ARCHIMEDES

in the Middle Ages

VOLUME THREE

THE FATE OF THE MEDIEVAL ARCHIMEDES
1300 to 1565

PART III: The Medieval Archimedes in the Renaissance, 1450–1565

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The *Praeparatio ad Archimedis opera* of Francesco Maurolico

/Francisci Maurolyci Messanensis Praeparatio ad Archimedis opera

Proemium

Archimedes Syracusanus acutissimus geometra, machinator praestantissimus, ac syderum speculator clarissimus extitit. Qui cum M. Marcellus Syracusas obsideret, machinis ingeniose inventis diu patriam tutatus est. Nam saxorum iactu, ferrea manu comprehensis navigiorum hostilium malis, telorum iaculatu per murorum rimas ad id factas, hostem admiratione pariter ac terrore concusserat, ut Livius ait. Is idem portentosae magnitudinis navim ab Hierone Rege constructam, solus, machinis sua una manu correptis, deduxit, ut Moschion multis refert. Sphaeram, in qua motus omnes astrorum repraesentarentur, fabricavit; de qua extat Claudiani Epigramma. Coronam auream a dicto Rege Diis dicatam, ab artifice vitiatam, novo miroque ingenio coarguit, ut Vitruvius refert. Sed non omnia eius inventa litteris mandata sunt, tam scilicet mechanica quam geometrica. Demum post triennium, captis Syracusis Philosophus illustrissimus ab imprudente milite, geometricis lineamentis in pulvere descriptis, intentus, dum interrogatus quis esset nomen suum edere differt, illumque ne lineas disturbaret, oraret, peremptus sanguine proprio deductas formas faedavit. Sic vir praeclarus, quem apprime incolumem Romanus ductor cupiebat, quemque servari praeceperat, ingenio et arte et salutem et interitum sibi comparavit. Non defuit tamen ductoris munificentia erga perempti cognatos, quippe quos et honore decoravit et praesidio iuvit, ut Livius, Valerius, et aliique historici prodidere. Sed intentum nostrum est commemorare huius egregii philosophi monumenta quibus praecipue nomen suum immortalitati mandavit, et quorum parcius mentio fit in historiis.

Eius itaque operum in ordine primum est τοῦ κύκλου μέτρησις, hoc est, circuli dimensio, in quo demonstrat circulum esse aequalem triangulo orthogonio cuius eorum quae circa rectum angulum laterum unum aequale est circuli semidiametro, alterum peripheriae. Secundo loco ponendum est opus περὶ σφαίραι καὶ κυλίνδρου, hoc est, de sphaera et cylindro, ad Dositheum, in quo demonstrat sphaerae superficiem quadruplam esse suo maximo circulo, et cylindrum eiusdem crassitudinis axisque cum sphaera esse ad eam sesquialterum, et alia circa sphaerica segmenta. Ex quo tantum gloriae sibi comparasse visus est Archimedes, ut qui rem in geometria praecipuam primus omnium demonstraverit ut eius sepulchro sphaeram et cylindrum insculpi mandatum sit. Cicero id sepulchrum, dum Siciliam peragraret, a se inter dumeta inventum memorat. Tertium in ordine ponimus opus περὶ ἰσορροπικῶν, / [2] hoc est, de aequiponderantibus, in quo praeceptum tradit inveniendi centra gravitatis in rectilineis planisque figuris. In quo quidem mirum est, ab eo centra solidorum fuisse praetermissa. Quartum erit τετραγονισμὸς (!) παραβολῆς, hoc est, Quadratura parabolae, ubi demonstrat parabolen esse sesquitertiam ad triangulum rectilineum eiusdem basis ac celsitudinis; utiturque in demonstratione tum doctrina aequiponderantium tum geometrico syllogismo. Quem libellum scribit ad Dositheum, faciens mentionem de morte Cononis, ad quem antea scribere consueverat, quorum uterque in geometria versatus, et Archimedis familiaris extiterat. Quinto his loco succedit opus περὶ ἑλίκων γραμμῶν, hoc est, de spiralibus lineis, ubi rectas quasdam spiram tangentes peripheriis circularibus aequales esse demonstrat; item primam spiram esse tertiam partem sui circuli, secundam spiram ad suum circulum esse sicut septem ad duodecim, itaque deinceps. Nam cum circuli sint in proportione quadratorum numerorum, ipsae spirae consistunt in proportione hexagonalium aequiangulorum. Hunc quoque libellum misit ad Dositheum, faciens item Cononis mentionem, qui morte praereptus ea inexplicata reliquerat. Sextum in ordine facio libellum περὶ κωνοειδέων καὶ σφαιροειδέων, hoc est, de conoidibus et sphaeroidibus figuris, quem et ad Dositheum praedictum misit; in quo multa de proportione circulorum et ellipsium inter se demonstrat. Item, solidum parabolicum esse sesquialterum ad suum conum. Et alia de proportione tam hyperbolici quam sphaeroidis solidi ad conum suum; acutissima quidem et tanto ingenio digna inventa. Septimum erit opus de numero arenae, in quo plus admirationis titulus affert quam liber ipse speculationis, ad Gelonem Regem inscriptus. Multa in eo de magnitudine terrae ac luminarium, quae quoniam ea tempestate nondum satis perspecta fuerant, culpa tempori, non Philosopho imputanda. Addendum postea opus περὶ ἰσοπεριμέτρων, hoc est, de figuris aequalis ambitus, quod ab aliquibus Archimedi, ab aliis vero et verius Theoni Alexandrino attribuitur. In quo ostensum est, inter planas figuras isoperimetras circulum, inter solidas vero sphaeram esse capacissimam. Et tandem nonum, et minime praetermittendum, περὶ κατοπτρῶν καυστικῶν, hoc est, de speculis comburentibus, quod aliqui Archimedi, alii verius Ptolomaeo adscribunt; in quo docet, speculo ad comburendum efficaci dandam esse formam concavam a parabola descriptam, quippe in quam solares radii ad aequidistantiam incidentes ad idem punctum reflectuntur; in quo collectus ex plurima luce calor potentissimus sit ad comburendum fomitem ibi appositum. Eutotius Ascalonita commentarios scripsit in opera de circuli dimensione, de sphaera et cylindro, de aequiponderantibus, ubi multa plus obscuritatis quam aut iucunditatis aut utilitatis habentia, et nihil ad intelligentiam authoris spectantia, intermiscuit.

Haec ego omnia cum vidissem, conatus sum ad faciliorem intellectum multa lemmata adiicere, multa ab Archimede omissa demonstrare, tum in aequalium momentorum negotio centra solidorum tractare, rem ab illo praetermissam. In libello de sphaera et cylindro usus sum faciliori via; in quo, ne quis arbitretur me inconcessibilia principia postulasse, si cuilibet superficiei aliquam sphaericam aut conicam aut sphaericae portionis superficiem aequalem esse supponam, aut conicam sive cylindricam sub data celsitudine, demonstrabimus et hic ipsa principia. Item datis duabus superficiebus, superficiem esse uni datarum similem et alteri aequalem. Datisque duobus solidis, aliquod solidum esse uni datorum simile et alteri aequale. Ad quod cum necessaria sit duarum mediarum proportionalium inventio, id ipsum problema ex veterum philosophorum traditione tractabimus, ut Eutotius memoratus in commentariis scripsit. Praemittemus autem principia quae ut facile concessibila postulavimus.

3 /Postulata.

1. Quibuslibet duabus eiusdem generis magnitudinibus esse duas lineas proportionales.

2. Perimetrum figurae planae circumscribentis aut includentis planam figuram perimetro circumscriptae aut inclusae esse maiorem, si tamen ad easdem partes cavae fuerint.

3. Figuram circumscribentem aut includentem circumscripta aut inclusa esse maiorem.

4. Superficiem figurae solidae circumscribentis aut includentis solidam quampiam figuram superficie inscriptae aut inclusae ad easdem partes cavae esse maiorem.

5. Figuram solidam circumscribentem aut includentem figura circumscripta aut inclusa esse maiorem.

6. Si duae magnitudines semper alia magnitudine sint simul maiores aut simul minores erunt inter se aequales.

Propositio I.

SUPERFICIEM CYLINDRICAM SUPERFICIE COLUMNARI CIRCUMSCRIPTA VEL INCLUSA ESSE MAIOREM.

Capiatur enim unum laterum columnae circumscriptae erectum super lineam rectam *AC* [Fig. III.5A.1], et portio superficiei cylindricae erecta super arcum *ABC*.

Et sic demonstrandum erit quod superficies cylindrica *ABC* maior est superficie plana *AC*; hoc enim ostenso, sequitur totam superficiem cylindricam tota superficie columnari inscripta maiorem esse.

Itaque sit, si possibile est, superficies cylindrica *ABC* minor superficie plana *AC*, ita ut cum superficie *Z* sit superficiei *AC* aequalis. Et diviso arcu *ABC* iterum atque iterum donec relictae portiones sint minus quam *Z*, erit iam cylindrica superficies *ABC* cum reliquis portionibus sumpta minor superficie plana *AC*, et a fortiori minor superficiebus planis erectis super lineis rectis *AD*, *DB*, *BE*, *EC*, quod est contra quartum postulatum. Non igitur minor est superficies cylindrica *ABC* superficie plana *AC*. Similiter nec aequalis esse eidem demonstrabitur. Superest ergo ut maior sit, quemadmodum proponitur.

Propositio II.

SUPERFICIEM CONICAM SUPERFICIE PYRAMIDALI INSCRIPTA VEL INCLUSA ESSE MAIOREM.

Idem processus eademque descriptio huic propositioni inserviet quae praecedenti. Unde si superficies tam cylindrica columnarem inscriptam, quam conica pyramidalem inscriptam excedit, multo magis et inclusam quam non tangit excedet, quandoquidem inscripta maior est quam inclusa. Constat ergo propositum.

4 /Propositio III.

SUPERFICIEM COLUMNAREM MAIOREM ESSE SUPERFICIE CYLINDRICA INSCRIPTA VEL INCLUSA.

