text B). There the proof rests upon Proposition IV of that work and hence ultimately on the Archimedean method of Proposition I of *On the Measurement of the Circle*. Maurolico was also to prove the same proposition in Proposition VIII of his *Praeparatio* (see Text A above), but there the proof was by what I have called the "converse way" rather than by the "easier way." Finally

it is worth noting that Legendre uses the "easier way" to prove the same proposition in his *Elements of Geometry*, Bk. IV, Prop. XI (*ed. cit.*, pp. 94–95). Since I have already given two instances of proofs by Legendre using the "easier way," from this point on I shall only cite the text without giving the full proofs.

31–37 "Manifestum. . . . circulorum." The substance of these first two corollaries was used in the *Liber de curvis superficiebus*, Prop. V (Vol. 1, p. 474, lines 75–79), where it is noted that they follow from Proposition III of the *Liber de curvis superficiebus* (=Maurolico's Proposition VI) and the penultimate proposition of Book V of Euclid. I am reasonably certain that Maurolico saw them to be necessary for the proof of Proposition IX of his work (see line 48) and so decided to present them here as corollaries.

34 "13ᵃ 5ⁱ" Campanus V.13 = Greek and Zamberti V.12. See the *Elementa* (Basel, 1546), pp. 123–24.

35–37 "Quare . . . circulorum." This corollary was also necessary for the proof of Proposition IX (see line 52).

38–40 "Item . . . dicendum." This corollary was drawn by Maurolico from the corollary to Proposition III of *Liber de curvis superficiebus* (Vol. 1, p. 466, lines 36–38). We have already seen how it was used in the proof of Corollary II to Proposition II and similarly in the proof of Corollary II to Proposition IV. It was also used in the proofs of Corollary I to Proposition IX and of the corollaries to Proposition XIV. Finally, it should be observed that this corollary became Proposition XLIV in Maurolico's *Praeparatio* (see Text A above).

41 "15ᵃ 6ⁱ" Campanus VI.15 = Greek and Zamberti VI.16. See the *Elementa* (Basel, 1546), pp. 149–50.

42–49 "Denique. . . . circuli." This corollary is important for the proofs of Corollary III to Proposition II, of Corollary III (unnumbered) to Proposition IV, of the corollaries to Proposition VII, of Corollary I to Proposition IX and of the corollaries to Propositions XIV and XV. Or to put it briefly, it was crucial in Maurolico's efforts to convert the rectangular measures of curved surfaces to circles, i.e., to pass from the rectangular measures of the *Liber de curvis superficiebus* to the circular measures of Archimedes' *On the Sphere and the Cylinder*.

50 "Alia . . . sextae." This demonstration with its corollaries is so close to the first demonstration with its four corollaries that I am sure that if Maurolico had been the editor he would have included only one of them. I have already suggested a possible preference on Maurolico's part for the first demonstration.

Proposition VII

2–3 "Coni-coluri . . . peripherias." Though the enunciation has

been shortened, its source is clearly Proposition IV of the *Liber de curvis superficiebus* (Vol. 1, pp. 466–68). The proofs are substantially the same. An equivalent proposition with a distinctly different proof was presented in the *Verba filiorum* (*ibid.*, pp. 302–06).

39–44 "Itaque. . . . primae (*!* sextae)." Cf. my Commentary, Proposition II (lines 71–73, 75–77), Proposition IV (lines 75–77, 78–81) and Proposition VI (lines 38–40, 42–49). The second corollary is equivalent to Proposition 1.16 of Archimedes' *On the Sphere and the Cylinder* (Vol. 2, 26rO). This corollary justifies the assumption in Propositions XXIII, XXIV and XXIX of cones with base circles equal to the curved surfaces of truncated cones.

Proposition VIII

2–8 "Si . . . descriptorum." This enunciation was drawn from the first half of the enunciation of Proposition V of the *Liber de curvis superficiebus* (Vol. 1, pp. 468–70, lines 1–7), with only minor terminological changes. Maurolico's next proposition contains the remainder of the enunciation.