Capiantur duo latera columnae erecta super rectas *AF*, *FC* [Fig. III.5A.2], et portio cylindricae superficiei super arcum *ABC*, quae superficies utraeque terminantur ad latera cylindri erecta super puncta *A*, *C*, quae latera sunt contactus superficierum planarum super *AF*, *FC* positarum cum superficie cylindrica posita super arcum *ABC*.

Itaque demonstrandum est quod plana superficies *AFC* maior est cylindrica superficie *ABC*; hoc enim ostenso, sequitur totam columnarem planam superficiem maiorem esse tota cylindrica superficie.

Itaque sit, si possibile est, plana superficies *AFC* minor cylindrica superficie *ABC*. Sectoque iterum arcu *ABC* atque iterum ducantur tangentes circulum donec relictae portiones inter tangentes et peripheriam sint minus spatio *Z* quo superficies cylindrica *ABC* excedere supponitur planam *AFC*, eritque a fortiori eadem superficies cylindrica *ABC* maior quam superficies plana posita super *AK*, *KL*, *LM*, *MN*, *NC* una cum portionibus relictis, quod est contra quartum postulatum. Non igitur minor est superficies plana *AFC* superficie cylindrica *ABC*. Similiter nec aequalis esse demonstrabitur. Maior igitur erit, quod est propositum.

Propositio IV.

SUPERFICIEM PYRAMIDALEM MAIOREM ESSE SUPERFICIE CONICA INSCRIPTA VEL INCLUSA.

Non alio processu quam praecedens demonstrabitur. Unde si superficies, tam columnaris cylindricam inscriptam, quam piramidalis conicam inscriptam excedit, multo fortius et inclusam excedet, quandoquidem inscripta sibi inclusam excedit.

Illud autem notandum quod in cylindrica superficie relictae portiones supradictae tam in praemissa quam in prima huius capiendae sunt in utraque basi quo superficies circumscribens eosdem terminos habeat cum inscripta, sicut quartum postulatum supponit. Constat ergo propositum.

Propositio V.

ESSE ALIQUAM LINEAM AD QUAM DATA LINEA DATAM HABET RATIONEM.

Sit data ratio magnitudinis *A* ad magnitudinem *B*, data linea *C* [Fig. III.5A.3], erunt utique per primum postulatum lineae ipsis magnitudinibus
5 *A*, *B* / proportionales quae sint *D*, *E*, ut scilicet *A* ad *B* sit sicut linea *D* ad lineam *E*. Sit per decimam sexti Euclidis sicut *D* ad *E*, sic *C* ad *F*. Eritque *C* ad *F*, sicut *A* ad *B*, quod est propositum.

Propositio VI.

QUIBUSLIBET DUABUS LINEIS ALIQUAM MEDIAM INTERESSE PROPORTIONALEM.

Ut si sint datae lineae rectae *AB*, *BC* [Fig. III.5A.4] in rectum coniunctae. Descripto super totam *AC* semicirculo, excitataque perpendiculari *AD* (*! BD*) usque ad peripheriam, erit per nonam sexti Euclidis *BD* inter ipsas *AB*, *BC* media proportionalis. Constat ergo propositum.

Propositio VII.

ESSE ALIQUOD QUADRATUM AD QUOD DATUM QUADRATUM DATAM HABEAT RATIONEM.

Sit data ratio quae magnitudinis *A* ad magnitudinem *B* [Fig. III.5A.5]; datum quadratum lineae *C*. Erit iam per quintam praemissam sicut *A* ad *B*, sic linea *C* ad aliquam lineam, quae sit *D*; intersit itaque ipsis *C*, *D* per praecedentem media proportionalis *E*; eritque sicut *C* ad *D*, hoc est, sicut *A* ad *B*, sic quadratum *C* ad quadratum *E*, quandoquidem dupla est ratio *C* ad *D* eius quae *C* ad *E*. Constat ergo propositum.

Propositio VIII.

CIRCULORUM PERIPHERIAE SUNT DIAMETRIS PROPORTIONALES.

Sint circuli *AIB* et *CKD* [Fig. III.5A.6], quorum diametri *AB*, *CD*. Sitque sicut peripheria *AIB* ad periferiam (*!*) *CKD* sic diameter *AB* ad lineam *EF*, per quintam huius.

Et demonstrandum erit quod linea *EF* aequalis erit lineae *CD*.

Nam si linea *EF* maior est quam linea *CD*, intelligatur circulus *ELF* concentricus circulo *CKD*, et inscribatur circulo *ELF* figura multiangula
6 *ENF* minime tangens circulum *CKD*, per / decimamtertiam duodecimi Elementorum Euclidis. Et ei similis figura *AMB* inscribatur circulo *AIB*. Eritque sicut linea *AB* ad lineam *EF*, sic perimeter figurae *AMB* ad perimetrum figurae *ENF*, et ideo sicut peripheria *AIB* ad peripheriam *CKD*; et permutatim sicut perimeter *AMB* ad peripheriam *AIB*, sic perimeter *ENF* ad peripheriam *CKD*. Sed per secundum postulatum maior peripheria *AIB* perimetro *AMB*, igitur et maior peripheria *CKD*

perimetro *ENF*, inclusa videlicet maior includente, quod est contra dictum postulatum. Non est ergo maior linea *EF* quam linea *CD*.

Si autem minor, tunc conversim erit sicut peripheria *CKD* ad peripheriam *AIB*, sic iam linea *EF* ad lineam *AB*. Sit itaque sicut linea *EF* ad lineam *AB*, sic linea *CD* ad lineam *GH*, per quintam huius. Eritque sicut linea *CD* ad lineam *GH*, sic peripheria *CKD* ad peripheriam *AIB*, et erit per 14^{am} quinti, quoniam *CD* maior quam *EF*, iam et *GH* maior quam *AB*. Unde sequitur idem impossibile quod prius, ut scilicet peripheriae primae ad peripheriam secundam ratio sit sicut prima diameter ad lineam maiorem secunda diametro. Non est igitur linea *EF* minor quam linea *CD*. Sed nec maior fuit. Aequalis erit ergo, quod fuit demonstrandum.

Propositio IX.

SIMILES CIRCULORUM ARCUS SUNT CHORDIS ATQUE ETIAM DIAMETRIS PROPORTIONALES.

Nam similes arcus per diffinitionem assumunt aequales tam ad centrum quam ad peripherias angulos. Ed idcirco per ultimam sexti Euclidis sunt proportionales integris suorum circulorum peripheriis. Sed per praecedentem peripheriae diametris sunt proportionales; igitur et similes arcus sint itidem diametris; quare propter similitudinem triangulorum et chordis proportionales, quod proponitur demonstrandum.

Propositio X.

CIRCULI SUNT QUADRATIS DIAMETRORUM PROPORTIONALES.

Sunto circuli *AIB*, *CKD* [Fig. III.5A.7], quorum diametri *AB*, *CD*; sitque sicut circulus *AIB* ad circulum *CKD*, sic quadratum *AB* ad quadratum lineae *EF*, per septimam huius.

Et demonstrandum erit quod linea *EF* aequalis est lineae *CD*.

Nam secus est maior aut minor. Si linea *EF* maior quam linea *CD*, tunc fiant eadem quae in praemissa 8^{a}. Eritque sicut quadratum *AB* ad quadratum *EF*, sic figura *AMB* ad figuram *ENF*, et ideo sicut circulus *AIB* ad circulum *CKD*; et permutatim sicut figura *AMB* ad circulum *AIB*, sic figura *ENF* ad circulum *CKD*; maior est autem per 3^{um} postulatum circulus *AIB* quam figura *AMB*. Ergo maior circulus *CKD* quam figura *ENF*, quod est impossibile per dictum postulatum. Non est ergo maior linea *EF* quam linea *CD*.

7 Si autem minor, tunc con/versim erit sicut circulus *CKD* ad circulum *AIB*, sic quadratum *EF* ad quadratum *AB*; sit ergo per septimam sicut quadratum *EF* ad quadratum *AB*, sic quadratum *CD* ad quadratum *GH*; eritque sicut quadratum *CD* ad quadratum *GH*, sic circulus *CKD* ad circulum *AIB*; et quoniam *CD* maior quam *EF*, iam per 14^{am} 5^{i} *GH* maior quam *AB*. Unde sequitur idem impossibile, ut scilicet circuli primi ad circulum secundum ratio sit ut quadratum primae diametri ad quadratum lineae maioris secunda diametro. Non est ergo minor linea *EF* quam

linea *CD*; sed nec maior fuit. Aequalis est ergo, quod fuit demonstrandum.

Propositio XI.

SIMILES CIRCULORUM SECTORES ET SIMILES PORTIONES SUNT QUADRATIS DIAMETRORUM PROPORTIONALES.

Similes enim sectores sunt per ultimam sexti Euclidis circulis integris proportionales. Quare per praecedentem sunt et quadratis diametrorum proportionales. Item in huiusmodi sectoribus triangula rectilinea ad centra super chordas portionum sunt et dictis quadratis proportionalia. Quare cum ablata, quae sunt ipsa triangula, sint totis, quae sunt sectores, proportionalia, erunt per decimam nonam quinti Elementorum et relicta, quae sunt portiones, totis proportionalia, quod fuit ostendendum.

Propositio XII.

CYLINDRORUM SIMILIUM SUPERFICIES SUNT QUADRATIS QUAE EX DIAMETRIS BASIUM PROPORTIONALES.

Fiant ea quae in x^{a} praecedenti, et super circulos *AB*, *CD*, *EF*, *GH* [Fig. III.5A.8] intelligantur similes cylindri erecti; et super figuras multiangulas *AB*, *CD*, *EF*, *GH* intelligantur columnae lateratae cylindris inscriptae, et eiusdem altitudinis cum cylindris; et pro circulis, cylindricae superficies quarum circuli sunt bases; pro figuris autem inter circulos descriptis sumantur columnares laterum superficies quorum figurae sunt bases; et pro tertio postulato citetur prima huius. Et idem processus in demonstratione servetur.

Propositio XIII.

CONORUM SIMILIUM SUPERFICIES SUNT QUADRATIS QUAE EX DIAMETRIS BASIUM PROPORTIONALES.

Fiant similiter ea quae in x^{a} praecedenti, et super circulos *AB*, *CD*, *EF*, *GH* [Fig. III.5A.9] intelligantur [similes coni erecti et intelligantur]
8 totidem laterum pyramides conis inscriptae et eiusdem celsitudinis / cum conis. Et pro circulis conicae superficies quarum figurae (/ circuli) sunt bases; pro figuris autem intra circulos descriptis sumantur laterales pyramidum superficies quarum figurae sunt bases, et pro tertio postulato citetur secunda huius. Nam idem penitus est demonstrationis syllogismus.

Propositio XIV.

DATAE CUILIBET SUPERFICIEI ALIQUEM CIRCULUM ESSE AEQUALEM.

Esto data quaelibet superficies *A* [Fig. III.5A.10]; describatur super quamvis lineam *BD* circulus *BED*, qui aut aequalis erit superficiei *A* aut minime aequalis. Si aequalis, constat iam propositum. Sin autem, tunc per VII^{am} huius sicut est circulus *BED* ad superficiem *A*, sic iam sit quadratum *BD* ad quadratum *CF*; eritque per decimam huius sicut quadratum *BD* ad quadratum *CF*, sic omnino circulus *BED* ad circulum *CGF*; quamobrem sicut circulus *BED* ad circulum *CGF*, sic circulus *BED* ad superficiem *A*; et idcirco per nonam quinti aequalis est superficiei *A* circulus *CGF*. Rursum ergo constat propositum.

Propositio XV.

CUIVIS DATAE LINEAE ALICUIUS CIRCULI PERIPHERIAM ESSE AEQUALEM.

Esto quaevis data linea *A* [Fig. III.5A.11]; describatur super quamvis lineam *BD* circulus *BED*, cuius peripheria si aequalis sit lineae *A* constat propositum. Sin vero, tunc per quintam huius sicut est peripheria *BED* ad
9 lineam *A*, sic sit diameter *BD* ad lineam *CF*; eritque per / octavam huius sicut diameter *BD* ad diametrum *CF*, sic peripheria *BED* ad peripheriam *CGF*; eandem igitur rationem cum habeat peripheria *BFD* ad peripheriam *CGF* quam ad lineam *A*, aequalis erit per nonam quinti lineae *A* peripheria *CGF*. Rursum ergo liquet propositum.

Ex quibus manifestum est et datum circulum alicui rectilineo esse aequalem, itemque dati circuli peripheriam alicui rectae lineae esse aequalem.

Similiter haec enim ostenduntur, mutato tantum supposito: ut scilicet pro superficie circulus, et pro data linea circuli peripheria dari supponatur.

[Vocat Maurolicus coni-colurum segmentum coni abscissum a plano parallelo basi coni comprehensumque inter duo praedicta plana.]

Propositio XVI.

CONORUM-COLURORUM SIMILIUM CURVAE SUPERFICIES SUNT QUADRATIS QUAE EX DIAMETRIS CORRELATIVARUM BASIUM PROPORTIONALES.

Nam cum coluri-coni segmenta sint conorum, et similes coni-coluri similium conorum similiter factae portiones erunt. Sed per decimamtertiam huius desectorum conorum, quoniam similes sunt, superficies sunt quadratis quae ex diametris basium proportionales; et integrorum conorum qui similes sunt superficies quadratis quae ex diametris basium proportionales; et ideo superficiebus desectorum conorum proportionales, quandoquidem basium diametri basium diametris proportionales. Igitur per decimamnonam quinti et relictorum conorum-colurorum superficies dictarum diametrorum correlativarum quadratis proportionales erunt, sicut proponitur demonstrandum.

Propositio XVII.

SOLIDORUM SIMILIUM A DIMIDIIS SIMILIUM PLANARUM MULTIANGULARUM FIGURARUM, STANTIBUS DIAMETRIS,

SEMEL CIRCUMDUCTIS DESCRIPTORUM SUPERFICIES SUNT IPSARUM DIAMETRORUM QUADRATIS PROPORTIONALES.

Namque talium solidorum superficies componuntur ex similibus numeroque aequalibus conicis superficiebus, inter quas binae ad extrema diametrorum sunt perfectorum conorum, caeterae conorum-colurorum, ex quibus quandoque binae cylindricae. Verum per duodecimam, decimamtertiam et decimamsextam praemissas tales superficies correlativae correlativis singulae singulis comparatae sunt sicut quadrata quae ex diametris correlativarum basium. Huiusmodi autem quadrata sunt quadratis diametrorum super quas dimidia figurarum circumducuntur proportionalia. Igitur per coniunctam proportionem, quoties opus fuerit assumptam, integrae solidorum superficies erunt et dictarum diametrorum quadratis proportionales, quemadmodum ponitur demonstrandum.

Propositio XVIII.

SPHAERARUM SUPERFICIES SUNT QUADRATIS DIAMETRORUM PROPORTIONALES.

Sint duae sphaerae *AIB* et *CKD* [Fig. III.5A.12], quarum diametri *AB*, *CD*. Sitque per primam (! septimam) huius sicut sphaerica superficies *AIB* ad sphaericam superficiem *CKD*, sic qua/dratum *AB* ad quadratum *EF*.

Et demonstrandum erit quod linea *EF* aequalis est lineae *CD*.

Nam, si possibile est, sit maior linea *EF* quam linea *CD*. Et circa diametrum *EF* sphaera *ELF* intelligatur concentrica sphaerae *CKD*, et intra sphaeram *ELF* solidum *ENF* tornatile ex conicis superficiebus sphaeram *CKD* minime tangentibus, et huic simile solidum *AMB* intra sphaeram *AIB*. Eritque per praecedentem sicut quadratum *AB* ad quadratum *EF*, sic superficies solidi *AMB* ad superficiem solidi *ENF*. Et ideo sicut sphaerica superficies *AIB* ad sphaericam superficiem *CKD*, et permutatim sicut superficies solidi *AMB* ad sphaericam superficiem *AIB*, sic superficies solidi *ENF* ad superficiem sphaericam *CKD*. Sed per quartum postulatum maior est superficies sphaerica *AIB* quam superficies solidi *AMB*. Ergo et superficies sphaerica *CKD* maior quam superficies solidi *ENF*, quod est impossibile per dictum postulatum. Non est ergo maior linea *EF* quam linea *CD*.

Si autem sit minor, tunc conversim erit sicut sphaerica superficies *CKD* ad sphaericam superficiem *AIB*, sic quadratum *EF* ad quadratum *AB*; sit ergo per septimam huius sicut quadratum *EF* ad quadratum *AB*, sic quadratum *CD* ad quadratum *GH*. Eritque sicut quadratum *CD* ad quadratum *GH*, sic iam superficies sphaerica *CKD* ad superficiem sphaericam *AIB*. Et quoniam *CD* maior quam *EF*, iam per decimam quartam quinti erit et *GH* maior quam *AB*. Unde sequitur idem impossibile quod prius, ut scilicet sphaericae superficiei primae ad sphaericam superficiem secundam ratio sit sicut quadratum primae diametri ad quadratum lineae maioris secunda diametro. Non est ergo minor linea

EF quam linea *CD*. Sed nec maior fuit: aequalis ergo erit, quod fuit demonstrandum.

Propositio XIX.

SIMILIUM SPHAERICARUM PORTIONUM SUPERFICIES SUNT QUADRATIS DIAMETRORUM PROPORTIONALES.

Eadem huic quae praecedenti demonstratio inserviet, hoc excepto, quod pro totalibus sphaerarum superficiebus capiantur portionum superficies, et pro integrorum solidorum tornatilium superficiebus capiantur partiales ipsorum superficies quae similibus sphaerarum portionibus inscriptae similes sunt. Unde concludetur propositum, quod scilicet talium portionum superficies erunt quadratis ex sphaerarum diametris, atque etiam ex diametris circulorum super quos portiones consistunt, proportionales, quandoquidem talium circulorum diametri sunt sphaerarum diametris, propter similitudinem, proportionales.

11 /Propositio XX.

CUILIBET DATAE SUPERFICIEI ALICUIUS SPHAERAE SUPERFICIEM ESSE AEQUALEM.

Est quaevis data superficies *A* [Fig. III.5A.13], describatur super quamvis lineam *BD* sphaera *BED*, cuius superficies aut aequalis est superficiei *A* aut non. Si aequalis, constat propositum. Secus autem, tunc per septimam huius sicut est sphaerica superficies *BED* ad superficiem *A*, sic iam esto quadratum *BD* ad quadratum *CF* [per septimam huius]. Eritque per decimam octavam praecedentem sicut quadratum *BD* ad quadratum *CF*, sic utique sphaerica superficies *BED* ad sphaericam superficiem *CGF*. Eandem igitur rationem habebit sphaerica superficies *BED* ad sphaericam superficiem *CGF* et ad superficiem *A*. Quare per nonam quinti Elementorum aequalis erit superficiei *A* sphaerica superficies *CGF*. Rursum ergo constat propositum. [Adverte quod figurae sphaericae compleri debent circa diametros *BD*, *CF*.]

ET SIMILITER CONSTABIT CUIUSLIBET DATAE SPHAERAE SUPERFICIEM ALICUI RECTILINEO ESSE AEQUALEM.

Data scilicet primum sphaera, et mox rectilineo descripto.

Propositio XXI.

DATIS DUABUS QUIBUSCUNQUE SUPERFICIEBUS ALIQUAM SUPERFICIEM ESSE UNI EARUM SIMILEM ET ALTERI AEQUALEM.

Haec propositio est tanto generalior decimaquarta et vigesima huius, quanto superficies generalior quam circularis aut sphaerica. Sit igitur, exempli gratia, superficies quaevis *A* [Fig. III.5A.14], et portio quaepiam circuli *BED* super chordam *BD*.

Ostendam aliquam circuli portionem esse superficiei *A* aequalem et portioni *BED* similem.

Sit enim per septimam huius sicut *BED* portio ad *A* superficiem, sic *BD* chordae quadratum ad quadratum *CF* lineae. Et per 31^{am} tertii Elementorum super *CF* lineam constituatur circuli portio *CGF* similis ipsi *BED* portioni. Et quoniam chordae similium portionum sunt diametris circulorum proportionales, et periede (! perinde) quadrata quadratis proportionalia. Atque per undecimam huius similes portiones circulorum sunt quadratis diametrorum proportionales. Idcirco erit sicut quadratum *BD* ad quadratum *CF* sic portio *BED* ad portionem *CGF*. Itaque cum eandem rationem habeat portio *BED* ad portionem *CGF* quam et ad superficiem *A*, aequalis erit per nonam quinti elementorum superficiei *A* portio *CGF*. Sed et similis fuit portioni *BED*; constat ergo propositum.