9–50 "Intra. . . . proponitur." Maurolico follows the author of the *Liber de curvis superficiebus* in giving two proofs, one where the semipolygon rotated has an odd number of sides and the other where it has an even number of sides. However, in the medieval tract the proof for the one with an even number of sides is given first.

Proposition IX

2–9 "Si . . . rectangulum." This enunciation was drawn by Maurolico from the second half of the enunciation of Proposition V of the *Liber de curvis superficiebus* (Vol. 1, p. 470, lines 7–11).

16 "per 29am 3ii" Campanus III.29 = Greek and Zamberti III.30. See the *Elementa* (Basel, 1546), p. 77.

37 "per 26am 3ii" Campanus III.26 = Greek and Zamberti III.27. See *ibid.*, p. 75.

64–69 "Unde. . . . primae (*!* sextae)." Compare my Commentary, Proposition II (lines 71–73, 75–77), Proposition IV (lines 75–77, 78–81), Proposition VI (lines 38–40, 42–49). The second part of the corollary appears within the proof of Proposition I.25 of Archimedes' *On the Sphere and the Cylinder* (Vol. 2, 27vK–L). It justifies the assumption in Proposition XXV of a cone whose base circle is equal to the surface of the solid of rotation considered in this proposition.

Proposition X

2–4 "Sphaerae . . . circuli." This proposition was taken by Maurolico from Proposition VI of the *Liber de curvis superficiebus* (Vol. 1, pp. 478–80). The proof is, of course, that of the "easie

way." A similar proof was given by Legendre, *Elements of Geometry*, Bk. VIII, Prop. X (*ed. cit.*, p. 206). V. Flauti also used the "easier way" in his *Corso di geometria elementare e sublime*, Vol. 2 (Naples, 1852), pp. 406–408.

15 "per 13^{am} 12ⁱ" Campanus XII.13 = Greek and Zamberti XII.16. See the *Elementa* (Basel, 1546), pp. 413–14.

60–61 "Manifestum . . . sphaerae." Maurolico presents a single circle as the measure of the surface of the sphere, as indeed did Archimedes in the course of his proof of Proposition I.33 of *On the Sphere and the Cylinder* (=Moerbeke I.31; see Vol. 2, 29rA). Maurolico does this (even though he will present the more conventional measure of four great circles of the sphere in the next proposition) because he wishes to justify the assumption in Proposition XXV of the existence of a cone whose base circle is equal to the surface area of a sphere.

Proposition XI

2–5 "Sphaerae . . . diametro." Maurolico has taken this proposition directly from the corollary to Proposition VI of the *Liber de curvis superficiebus* (Vol. 1, pp. 480–82). His proof was also taken from the medieval text, except that he has adapted it to a particular sphere and a particular cylinder. This proposition appears as Proposition I.33 of Archimedes' *On the Sphere and the Cylinder* (see Vol. 2, 29rA), with an entirely different proof.

12 "per 18^{am} sexti" Campanus VI.18 = Greek and Zamberti VI.20. See the *Elementa* (Basel, 1546), pp. 152, 154.

28–30 "Manifestum . . . sesquialtera." This corollary was taken from the supplementary corollary to Proposition VI of the *Liber de curvis superficiebus* (Vol. 1, p. 482, lines 54–56).

Proposition XII

2–9 "Si . . . superficies." This proposition with its scholium and corollary was independently added by Maurolico, it being in neither the *Liber de curvis superficiebus* nor Archimedes' *On the Sphere and the Cylinder*. With it, the section concerned with, and leading up to, the determination of the surface of a sphere is terminated.

14 "per 15^{am} 3ⁱⁱ" Campanus III.15 = Greek and Zamberti III.16. See the *Elementa* (Basel, 1546), pp. 66–68.

Proposition XIII

2–8 "Conicae . . . abscindentis." This proposition and the next were constructed by Maurolico on the model of Propositions VIII and IX (and thus ultimately on the model of Proposition V of the *Liber de curvis superficiebus*). This illustrates the continued dominance of the medieval work even over the proofs of propositions not included in the medieval tract.