Similiter data qualibet superficie et alia quavis superficiali figura, utpote cylindrica, conica, demonstrabimus aliquam superficiem esse figurae propositae similem et date superficiei aequalem, pro undecima tamen duodecimam, XIII^{am}, aut XVI^{am} huius adducentes.

12 /Item data superficie ac tornatili solido, non aliter per decimamseptimam ostendemus esse quodpiam tornatile solidum cuius superficies datae sit aequalis ac tornatilis propositi superficiei similis etc.

Propositio XXII.

CUILIBET DATAE SUPERFICIEI ESSE ALIQUAM SUPERFICIEM CYLINDRICAM AEQUALEM CIRCA DATUM AXEM.

Sit data superficies *A* [Fig. III.5A.15], datus axis *BC* perpendicularis plano in quo iacet linea *DE* indefinita. Iam, quoniam circum axem *BC* constitui possunt infinitae columnae lateratae quarum superficies laterales singulae minores sint superficie *A*, tales autem superficies maiores sunt superficiebus cylindricis circa eundem axem sibi inscriptis per tertiam huius, idcirco et infinitae superficies cylindricae circum axem *BC* fieri possunt singulae minores data superficie *A*. Item, quoniam circum axem *BC* construi possunt infinitae columnae lateratae quarum superficies laterales singulae maiores sint superficie *A*, tales autem superficies minores sunt superficiebus cylindricis quibus inscribuntur, per primam huius, ideo et infinitae superficies cylindricae circum axem *BC* locari queunt singulae maiores data superficie *A*. Erit itaque in linea *DE* aliquis terminus intra quem cylindricae superficies sint minores singulae superficie *A* et extra quem sint maiores. Esto talis terminus punctum *D*, eritque per primam et tertiam ex dictis superficiebus cylindricis quaecunque intra terminum *D* consistit minor superficie cylindrica cuius semidiameter basis *CD*, quaecunque autem extra terminum *D* maior eadem. Cum itaque superficies data *A* et superficies cylindrica cuius basis semidiameter *CD* axisque *BC* collatae ad omnem aliam superficiem intra extraque punctum *D* sint ea simul maiores aut simul minores, iam per ultimum postulatum aequales erunt. Constat ergo propositum.

Propositio XXIII.

CUILIBET DATAE SUPERFICIEI ESSE ALIQUAM SUPERFICIEM CONICAM AEQUALEM CIRCA DATUM AXEM.

Sit data superficies *A* [Fig. III.5A.16], datusque axis *BC* perpendicularis ad planam in quo recta *DE* indefinita. Iam, quoniam circum axem *BC* poni possunt infinitae pyramides lateratae quarum superficies laterales singulae minores sint superficie *A*, tales autem superficies maiores sunt superficiebus conicis circa eundem axem sibi inscriptis, per quartam huius, propterea et infinitae superficies conicae circum axem *BC* possunt erigi singulae minores data superficie *A*. Item, quoniam circa axem *BC* esse possunt infinitae pyramides quarum laterales superficies singulae maiores
13 sint superficie *A*, tales / autem superficies minores sunt superficiebus conicis quibus inscribuntur, per secundam huius, ob id et infinitae superficies conicae circum axem *BC* stabunt singulae maiores data superficie *A*. Erit itaque in linea *DE* aliquis terminus intra quem conicae superficies sint minores singulae superficie *A* et extra quem sint maiores. Esto talis terminus punctum *D*, eritque per secundam et quartam ex dictis conicis superficiebus quaecunque intra punctum *D* minor superficie conica cuius semidiameter basis *CD*, quecunque autem extra punctum *D* maior eadem. Cum itaque superficies data *A* et superficies conica cuius basis semidiameter *CD* axisque *BC* collatae ad omnem aliam superficiem intra extrave punctum *D* sumptam sint simul maiores aut simul minores, iam per ultimum postulatum aequales esse convincuntur. Constat ergo positum.

Propositio XXIV.

DUABUS SPHAERICIS PORTIONIBUS SUPER UNUM PLANUM EXISTENTIBUS, MAIOR EST SUPERFICIES EXTERIORIS.

Ut si portiones sphaericae *ABC*, *DEF* [Fig. III.5A.17], quarum exterior *ABC*, stent super planum in quo recta *ADFC*, maior erit superficies *ABC*. Stent enim ex altera parte eiusdem plani portiones sphaericae *AGC*, *DHF* singulae suis conterminis aequales et similes. Eritque per quartum postulatum totalis superficies *ABCG* maior totali superficie *DEFH*, includens videlicet inclusa. Ergo et dimidium dimidio maius, superficies scilicet *ABC* superficie *DEF*, quod fuit demonstrandum.

Propositio XXV.

CUILIBET DATAE SUPERFICIEI ESSE AEQUALEM ALICUIUS SPHAERICAE PORTIONIS SUPER DATUM CENTRUM ET A DATO PLANO ABSCISSAE SUPERFICIEM.

Si data superficies *A* [Fig. III.5A.18], datum centrum *B*, datum planum in quo iacet recta *CD* ita ut *BC* sit perpendicularis ipsi plano indefinito. Iam, ut in 22ª et 23ª huius patuit, infinitae cylindricae sive conicae superficies habentes axem in rectum ipsi *BC* consistent tum minores, tum

maiores singulae data superficie *A*. Cumque per quartum postulatum cylindrica sive conica superficies maior quidem sit sphaericae portionis
14 sibi / inscriptae superficie, minor vero sphaericae portionis cui inscribitur superficie, propterea et infinitae sphaericarum portionum circa centrum *B* et a plano *CD* abscissarum superficies erunt tum minores, tum maiores superficie *A*. Erit itaque aliquis terminus in linea *CD* intra quem sphaericarum portionum superficies minores erunt superficie *A* et extra quem maiores. Esto talis terminus punctum *D*. Eritque per praecedentem ex dictarum sphaericarum portionum superficiebus quaecunque intra punctum *D* minor superficie sphaericae portionis cuius basis semidiameter *CD*, quaecunque autem extra punctum *D* maior eadem. Quando itaque superficies data *A* et superficies sphaericae portionis cuius basis semidiameter *CD* collatae ad omnem aliam superficiem intra extrave punctum *D* sumptam sint simul maiores aut simul minores, ideo per ultimum postulatum aequales erunt, quod est propositum.

Propositio XXVI.

QUIBUSLIBET DUABUS LINEIS DUAS MEDIAS ESSE PROPORTIONALES.

Sint quaelibet datae lineae rectae *AB*, *BC* [Fig. III.5A.19], compleatur rectangulum *ABCD*, et *BC*, *BA* producantur indefinitae, centrumque rectanguli sit *E*, in quo se vicissim secant diametri *AC*, *BD*; productis autem *BA*, *BC* applicetur per *D* ducta recta *FDG*, hac conditione, ut coniunctae *EF*, *EG* sint aequales.

Aio tunc *CF*, *AG* medias proportionales interiacere ipsis *AB*, *BC*.

Ducantur enim *EH*, *EK* perpendiculares ad *AB*, *BC*; eritque per sextam secundi Euclidis rectangulum *BF*, *FC* cum quadrato *KC* sumptum aequale quadrato *KF*; commune ponatur quadratum *EK*. Eritque rectangulum *BF*, *FC* cum quadrato *KC* et quadrato *KE*, hoc est cum quadrato *EC* sumptum, aequale quadrato *KF* cum quadrato *EK*, hoc est quadrato *EF*. Similiter omnino demonstrabimus quod rectangulum *BG*, *GA* cum quadrato *EA* sumptum aequale est quadrato *EG*; aequalia vero sunt per hypostesim (! hypothesim) quadrata *EF*, *EG*. Igitur rectangulum *BFC* cum quadrato *EC* aequale est rectangulo *BGA* cum quadrato *EA*. Item quadrata *EC* et *EA* sunt aequalia quoniam dimidiis eiusdem *AC* lineae debentur.
15 Supe/rerunt ergo rectangulum *BF*, *FC* et rectangulum *BG*, *GA* inter se aequalia. Quare per 15^{am} sexti Euclidis sicut *BG* ad *BF*, sic *FC* ad *GA*. Et propter similitudinem triangulorum *GBF*, *DCF*, sicut *DC*, hoc est *AB*, ad *CF*, sic *FC* ad *GA*. Item propter similitudinem triangulorum *GAD*, *DCF*, sicut *DC* ad *CF*, sic *GA* ad *AD*, hoc est ad *BC*. Itaque quatuor lineae *AB*, *CF*, *GA*, *BC* sunt continuae proportionales. Et hoc proponebatur demonstrandum.

Et hic quidem est modus Heronis, qui Mechanica scripsit. Subiiciemus nunc aliorum philosophorum circa idem problema ex Eutocio sumptas traditiones.

Propositio XXVII.

IDEM ALITER DEMONSTRARE.

Sunto datae lineae *AB*, *BC* [Fig. III.5A.20], et compleatur rectangulum *ABCD*, cui per nonam 4^{i} Euclidis circulus circumscribatur. Mox inter *BA*, *BC* indefinitum productas deducatur recta *FDHG*, hac conditione, ut ipsae *FD*, *HG* sint inter se aequales; tunc enim *CF*, *AG* erunt inter ipsas *AB*, *BC* datas mediae proportionales. Nam per 35^{am} tertii Euclidis rectangulum *AG*, *GB* aequale est rectangulo *HG*, *GD*, et rectangulum *CF*, *FB* aequale est rectangulo *DF*, *FH*. Sed rectangulum *DF*, *FH* aequale est rectangulo *HG*, *GD*, quandoquidem *DF*, *GH* lineae supponuntur aequales. Igitur rectangulum *CF*, *FB* aequale est rectangulo *AG*, *GB*. Quare per xv^{am} sexti, non aliter quam in praecedenti, ex similitudine ipsa triangulorum demonstrabuntur *AB*, *CF*, *GA*, *BC* continue proportionales.

Et hic est modus Apollonii et Philonis Byzantii, ut testatur Ioannes Philoponus Alexandrinus. Et est idem ferme cum modo Heronis, quamvis demonstrationes aliquantum differant.

Propositio XXVIII.

ALITER IDIPSUM OSTENDERE.

Sunto datae rectae *AB*, *BG* ad angulum rectum positae [Fig. III.5A.21], quae producantur, sintque *BE*, *BD* singulae ipsi *AB* aequales, ad cuius spatium super centro *B* describatur semicirculus *ADE*, coniunctaque *AG* producatur ad peripheriam in punctum *Z*, et circa punctum *E* moveatur canon *ET* (*! EH*) donec portio canonis *TK* inter peripheriam
16 et *AZ* per aequalia secetur ab ipsa *BD* in puncto *L*. / Et per punctum *K* ipsi *BD* parallelus agatur *MN*, sitque sicut *MA* ad *AB*, sic *EM* ad *MX*. Itemque sic *NM* ad *MO*.

Dico itaque quod *BG*, *MX*, *MO*, *AB* lineae sunt continuae proportionales.

Agatur enim ipsi *BD* parallelus *TP*; eruntque *BP*, *BM* aequales, quandoquidem *TL*, *LK* iisdem parallelis interpositae fuerunt inter se aequales. Quare *KM* ad *ME*, sicut *TP* ad *PE*; et ideo sicut *NM* ad *MA*; et ideo sicut *ME* ad *MN*: quandoquidem *MN* media proportionalis est inter ipsas *AM*, *ME*; igitur et *ME* ipsis *KM*, *MN* media proportionalis est. Non dubium ergo quin *KM*, *ME*, *MN*, *MA* sint in proportione continua. Sed sicut *BG* ad *KM*, sic *AB* ad *MA*, propter similitudinem triangulorum. Itemque per hypothesim sicut *MX* ad *ME*, et sic *MO* ad *MN*. Ergo ex permutata proportione sequitur ut ipsae quoque lineae *BG*, *MX*, *MO*, *AB* sint in eadem continua proportione, quod erat demonstrandum.

Est autem haec traditio Pappi in Mechanicis. Et hac eadem uti videntur Diocles et Porus (*!*).

Propositio XXIX.

ID IPSUM ALITER DEMONSTRARE.

Sunto datae rectae lineae *AB*, *BC* ad rectos angulos positae [Fig.

III.5A.22], quae producantur in infinitum, sintque ABD, CBE; et fabricetur rectus angulus FGH, et in uno crure, ut pote FG, moveatur regula KL in canali quopiam qui sit in ipso FG, utque KL regula parallelus sit semper ipsi GH; alterum regulamentum ipsi GH cruri insertum, scilicet HM, ipsique FG parallelum aptetur; ut KL regula ipsarum FG, HM aptata canalibus, ipsique GH parallelus ultro citroque semper ad aequidistantiam ipsius GH moveatur. Huic demum structurae ipse ABC angulus ita erit accomodandus ita, inquam, rectae ABD, CBE regulamentis interponendae, ut A punctum contingat regulam KL, et ipsum C punctum contingat regulam GH, utque ABD eat per angulum G et linea CBE per angulum K. Hoc enim pacto fient nimirum ipsi AED, EDC anguli recti. Quare per 7^{am} sexti elementorum bis assumptam erunt lineae AB, BE, BD, BC continuae proportionales. Itaque datis duabus AB, BC totidem interiacent lineae proportionales, quod erat ostendendum.

Fuit autem haec inventio Platonis, cum, Deliis pestilentia laborantibus, consultus Apollo respondisset, ut lues cessaret, aram esse duplicandam; quae cum cubi formam haberet, cubusque aliter duplicari non posset nisi per duarum mediarum proportionalium interpositionem, proposita fuit huiusmodi quaestio.

Propositio XXX.

ALITER ILLUD IDEM DEMONSTRARE.

Sint datae rectae lineae AB et CD [Fig. III.5A.23], et intra parallelogrammata rectangula inter se aequalia et similia quorum maiora latera
17 aequalia sint AB maiori datarum / disponendaque sunt parallelogrammata in eodem plano ut eorum bases minores super eadem recta BK iaceant et eorum diametri aequidistent inter se; et fixo manente parallelogrammo medio compellatur unum supra, alterum infra eundem, aptata in eodem plano quousque puncta A, E, H extrema diametrorum in una linea AK cadant, quae praeterea cum recta BK extremi parallelogrami (!) lateris portionem CD abscindat aequalem minori datarum. Hoc enim facto ipsae EF, GH mediae proportionales erunt inter ipsas AB, CD; nam propter triangulorum similitudinem, sicut AB ad EF, sic AK ad KE; sicut autem AK ad KE, sic FK ad KG, sed et FK ad KG sicut EF ad HG. Igitur sicut EF ad HG, sic AB ad EF. Rursum, sicut EF ad HG, sic EK ad KH; sicut autem EK ad KH, sic GK ad KC. Sed et GK ad KC, sicut HG ad CD. Igitur EF ad HG, sicut HG ad CD. Constat ergo lineas AB, EF, HG, CD esse in proportione continua, sicut fuit demonstrandum. Et haec est Eratosthenis traditio.

Propositio XXXI.

IDEM ALITER OSTENDERE.

Sunto datae rectae AB, BG (! AG) in rectum positae [Fig. III.5A.24], describatur circa axem AG parabole ADH, quam in puncto D secet recta BD ipsi AG perpendicularis; et per punctum D ipsi AG parallelus agatur EDZ, cui a punctis A, G lineae ipsi BD parallelae occurrant ad puncta

E, *Z* lineae *AE*, *GZ*. Item per punctum *Z* circa non coincidentes *EA*, *AG* describatur hyperbole *ZH*, quae secet parabolam in puncto *H*, a quo ad lineas *AE*, *AG* perpendiculares cadant lineae *HT*, *HK*. Et per sextam huius ipsis *AB*, *AT* media proportionalis intersit *AL*.

Aio itaque quod *AL*, *AT* mediae proportionales interiacent ipsis *AB*, *AG*.

Nam cum hyperbole sit *HZ*, non coincidentes autem *KAG*, iam per 12^{am} secundi Conicorum Elementorum rectangulum *AH* aequale erit rectangulo *AZ*. Quare per 15^{am} sexti Elementorum Euclidis erit sicut *AT* ad *AG*, sic *GZ*, hoc est *BD*, ad *TH*. Sed per 20^{am} primi Conicorum Elementorum ratio *AB* ad *AT* dupla est rationis *BD* ad *TH*, quandoquidem parabole est *ADH*; ergo et ratio *AB* ad *AT* dupla est rationis *AT* ad *AG*. Quare fiet sicut *AB* ad *AL*, vel sicut *AL* ad *AT* (eadem enim ratio per hypothesim) sic *AT* ad *AG*. Igitur *AB*, *AL*, *AT*, *AG* continuae proportionales sunt, quod fuit demonstrandum.

Et haec est inventio Menaechmi aliis modis commendatior, quoniam innititur geometricis et absolubilibus praeceptis quando alii fortuito casu
18 videantur quodammodo intentum adipisci. Sed huic adde, quod si / *AB*, *BD* ponantur aequales, et mox per *A*, *D* puncta parabole *ADH* describatur, et reliqua, ut prius, erit per iam demonstrata sicut *BD*, hoc est *AB*, ad *TH*, sic *AT* ad *AG*. Sed per 20^{am} primi Conicorum ratio *AB* ad *AT* dupla est rationis ipsius *BD*, hoc est *AB*, ad *TH*. Igitur hac via fient *AB*, *TH*, *AT*, *AG* continue proportionales. Nec opus est ipsius *AL* interpositione, quemadmodum Menaechmus docet.

Propositio XXXII.

ADHUC IDIPSUM ALITER DEMONSTRARE.

Sunto datae rectae *AB*, *BG* [Fig. III.5A.25], quarum maior *AB*, describatur super *AB* diametrum circulus *BGA* intra quem per secundam (! primam) quarti Euclidis coaptetur *BG*, quae producta occurrat ipsi *AD* tangenti circulum in puncto *D*. Item *EG* secet ad rectos angulos diametrum *AB* in puncto *Z*, et super *EG* diametrum fiat semicirculus *EHG* rectus ad circulum *BGA*; hinc super semicirculum *BGA* erigatur hemicylindrus rectus; et in rectangulo quod per axem cylindri describatur hemicirculus super *AB* diametrum, qui semicirculus moveatur super planum circuli *BGA* semper rectus ad idem plano, moto scilicet diametro *AB* versus *G*, manente termino *B* immoto, sitque semicirculus iam motus *TKB* super diametro *BT* secante periferiam *BGA* in puncto *L*; quo quidem motu peripheria *TKB* describet in superficie cylindrica lineam quandam curvam; inde moveatur triangulum *BDA* super axem *AB*, quo motu punctum *G* circumferetur in peripheria *EHG*, et linea *BD* sic circulata describet conicam superficiem, et secabit lineam curvam in superficie cylindrica descriptam in puncto quodam, quod sit *K*, in quo peripheria *TKB* latus ipsum *BD* motum secabit. Sit ergo *BDA* triangulum ad talem situm translatum ipsum *BMA* triangulariter, latere scilicet *BM* ipsam *TKB* peripheriam secante in puncto *K* in ipsa cylindrica superficie. Et coniugatur *KL* recta, quae erit communis sectio plani *TKB* et cylindricae superficiei, quoniam scilicet cylindrus rectus est, atque ideo planum ipsum *TKB* basi cylindricae rectum aequidistat axi cylindrico. Quare *KL* eidem axi parallelus erit circulo *BGA*, et ideo rectae *TB* perpendicularis; praeterea communis sectio circuli *TKB* et circuli *EHG* sit recta *HN*, quae per 19^{am} undecimi Euclidis circulo *BAG* et ideo rectae *TB* perpendicularis erit; et coniungantur *TK*, *LH*.

Aio itaque quod *BL*, *BK* ipsis *BG*, *AB* interiacent mediae proportionales, quod sic demonstratur.