Proposition XIV

22 "per 13ᵃᵐ 5ⁱ" Campanus V.13 = Greek and Zamberti V.12. See
the *Elementa* (Basel, 1546), pp. 123–24.

33–34 "per 15ᵃᵐ sexti" Campanus VI.15 = Greek and Zamberti
VI.16. See *ibid.*, pp. 149–50.

53–55 "Nec . . . primae (! sextae)." This corollary is completely
parallel to Corollary I of Proposition IX. It does not appear as
a separate proposition in Archimedes' *On the Sphere and the
Cylinder* but is developed in the course of Proposition I.37
(=I.35 in the Moerbeke text; see Vol. 2, 29vN). It provides
authority for the assumption in Proposition XXIX of a cone
whose base circle is equal to the surface of the segment of this
kind of solid of rotation. See the Commentary, Proposition IX,
lines 64–69.

Proposition XV

2–5 "Si . . . segmenti." This proposition was constructed entirely
on the model of Proposition X (and thus ultimately on that of
Proposition VI of the *Liber de curvis superficiebus*), except that
it is concerned with the surface of a spherical segment rather
than that of the whole sphere. It could be that Proposition
III(XI) in the Florentine version of the *Liber de curvis super-
ficiebus* provided Maurolico with the suggestion to construct a
proof like that of Proposition VI of the medieval work, since
the proof of Proposition III(XI) specifically depended on
Proposition V of the medieval work (see Vol. 1, p. 534, lines
15–16) and of course Proposition VI itself depended on Proposi-
tion V.

111–14 "Item . . . primae (! sextae)." This corollary reducing the
rectangular measure of a spherical segment to a circle is not
in Archimedes' *On the Sphere and the Cylinder* but it leads
directly to Proposition XVI, which is equivalent to Proposition
I.42 of Archimedes' work (=I.40 of Moerbeke's text; see Vol. 2,
30vL). The close relationship between this corollary and the next
proposition is asserted by Maurolico (lines 114–15).

Proposition XVI

2–8 "Si . . . semidiameter." The second part of the enunciation is
equivalent to the above-noted Proposition I.42 of Archimedes'
work. I have suggested in Section III of this chapter that Mauro-
lico may have learned of the Archimedean proposition from
Johannes de Muris' *De arte mensurandi*, Chap. X. It also
circulated in a brief fragment of *On the Sphere and the Cylinder*
perhaps translated by Gerard of Cremona (see Vol. 1, p. 436,
Prop. 2). The proof of the first half of the proposition, which
presents another rectangular measure of a curved surface, de-
pends on the preceding proposition. Furthermore, the second

half of the proof is simply proved by the simultaneous application of the first half and Proposition IV of Maurolico's *De circuli dimensione*. While its proof is given in the second half of the proposition, it is also reflected in the corollary (lines 47–59). The corollary is crucial for the proof of the next proposition, just as Archimedes' Proposition I.42 was crucial for the proof of Proposition II.3.

37–38 "per 29ᵃᵐ (! 30ᵃᵐ) 3ⁱⁱ" Campanus III.30 = Greek and Zamberti III.31. See the *Elementa* (Basel, 1546), pp. 77–78.

Proposition XVII

2–3 "Si . . . segmentorum." This proposition is present in the corollary to Proposition III(XI) of the Florentine version of the *Liber de curvis superficiebus* (Vol. 1, p. 540, lines 103–04). It also appears as a part of the proof of Archimedes' *On the Sphere and the Cylinder*, Proposition II.3 (Vol. 2, 32rB-C).

Proposition XVIII

2–4 "Si . . . segmenti." This proposition is equivalent to Proposition III(XI) of the Florentine version of the *Liber de curvis superficiebus* (Vol. 1, p. 532). Maurolico's proof depends on Proposition XVI and thus does not need to employ the method of the "easier way," which the author of the Florentine version used. Maurolico had already used that method in Proposition XV on which Proposition XVI depended. Indeed, since Proposition III(XI) of the Florentine version was cast in the form of Proposition VI of the *Liber de curvis superficiebus*, Maurolico's alleged use of the Florentine version may well have suggested to him his proof of Proposition XV, which was also constructed on the model of Proposition VI of the *Liber de curvis superficiebus* (see above, Commentary, Proposition XV, lines 2–5).

Proposition XIX

2–6 "Si . . . segmenti." This proposition does not occur in either the *Liber de curvis superficiebus* or in Archimedes' *On the Sphere and the Cylinder*. But the determination of the surface areas of spherical zones was a popular exercise in the manuals of Piero della Francesca (see his *Trattato*, described in Part III, Chap. 2, Sect. III, n. 14; repeated in his *De corporibus regularibus*, ed. of G. Mancini, p. 555, and the Italian version published by Pacioli with his *Divina proportione* [see casus 28, 19v]) and Luca Pacioli (see *Summa*, II, 73r, Prob. 53, which reflects Piero della Francesca's *Trattato*). Both the *De corporibus regularibus* and the *Summa* would have been easily available to Maurolico, since both were printed and widely read.

Proposition XX

2–7 "Si . . . aequales." So far as I know, this proposition was entirely original with Maurolico. It marks the end of the section

on the areas of segments of solids of rotation and spherical segments.

Proposition XXI

2–4 "Conus . . . illorum." This and the succeeding proposition are ancillary to the proof of Proposition XXIII. Proposition XXI was probably constructed out of lines 163–70 of Proposition VII of the *Liber de curvis superficiebus* (Vol. 1, p. 494).

Proposition XXIII

2–21 "Solidum. . . . delapsae." This proposition is preliminary to Proposition XXIV (and thus to Proposition XXV dependent on Proposition XXIV). It reveals the procedures of Proposition VII of the *Liber de curvis superficiebus* (Vol. 1, pp. 482–94) and thereby concerns the measures of the solids described by the rotation of triangles applied laterally or angularly to an axis. In each case the measure is shown to be a cone whose base is equal to the conical or cylindrical surface described by the side opposite the fixed terminus (i.e. the angle applied to the axis) and whose altitude is equal to the perpendicular drawn from that fixed terminus or angle to its opposite side. In the first part of the proposition the triangle rotated is laterally applied to the axis (i.e., one of its sides is the axis). There are three possible cases of such application depending on whether the angle at the fixed terminus is a right angle, an acute angle or an obtuse angle. In the second part of the proposition the triangle in rotation is imagined to have an angle other than a right angle with the axis. In the solids fashioned in the first and second parts of the proposition the side opposite the fixed terminus describes a conical surface. Finally, in the third part, the side opposite the fixed terminus of the rotating triangle is parallel to the axis and the surface described by that side is a cylindrical surface (and the perpendicular from the fixed terminus to the opposite side either falls on the terminus of the opposite side or it falls within or without the termini of the opposite side).

I have suggested earlier that the corollaries to Propositions II, VII and IV justify the assumption of cones whose base circles are equal to the surfaces of cones (Part I), truncated cones (Part II) and cylinders (Part III). I should add that the third part of the proposition does not directly reflect the procedure of the *Liber de curvis superficiebus* but was added by Maurolico because in his next proposition he used a solid of rotation described by a semipolygon having an odd number of sides (and thus where the middle side describes a cylindrical surface) while the author of the *Liber de curvis superficiebus* in Proposition VII employs only a solid described by a semipolygon with

an even number of sides all of which sides describe conical
surfaces (see the Commentary, Proposition XXIV, line 9).
37 "per 12^{am} 12^i" Campanus XII.12 = Greek and Zamberti
XII.15. See the *Elementa* (Basel, 1546), pp. 412–13.
104 "per 9^{am} 12^i" Campanus XII.9 = Greek and Zamberti XII.10.
See *ibid.*, pp. 402–03.