Nam per octavam sexti Elementorum Euclidis quadratum *HN* aequale est rectangulo *EN*, *NG*, et ideo adducta 34^{a} tertii rectangulo *BN*, *NL*: quo fit ut angulus *BHL* sit rectus. Cum anguli *BLK*, *BKT* recti sint, erunt ob id triangula *BHL*, *BLK*, *BKT* similia, quandoquidem aequiangula. Unde sequitur ut ipsae *BH*, *BL*, *BK*, *BT* sint continuo processu propor-
19 tionales. Verum *BH* aequalis ipsi *BG*, / quia sunt latera coni recti cuius axis *BZ*, vertex autem *B*. Item *BT* aequalis ipsi *BA*, per hypothesim. Igitur et *BG*, *BL*, *BK*, *BA* continuae proportionales erunt, quod fuit demonstrandum.

Est autem inventio Archytae Tarentini, ut refert Eudemus et Eutotius; ingeniosa quidem et tali viro digna speculatio, cuius praxis etsi difficilis sit, facillime tamen demonstratur.

Propositio XXXIII.

ESSE ALIQUEM CUBUM QUI AD DATUM CUBUM DATAM HABEAT RATIONEM.

Sit data ratio quae magnitudinis *A* ad magnitudinem *B*, datus cubus lineae *C* [Fig. III.5A.26]. Erit iam per quintam huius sicut *A* ad *B*, sic linea *C* ad aliquam lineam quae sic (! sit) *D*. Itaque ipsis *C*, *D*, per quamvis sex (! septem) praecedentium, interiacebunt duae mediae proportionales, quae sint *E*, *F*. Eritque per 36^{am} undecimi Euclidis sicut *C* ad *D*, hoc est sicut *A* ad *B*, sic cubus *C* ad cubum *E*. Igitur cubus datus *C* ad cubum *E* datam habet rationem quae *A* ad *B*, quod est propositum.

Propositio XXXIV

SIMILES CYLINDRI SUNT CUBIS QUI EX BASIUM DIAMETRIS PROPORTIONALES.

Sunto similes cylindri *XAIB* et *ZCKD* [Fig. III.5A.27], quorum basium diametri *AB*, *CD*, sitque sicut cylindricus *XAIB* ad cylindrum *ZCKD*, sic cubus *AB* ad cubum lineae *EF* per praecedentem.

Et demonstrandum erit quod linea *EF* aequalis erit lineae *CD*.

Nam secus erit aut maior aut minor. Si linea *EF* maior est quam linea *CD*, intelligatur circulus *ELF* concentricus circulo *CKD*, et inscribatur circulo *ELF* figura multiangula *ENF* minime tangens circulum *CKD* per 13^{am} duodecimi Euclidis. Et ei similis figura *AMB* inscribatur circulo *AIB*, et super tales figuras erigantur lateratae columnae *QENF* et *XAMB* ipsis iam cylindricis *QELF*, *XAIB* similibus inscriptae; eritque per octavam 12^{i} Euclidis sicut cubus *AB* ad cubum *EF*, sic columna *XAMB* ad columnam *QENF*, quandoquidem similes sunt columnae sicut et cylindri; quare sicut columna *XAMB* ad columnam *QENF*, sic cylindrus *XAIB* ad cylindrum *ZGKD* (! *ZCKD*) et permutatim, sicut columna *XAMB* ad cylindrum *XAIB*, sic columna *QENF* ad / cylindrum *ZCKD*; sed maior est cylindrus *XAIB* quam columna *XAMB* inscripta per quintum postulatum; ergo et cylindrus *ZCKD* est maior quam columna *QENF*, quod est impossibile per dictum postulatum. Non est ergo maior linea *EF* quam linea *CD*.

Si autem minor, tunc conversim erit sicut cylindricus *ZCKD* ad cylindrum *XAIB*, sicut cubus *EF* ad cubum *AB*; sit ergo sicut cubus *EF* ad cubum *AB*, sic cubus *CD* ad cubum *GH*, per praecedentem; eritque sicut cubus *CD* ad cubum *GH*, sic cylindrus *ZCKD* ad cylindrum *XAIB*; et quoniam *CD* maior est quam *EF*, ideo per 8^{am} 12^{i} et 14^{am} quinti erit *GH* maior quam *AB*. Unde sequitur idem impossibile, ut scilicet cylindri primi ad cylindrum secundum ratio sit sicut columna diametri primae basis ad columnam lineae maioris diametro secundae basis. Non est ergo minor linea *EF* quam linea *CD*. Sed nec maior fuit; aequalis ergo. Et hoc fuit demonstrandum.

Propositio XXXV.

SIMILES CONI SUNT CUBIS QUI EX BASIUM DIAMETRIS PROPORTIONALES.

Omnino eadem demonstratio, si pro columnis pyramides lateratas conis includas et iisdem argumentationibus utaris.

Propositio XXXVI.

SIMILES CONI-COLURI SUNT CUBIS QUI EX CORRELATIVARUM BASIUM DIAMETRIS FIUNT PROPORTIONALES.

Nam similes coni-coluri sunt similium conorum similiter factae portiones, sed per praecedentem integri coni sunt cubis qui ex basium diametris proportionales, et abscissi coni sunt cubis qui ex basium sectricium diametris fiunt proportionales; sed hi cubi sunt illis proportionales, quoniam basium harum diametri basium illarum diametris sunt proportionales propter similitudinem portionum. Igitur abscissi coni sunt integris conis proportionales. Quare per 19^{am} quinti et relictae portiones, qui sunt ipsi coni-coluri, sunt tam abscissis quam integris conis, et ideo correlativarum basium cubis proportionales, quod fuit ostendendum.

Propositio XXXVII.

SIMILIA SOLIDA TORNATILIA, HOC EST, A DIMIDIIS SIMILIUM PLANARUM MULTIANGULARUM FIGURARUM SUPER FIXAS

DIAMETROS SEMEL CIRCUMDUCTIS DESCRIPTA, SUNT CUBIS IPSARUM DIAMETRORUM PROPORTIONALIA.

Namque huiusmodi solida compacta sunt ex similibus numeroque aequalibus conis, inter quos bini et bini qui sunt ad extrema diametrorum sunt perfecti, caeteri vero coluri-coni, inter quos tamen sunt quandoque bini cylindri; verum per 34^{am}, 35^{am}, et 36^{am} praecedentes tales coni, correlativos correlativis comparando, sunt cubis qui ex diametris correlativarum basium fiunt diametris proportionales; hi autem cubi sunt cubis diametrorum super quas dimidia figurarum circumducuntur proportionales, quandoquidem diametri diametris proportionales sunt propter similitudinem figurarum. Igitur per 13^{am} quinti elementorum et aggregata
21 omnium conorum, hoc est ipsa / solida tornatilia integra erunt cubis earundem diametrorum proportionalia, quod fuit demonstrandum.

Propositio XXXVIII.

SPHAERAE SUNT CUBIS DIAMETRORUM PROPORTIONALES.

Sint duae sphaerae *AIB* et *CKD* [Fig. III.5A.28], quarum diametri *AB*, *CD*; sitque per 33^{am} huius sicut sphaera *AIB* ad sphaeram *CKD*, sic cubus *AB* ad cubum alicuius lineae *EF*.

Et demonstrandum erit quod linea *EF* aequalis erit lineae *CD*.

Nam secus erit aut maior aut minor. Si linea *EF* maior sit quam linea *CD*, intelligatur sphaera *ELF* concentrica sphaerae *CKD*; et per 13^{am} 12^{i} Euclidis et per figuram planam multiangulam inscribatur sphaerae *ELF* solidum tornatile *ENF* cuius superficies minime tangat sphaeram *CKD*; et ei simile solidum *AMB* inscribatur sphaerae *AIB*; eritque per praecedentem sicut cubus *AB* ad cubum *EF*, sic solidum tornatile *AMB* ad solidum tornatile *ENF*; quare sicut solidum *AMB* ad solidum *ENF*, sic sphaera *AIB* ad sphaeram *CKD*: et permutatim, sicut solidum *AMB* ad sphaeram *AIB*, sic solidum *ENF* ad sphaeram *CKD*. Sed maior sphaera *AIB* solido *AMB* per quintum postulatum; ergo et sphaera *CKD* maior solido *ENF*, quod est impossibile per dictum postulatum. Non ergo est maior linea *EF* diametro *CD*.

Si autem minor, tunc conversim erit sicut sphaera *CKD* ad sphaeram *AIB*, sic cubus *EF* ad cubum *AB*. Sit ergo per 33^{am} huius sicut cubus *EF* ad cubum *AB*, sic cubus *CD* ad cubum *GH*; eritque sicut cubus *CD* ad cubum *GH*, sic sphaera *CKD* ad sphaeram *AIB*. Et quoniam *CD* est maior quam *EF*, ideo per praemissam et 14^{am} quinti *GH* maior erit quam *AB*. Unde sequitur idem impossibile, ut scilicet sphaerae primae ad sphaeram secundam ratio sit sicut solidum inscriptum primae ad solidum lineae maioris secunda diametro. Non est ergo minor linea *EF* diametro *CD*; sed nec maior fuit: aequalis ergo, quod fuit demonstrandum.

Propositio XXXXIX (/ XXXIX).

SIMILES SPHAERALES CUNEI SUNT CUBIS DIAMETRORUM PROPORTIONALES.

Sphaeralem cuneum intelligo corpus sub conica superficie verticem in

centro sphaerae habente et sub assumpta sphaerica superficie comprehensum, quod quidem componitur ex cono et sphaerica portione communem basim habentibus circulum qui sphaeram per inaequalia secat. Demonstratur autem haec non aliter quam praemissa, hoc excepto, quod pro totalibus sphaeris cunei sphaerales similes capiantur, et pro tornatilibus solidis tornatiles cunei capiantur. Componitur autem tornatilis cuneus ex cono supradicto cunei sphaeralis et portione tornatilis solidi cadente intra sphaericam portionem. Nam huiusmodi
22 tornatiles cunei similes sunt, quandoquidem / intra similes sphaerales cuneos cadunt; et ideo per antepraemissam sunt cubis diametrorum sphaeralium proportionales. Atque etiam cubis dictarum basium diametrorum, quippe quae sphaeralibus diametris sint proportionales propter portionum similitudinem.