Proposition XXIV

2–7 "Solidum . . . egreditur." This proposition was taken from
Proposition VII of the *Liber de curvis superficiebus* (Vol. 1,
pp. 482–94). Its proof is very much shorter than that found
in the medieval tract, since the step-by-step use of the rotation
of triangles laterally or angularly applied to an axis was already
a part of the demonstration of the preceding proposition. This
proposition is also equivalent to Proposition I.26 of Archimedes'
On the Sphere and the Cylinder (see Vol. 2, 27vN–O) and in
fact the enunciation of Maurolico's proposition is closer to that
of Archimedes than that of the medieval tract, though the
proof obviously was constructed from the *Liber de curvis
superficiebus*. I do not know whether the closer agreement of
this proposition's enunciation with Archimedes' is accidental or
whether it is an indication that Maurolico was also following
the Archimedean text as he composed his elaboration of the
Liber de curvis superficiebus. Needless to say, I have already
discussed in Section III of this chapter the difficulties involved
in assuming that Maurolico had a copy of *On the Sphere and
the Cylinder* in 1534.

9 "decagonum *AB*" It is of interest that Maurolico here uses
a decagon as his regular polygon, though both Archimedes and
the author of the *Liber de curvis superficiebus* used a polygon
with 4n sides. Maurolico hearkens back to his use of the same
polygon in the first part of his Proposition VIII, where like the
author of the medieval tract he specifies that the proposition is
true for a solid described by a semipolygon with an odd
number of sides and also for one with an even number of sides.
But in Proposition XXIV, Maurolico proves that proposition
only for a solid described by a decagon, that is, by a semipolygon
of an odd number of sides. On the other hand, the author of
the *Liber de curvis superficiebus* in his Proposition VII
proves his proposition only for a solid described by a semi-
polygon with an even number of sides, at the same time
expressing some doubt whether it can be proved for a semi-
polygon with an odd number of sides (Vol. 1, p. 494, lines
174–80). He says he will leave the demonstration of the
latter case to a diligent posterity. This challenge was accepted

by the author of a version of the *Liber de curvis superficiebus* found in a manuscript of the British Museum (Harleian 625, 139v), as I have indicated in Vol. 1, pp. 547–57. Whether Maurolico's copy of the *Liber de curvis superficiebus* had this added commentary I do not know.

Proposition XXV

2–4 "Sphaera . . . est." This enunciation was taken by Maurolico from Proposition IX of the *Liber de curvis superficiebus* (Vol. 1, p. 502). However it was Proposition VIII of the medieval tract that included a proof by means of the "easier way" (*ibid.*, pp. 496–500), and which thus presented Maurolico with a model for his proof, which he adapts to a cone having as its base a circle equal to the surface of the sphere and as its altitude the radius of the sphere. Hence, in the medieval tract the method of the "easier way" was used to prove that a cylinder with axis and base diameter equal to the diameter of the sphere is 3/2 the sphere and in Maurolico's work that the sphere is equal to the above noted case. Finally, in Proposition I.34 of Archimedes' *On the Sphere and the Cylinder* (=I.32 in the Moerbeke text; see Vol. 2, 29rH) a different kind of proof was used to demonstrate that a sphere is equal to four times the cone whose base is equal to the greatest circle in the sphere and its altitude to the radius of the sphere. Thus the primary proposition proved in each of the three works differed from those of the other two. But even so, Maurolico followed the general method of the *Liber de curvis superficiebus* rather than that of Archimedes' work. Legendre later used the "easier way" to prove a proposition whose enunciation was more inclusive than those of his predecessors (*Elements of Geometry*, Bk. VIII, Prop. XV, *ed. cit.*, pp. 212–14): "Every spherical sector is measured by the zone which forms its base, multiplied by a third of the radius; and the whole sphere has for its measure a third of the radius, multiplied by its surface." Legendre also gave preliminary propositions (his Propositions XII–XIV, pp. 210–12) that essentially resemble Maurolico's Propositions XXIII and XXIV. We should further note that V. Flauti used the "easier way" to prove the same theorem as Maurolico in his *Corso di geometria elementare*, Vol. 2, pp. 409–10.

15 "per 13^{am} 12^{1}" Campanus XII.13 = Greek and Zamberti XII.16. See the *Elementa* (Basel, 1546), pp. 413–14.

Proposition XXVI

23–25 "Manifestum . . . sesquialter." This corollary was the main proposition proved in the *Liber de curvis superficiebus*, Prop.

VIII (Vol. 1, p. 496, lines 1–3). It was the corollary to Proposition I.34 of Archimedes' *On the Sphere and the Cylinder* (Cor. to Prop. I.32 in the Moerbeke text; see Vol. 2, 29rT). Maurolico repeats this corollary as a separate Proposition XXVIII, where it has a different proof.

25 "per 9ᵃᵐ 12ⁱ" Campanus XII.9 = Greek and Zamberti XII.10. See the *Elementa* (Basel, 1546), pp. 402–03.

Proposition XXVII

2–5 "Si . . . cono." This and the succeeding proposition constitute little more than manipulations of the predetermined measures of cylinders, cones and spheres. They need no commentary.

Proposition XXVIII

28–29 "Itaque . . . transibimus." This is an announcement of the end of the section on the volumes of spheres, cylinders and solids of rotation. He also announces the beginning of the section on the volumes of segments of such solids and of spheres.

Proposition XXIX

2–11 "Si . . . egreditur." This proposition was modeled on Proposition XXIV and hence ultimately on Proposition VII of the *Liber de curvis superficiebus*, except that here Maurolico was concerned with a segment of the solid of rotation instead of the whole solid. Like that of Proposition XXIV, its proof depends fundamentally on Proposition XXIII.

Proposition XXX

2–6 "Si . . . sphaerae." This proposition was modeled on Proposition XXV and has the same basic proof by the "easier way." It is equivalent to Proposition I.44 of Archimedes' *On the Sphere and the Cylinder* (=Proposition I.42 of Moerbeke's text; see Vol. 2, 31rA). I have discussed Maurolico's possible knowledge of this proposition from Johannes de Muris' *De arte mensurandi* in Section III above. The need for such a proposition would have occurred to Maurolico even if he had not read Archimedes' *On the Sphere and the Cylinder* but only the various treatments of the volumes of spherical segments in Fibonacci's *Practica*, Piero della Francesca's *De corporibus regularibus* and/or Pacioli's *Summa*. For all of these authors solved for such volumes by subtracting from a spherical sector a cone whose base is equal to the cutting circle and whose altitude is equal to the segment of the radius between the center of the sphere and the cutting circle.

Proposition XXXI

2–7 "Si . . . axem." This proposition is not in the *Liber de curvis superficiebus* in any form but was drawn either directly or indirectly from Proposition II.2 of Archimedes' *On the Sphere*

and the Cylinder (Vol. 2, 31vA–B). One possible source of its enunciation was Eutocius' *Commentary on the Sphere and the Cylinder* as partially translated by Valla, for in giving the beginning of Diocles' solution of the problem of cutting a sphere into a given ratio Valla includes Diocles' statement of the enunciation of Proposition II.2 (*De expetendis*, sign. x iii recto):

Scribens Pyria Diocles inquit demonstravit Archimedes quod omne segmentum sphaerae aequale est cono, basim quidem habenti cono quam ipsi segmento, fastigium autem quandam rationem habentem ad eum qui est ex segmento (! segmenti) verticem (! vertice) ad basim perpendicularem quam habet simul utrumque ex centro sphaerae et vicissim segmenti perpendicularis ad vicissim segmenti perpendicularem.

Though this is an awkward translation, Diocles gives the enunciation in terms of a specific figure immediately after it and there the purport of the enunciation is perfectly clear. The enunciation was also given by Johannes de Muris in the proem to Chapter X of his *De arte mensurandi*, as I have already said in Section III of this chapter. In that section I discussed at some length the difficulty that this proposition and its corollary poses for the view that Maurolico had not yet in 1534 seen the genuine text of Archimedes' *On the Sphere and the Cylinder*.