Propositio XL.

SIMILES SPHAERICAE PORTIONES SUNT CUBIS DIAMETRORUM PROPORTIONALES.

Nam cum sphaerica portio (per definitionem in praemissa assignatam) sit relictum solidum post coni a sphaerali cuneo abscissionem, atque per praecedentem tales cunei, per 35^{am} aut tales coni (quia similes), sint diametrorum basium et etiam sphaerarum cubis proportionales; hoc est, cum tota sint abscissis proportionalia, erunt et relicta totis proportionalia, hoc est, sphaericae portiones dictarum diametrorum cubis proportionales, quod proponitur demonstrandum.

Cum igitur similes sphaericae portiones hemisphaeris minores sint diametrorum cubis proportionales, erunt et relictae sphaericae portiones hemisphaeriis maiores iisdem cubis proportionales.

Propositio XLI.

CUILIBET DATO SOLIDO ALIQUAM SPHAERAM ESSE AEQUALEM.

Esto quodvis datum solidum *A* [Fig. III.5A.29], exponatur quaelibet linea *BD*, super quam diametrum sphaera *BED* intelligatur. Aut igitur sphaera *BED* aequalis erit solido *A* aut minime. Si aequalis, constat propositum. Sin vero, tunc sicut est sphaera *BED* ad solidum *A*, sic sit cubus lineae *BD* ad cubum lineae *CF* per 33^{am} huius; eritque per 35^{am} (! 38^{am}) huius sicut cubus *BD* ad cubum *CF*, sic sphaera *BED* ad sphaeram *CGF*; itaque eandem rationem cum habeat sphaera *BED* ad solidum *A* quam et ad sphaeram *CGF*, aequalis erit iam sphaera *CGF* solido *A*. Rursum ergo constat propositum.

Propositio XLII.

DATIS DUOBUS QUIBUSCUNQUE SOLIDIS ESSE ALIQUOD SOLIDUM UNI EORUM SIMILE ET ALTERI AEQUALE.

Esto, exempli gratia, solidum quodlibet *A* [Fig. III.5A.30], et conus *BED* super basim cuius diameter *BD*.

Demonstrabo iam conum aliquem esse cono *BED* similem et solido dato *A* aequalem.

Sit enim per 33^{am} huius, sicut conus *BED* ad solidum *A*, sic cubus lineae *BD* ad cubum lineae cuiuspiam *CF*, et super circulum *CF* conus *CGF* erigatur similis cono *BED*; eritque per 35^{am} huius conus *BED* ad
23 conum *CGF*, sicut cubus *BD* ad cubum *CF*, et / ideo sicut conus *BED* ad solidum *A*; eandem ergo rationem cum habeat conus *BED* ad solidum *A* quam et ad conum *CGF*, aequalis erit iam conus *CGF* solido *A*. Constat igitur propositum.

Quod si super basim cuius diameter *BD* cylindrus supponatur, similiter et per 30^{am} (! 33^{am}) et 33^{am} (! 34^{am}) huius cylindrum *GCF* similem cylindro *EBD* et aequalem solido *A* esse demonstrabimus. Idem quoque per 33^{am} et 36^{am} pro conis-coluris, per 37^{am} pro tornatilibus solidis, per 39^{am} pro sphaeralibus cuneis, per 40^{am} pro sphaericis portionibus efficiemus. Non aliter de cubis, aut prismatibus, aut pyramidibus, aut parallelepipedis, aut polyhedris regularibus, sive irregularibus solidis propositum constabit, quandoquidem ut in Elementorum 12° ostensum est, similia solida semper sunt cubis correlativorum laterum proportionalia.

Propositio XLIII.

CIRCULUS AEQUALIS EST EI QUOD PRODUCITUR EX SEMIDIAMETRO IN SEMIPERIPHERIAM.

Sit circulus *AGH* [Fig. III.5A.31], cuius semidiameter *AB*, superficies autem *C* sit id quod fit ex semidiametro *AB* in semiperipheriam circuli *AHG*.

Demonstrandum est quod circulus *AHG* aequalis est superficiei *C*.

Nam per 14^{am} huius erit circulus quispiam aequalis superficiei *C*; sit ergo circulus *DIK*, cuius semidiameter *BD*, aequalis superficiei *C*; et ostendendum erit quod *BD* linea aequalis erit ipsi *BA*. Nam secus erit utravis maior; et circulus *DIK* describatur concentricus circulo *AGH*; et per 12^{am} 12^{1} Euclidis intra maiorem horum circulorum describatur figura multiangula circulum minorem minime tangens, fietque per 1^{am} sexti Euclidis ut superficies talis figurae inscriptae sit aequalis ei quod fit ex aliqua linea inter ipsas *AB*, *BD*, quae sit *BE*, in *ELM* semiperimetrum ipsius figurae; quod productum sit *F*; itaque si *AB* sit maior quam *BD*, erit tunc *BE* minor quam *AB* linea; et proinde superficies *C* maior superficie *F*; igitur tunc circulus *DIK* (qui aequalis figurae *C* positus fuit) maior erit figura *F* seu *ELM*, quod est impossibile per tertium postulatum.

Si autem *AB* sit minor quam *BD*, tunc *BE* maior erit quam *AB* linea; quare superficies *F* maior tunc est superficie *C*; et idcirco figura *F* seu *ELM* erit maior spatio *C* seu circulo *DIK* intra quem inscribitur, quod est impossibile per dictum postulatum. Cum ergo linea *BD* neque minor neque maior possit esse quam linea *AB*, erit aequalis, quod erat demonstrandum.

Similiter autem ostendemus quod superficies cylindrica aequalis est ei
24 quod fit / ex latere cylindrico in peripheriam basis. Idque per columnam inscriptam maiori cylindro, ita ut minorem non tangat. Item quod

superficies conica aequalis est ei quod fit ex latere conico in semiperipheriam basis. Idque per pyramidem inscriptam maiori cono, ita ut minorem similem vel sub eodem axe positum non tangat.

Propositio XLIV.

IN DUOBUS CIRCULIS INAEQUALIBUS, ID QUOD FIT EX DIAMETRO MAIORIS IN PERIPHERIAM MINORIS AEQUALE EST EI QUOD FIT EX DIAMETRO MINORIS IN PERIPHERIAM MAIORIS. IDEMQUE DE SEMIDIAMETRIS ET SEMIPERIPHERIIS EST DICENDUM.

Sunto duo circuli inaequales *AGE* et *BHF* [Fig. III.5A.32], quorum diametri *AE* et *BF*, et peripheriae earum aequales sint lineis *C*, *D*, hoc est, ut per 15^{am} huius linea *C* sit aequalis peripheriae circuli *AGE* et linea *D* aequalis peripheriae circuli *BHF*.

Demonstrandum ergo est quod rectangulum *AED* aequale est rectangulo *BFC*.

Nam cum per 8^{am} huius sicut est *AE* ad *BF*, sic sit *C* ad *D*, iam per 8^{am} sexti Elementorum Euclidis constat prima propositi pars. Quod si *AI* et *BL* supponantur semidiametri, iam per eandem 8^{am} lineae *C* et *D* tunc erunt semiperipheriae. Unde constabit quod superat demonstrandum.

Propositio XLV.

IN TRIBUS CIRCULIS QUORUM DIAMETRI SUNT IN PROPORTIONE CONTINUA, ID QUOD FIT EX DIAMETRO PRIMI IN PERIPHERIAM POSTREMI AEQUALE EST EI QUOD FIT EX DIAMETRO MEDII IN PERIPHERIAM SUAM. IDEMQUE DE SEMIDIAMETRIS ET SEMIPERIPHERIIS DICENDUM.

Intersit enim per sextam huius ipsis *AG* et *BL* diametris proportionalis linea *EI* [Fig. III.5A.33], eritque per octavam huius *F* peripheria circuli cuius diameter *EI* media proportionalis inter *C* et *D* peripherias, quandoquidem peripheriae sunt diametris proportionales.

Et demonstrandum erit quod rectangulum *AGD*, sive rectangulum *BLC* (aequalia enim sunt per praecedentem), aequale est rectangulo *EIF*.

Nam cum *AG* ad *EI* sit sicut *F* ad *D*, iam per 15^{am} sexti Euclidis constat prima propositi pars. Et reliqua similiter; nam si *AN*, *EO*, et *BR* lineae supponantur semidiametri, erunt per 8^{am} huius ipsae *C*, *F*, *D* semiperipheriae. Constat ergo propositum.

25 / Propositio XLVI.

IN TRIBUS CIRCULIS QUORUM SEMIDIAMETRI CONTINUAE PROPORTIONALES, QUOD FIT EX SEMIDIAMETRO PRIMI IN SEMIPERIPHERIAM POSTREMI AEQUALE EST CIRCULO MEDIO. IDEMQUE DICENDUM, SI IN PRIMO ET POSTREMO INTEGRAE DIAMETRI ET INTEGRAE PERIPHERIAE LINEAE IPSAE APPELLENTUR.

Namque cum per praecedentem, in eadem descriptione, tam rectangulum *AND* quam rectangulum *BRC* aequale sit rectangulo *EOF*, quod scilicet fit ex *EO* semidiametro in suam peripheriam *F*, quodque per 43^{am} praecedentem aequale est ipsi circulo cuius semidiameter *EO*, iam et ipse circulus cuius semidiameter *EO*, aequalis erit tam rectangulo *AND* quam rectangulo *BRC*; quorum videlicet utrumvis fit ex semidiametro alterius extremorum in semiperipheriam reliqui, sive maius ex diametro tota in peripheriam totam iuxta suppositionem. Constat ergo id quod proponitur demonstrandum.

Expletum Thermis hora noctis prima diei Iovis in extremo carnisprivio, qui fuit mensis Februarii tredecimus VIII. Ind. MDL.