Proposition XXXII

2–6 "Si . . . segmenti." The source of this proposition and its proof was the last part of Dionysodorus' solution of the problem of cutting a sphere into a given ratio as presented in the above-noted partial translation by Valla of Eutocius' *Commentary on the Sphere and the Cylinder* (*De expetendis*, sign. x iii recto; cf. Vol. 2, 41vK–N). Since the proposition is necessary for Dionysodorus' solution, which Maurolico gives as his "Aliter" to this proposition, Maurolico removes the proposition and its proof from the actual solution and presents it as a prior proposition.

Proposition XXXII-Aliter

2–41 "Propositam. . . . Archimedis." As I have already noted in the previous account, Dionysodorus' solution of the problem of Proposition II.4 was drawn by Maurolico from Valla's partial translation of Eutocius (*De expetendis*, sig. x ii verso–x iii recto; cf. Vol. 2, 41rT–41vN).

17–18 "per 39^{am} (! 20^{am}) primi con[ic]orum elementorum" This runs in Maurolico's *Emendatio et restitutio conicorum Apollonii Pergaei* (Messina, 1654), p. 19: "Si in parabola a sectione

ducantur duae lineae ad diametrum ordinate: erunt ut quadrata quae ab ipsis fiunt ad invicem sic secatae (!) sub ipsis ex diametro ad summitatem.'' (*Punctuation slightly changed*.)

Proposition XXXIII

2–3 "Pyramis . . . bases." The proof is by the "easier way."

16 "per 12^{am} 5^{i}." Campanus V.12 = Greek and Zamberti V.13. See the *Elementa* (Basel, 1546), pp. 122, 124.

42 "per 8^{am} 12^{i}" This reference is to one of the additions to Proposition XII.8 of Campanus (*ibid.*, p. 400): "Omnis laterata columna tripla est ad suam pyramidem."

Proposition XXXIV

13 "per 11^{am} 12^{i}" This is incomplete, for XII.11 in all the versions of Euclid indicates rather that cones or cylinders of equal height are proportional to their bases. In the Greek and Zamberti text the proper proposition is XII.14 (*ibid.*, 412). But this proposition was not given in the Campanus version. True, the proof of XII.14 in the Greek text does use XII.11 but it also crucially uses Proposition XII.13, which is missing in the Campanus version. Hence the reference is at the least incomplete.

Proposition XXXV

11 "per 8^{am} 12^{i}" The reference here assumes two of the enunciations added to XII.8 in the Campanus version (*ibid.*, pp. 400–01), namely that which asserts that prismatic columns of equal altitude are as their bases and that which asserts that a prismatic column is triple its pyramid (see above, Commentary, Prop. XXXIII, line 42). The further citation of Euclid XII.11 is quite correct, for here it refers to cones of the same altitude.

Proposition XXXVI

2–4 "Sphaera . . . semidiametro." This and the succeeding proposition adapt Propositions XXV and XXX to pyramids with bases respectively equal to the surfaces of a sphere and its sector and altitudes each equal to the radius of the sphere. This is possible since such pyramids will be equal to the cones of the same bases and altitudes and in the above-noted propositions the cones were shown to be equal respectively to a sphere and its sector. The corollaries to Propositions XXXVI and XXXVII relate the sphere and its sector to prisms and parallelepipeds and also present their measures as products. The product that covers both the sphere and its sector is $V = (r/3) \cdot A$, where V is the volume of the sphere or sector, r is the radius of the sphere or sector and A is the surface area of the sphere or sector. Again I remind the reader

of Maurolico's proclivity for reducing the measures of curved figures to products, i.e. to algebraic forms.

Proposition XXXVIII

2–3 "Sphaera . . . fere." Maurolico has drawn this enunciation from Proposition X of the *Liber de curvis superficiebus* (Vol. 1, p. 504). It is fitting that Maurolico should make the last proposition of the medieval work his last proposition, since, as I have everywhere emphasized, Maurolico's work constitutes a reworking and expansion of the medieval tract.