Prolegomena to the Works of Archimedes by Francesco Maurolico

Proem

Archimedes of Syracuse was a most acute geometer, a very outstanding engineer, a most famous observer of the stars. When Marcellus besieged Syracuse, he [Archimedes] long defended his native land with his ingeniously contrived machines. For, as Livy says,[1] by catapulting rocks, by seizing the masts of hostile ships in an iron grapnel, by hurling bolts through slits in the walls made for this, he shook the enemy equally with wonder and terror. This same man, by himself, and with machines grasped in his hand alone, drew along a ship of unusual size built by King Hieron, as Moschion reports at length.[2] He [Archimedes] built a [planetary] sphere in which all the motions of the stars were to be represented; the *Epigramma* of Claudianus speaks of this.[3] By an ingenious and marvelous device, he [Archimedes] proved as faulty a golden crown dedicated by the said king to the gods but corrupted by artifice, as Vitruvius reports.[4] But not all of his discoveries—mechanical as well as geometrical—have been reported in books. Finally after three years, with Syracuse taken, this most illustrious philosopher was killed by an ignorant soldier when Archimedes, intent upon geometrical drawings that had been described in sand and asked [by the soldier] who he was, delayed in giving his name and implored the soldier not to disturb his lines [in the sand]. [And thus killed] he besmirched his drawing with his own blood. So this famous man, whom the Roman general especially had wished unharmed and had ordered protected, gained his death as well as his well-being through his genius and art. Nor was the general's generosity toward the relatives of the slain man lacking, inasmuch as he

Proem

[1] *Ab urbe cond. libri*, XXIV, Cap. 34.

[2] Quoted by Athenaeus, *Deipnosophistae*, V, 206d, 207a–b.

[3] *Carmina minora*, Poem 51.

[4] *De re architectura*, IX, pref. 9–12.

adorned them with his protection, as Livy, Valerius, and other historians have recorded.[5] But our intention is to commemorate the [literary] monuments of this great philosopher by which he has commended his name to immortality and of which there has been scant mention by the historians.

And so the first in order of his works is τοῦ κύκλου μέτρησις, i.e., *The Measurement of the Circle*, in which he demonstrates that a circle is equal to the right triangle one of whose sides about the right angle is equal to the radius of the circle and the other to the circumference. In the second place is to be put the work περὶ σφαίραι καὶ κυλίνδρου, i.e., *On the Sphere and the Cylinder*, [addressed] to Dositheus, in which he demonstrates that the surface of a sphere is quadruple its maximum circle and that a cylinder of the same width and axis as the sphere is 3/2 the sphere; and he demonstrates [there] other things concerning segments of a sphere. From this [work] it has been seen that Archimedes gained for himself so much glory (as he was the first of all to have demonstrated this important matter in geometry) that a sphere and a cylinder were ordered engraved on his tomb. Cicero mentions that this tomb was found by him among thickets while he was traveling in Sicily. Third in order we place the work περὶ ἰσορροπικῶν, i.e., *On Things Weighing Equally* (=*On the Equilibrium of Planes*), in which he presents the rule for finding the centers of gravity of rectilinear, plane figures. It is remarkable that he omitted in this book the centers of solids. The fourth will be τετραγονισμὸς (!) παραβολῆς, i.e., *The Quadrature of the Parabola*, where he demonstrates that a parabola is 4/3 the rectilinear triangle of the same base and altitude. In the demonstration he uses both the doctrine of *On the Equilibrium* [*of Planes*] and geometrical argument. He writes this book to Dositheus, making mention of the death of Conon, to whom he had previously been accustomed to write. Both these men were versed in geometry and were friends of Archimedes. In the fifth place follows the work περὶ ἑλίκων γραμμῶν, i.e., *On Spiral Lines*, where he demonstrates that certain straight lines tangent to the spiral are equal to the circumferences of circles; also that the spiral of first revolution is one-third part of its circle; that the second spiral is to its circle as seven is to twelve, and so on. For since the circles are in the proportion of the square numbers, these spirals are in the proportion of regular hexagons. He also sent this book to Dositheus, again making mention of Conon, who seized by death had left them [i.e. the propositions concerning the spiral] unexplained. I put sixth in order the book περὶ κωνοειδέων καὶ σφαιροειδέων, i.e., *On Conoidal and Spheroidal Figures*, which he also sent to the aforementioned Dositheus. In it he demonstrates many things concerning the mutual proportion of circles and ellipses; also that a parabolic solid is 3/2 its cone, and other [propositions] concerning the proportion of both hyperbolic and spheroidal solids to their respective cones. Indeed

[5] See below, Appendix III, notes 1–4.

these are most acute discoveries and are worthy of such genius. The seventh is *The Sandreckoner* addressed to King Gelon. Its title offers more to admire than the book itself does to ponder. There are many things in it concerning the size of the earth and the celestial luminaries for which blame must be imputed to the times [in which they were written] rather than to the philosopher, since in that age such things were not yet adequately perceived. Afterwards there is to be added the work *περὶ ἰσοπεριμέτρων*, i.e., *On Figures of Equal Perimeter*, which is attributed by some to Archimedes and by others, more correctly, to Theon of Alexandria.[6] In this work it has been demonstrated that the circle is the most capacious of isoperimetric plane figures and the sphere of solid [isoperimetric figures]. And finally the ninth—and not at all to be excluded—is the *περὶ κατοπτρῶν καυστικῶν*, i.e., *On Burning Mirrors*, which some attribute to Archimedes and others, more correctly, to Ptolemy.[7] In it he teaches that the concave form described by a parabola is the one to be given to a mirror for it to be most effective in burning, inasmuch as the parallel solar rays falling on it are reflected to the same point. It is in this point where the most light is gathered that the heat is most potent for burning fuel placed there. Eutocius of Ascalon wrote commentaries on the works *On the Measurement of the Circle*, *On the Sphere and the Cylinder* and *On Things Weighing Equally* (=*On the Equilibrium of Planes*), where he has intermingled many things which have more obscurity than either pleasure or utility and nothing relevant for understanding the author [i.e. Archimedes].

When I had seen all of these works, I was forced to add many lemmas for their easier understanding, to demonstrate many things omitted by Archimedes, e.g., in the matter of equal moments, to treat of the centers of gravity of solids, a topic which he omitted. In the work *On the Sphere and the Cylinder* I have used an easier way [than Archimedes']. Lest any one judge that I have postulated in it principles that are not to be conceded when I suppose that to any surface there exists some equal spherical or conical surface or some surface of a spherical segment—or an equal conical or cylindrical surface of a given altitude, we shall demonstrate these principles here [in the *Praeparatio*].[8] In addition [we shall demonstrate] that with two surfaces given, there exists a surface similar to one of the given [surfaces] and equal to the other; and that with two solids given there exists a solid similar to one of the given [solids] and equal to the other. Since it is necessary for the demonstration of this last proposition to find two mean proportionals, we shall treat this problem out of the tradition of the old philosophers, as the above-

[6] See above in this chapter, Sect. II, notes 3–5.

[7] *Ibid.*, n. 6.

[8] In quoting this whole passage above in Section III of this chapter, I have added the numbers of the propositions of the *Praeparatio* where Maurolico proves these basic assumptions.

mentioned Eutocius wrote in his commentaries [*On the Sphere and the Cylinder*]. But we shall premise [without demonstrations] those principles which we have postulated [below] as being easily conceded.

Postulates[1]

1. There exist two lines proportional to any two magnitudes of the same kind.
2. The perimeter of a plane figure that circumscribes or includes a plane figure is greater than the perimeter of the circumscribed or included figure, so long as the figures are concave in the same direction.
3. A circumscribing or including figure is greater than the circumscribed or included figure.
4. The surface of a solid figure that circumscribes or includes any solid figure is greater than the surface of the inscribed or included figure concave in the same direction.
5. A circumscribing or including solid figure is greater than the circumscribed or included figure.
6. If two magnitudes are always at the same time greater or at the same time less than another magnitude, they are equal to one another.

Proposition I.

A CYLINDRICAL SURFACE IS GREATER THAN A COLUMNAR SURFACE IT HAS CIRCUMSCRIBED[1] OR INCLUDED.

For let there be taken one of the sides [i.e. faces] of the [prismatic] column that has been circumscribed, the side having been erected on straight line *AC* [see Fig. III.5A.1] and [let there be taken] that part of the cylindrical surface erected on arc *ABC*.

And so it is to be demonstrated that cylindrical surface *ABC* is greater than plane surface *AC*, for with this shown it follows that the whole cylindrical surface is greater than the whole columnar surface.

And so, if it is possible, let cylindrical surface *ABC* be less than plane surface *AC*, so that with surface *Z* [added to it] it is equal to surface *AC*. And if arc *ABC* is continually bisected until the portions [of the cylindrical surface] remaining are less than *Z*, then at that point the cylindrical surface *ABC* plus the remaining portions will be less than plane surface *AC*, and *a fortiori* less than the plane surfaces erected on straight lines *AD*, *DB*, *BE* and *EC*, which is against the fourth postulate. Therefore cylindrical surface *ABC* is not less than plane surface *AC*. It will be demonstrated in the same way that it is not equal to it. Therefore, it remains that it is greater than it, as proposed.

Post.
[1] See Commentary to Text A, Postulates.

Prop. I
[1] See Com., Prop. I, lines 2–3.

Proposition II.

A CONICAL SURFACE IS GREATER THAN AN INSCRIBED OR INCLUDED PYRAMIDAL SURFACE.

The same procedure and construction will serve for this proposition as for the preceding one. Whence, if the cylindrical or conical surface exceeds an inscribed columnar surface, even more does it exceed an included surface which it does not touch, inasmuch as the inscribed surface is greater than the [non-touching] included surface. Therefore, the proposition is evident.

Proposition III.

A COLUMNAR SURFACE IS GREATER THAN AN INSCRIBED OR INCLUDED CYLINDRICAL SURFACE.

Let the sides of a [prismatic] column be erected on lines *AF* and *FC* [see Fig. III.5A.2], and let a portion of the cylindrical surface be erected on arc *ABC* and both surfaces are terminated by the lines of the cylinder erected on points *A* and *C*. These lines are the contacts between the plane surfaces erected on *AF* and *FC* and the cylindrical surface erected on arc *ABC*.

And so it is to be demonstrated that plane surface *AFC* is greater than cylindrical surface *ABC*, for with this shown it follows that the columnar plane surface is greater than the whole cylindrical surface.

And so, if it is possible, let plane surface *AFC* be less than cylindrical surface *ABC*. And if arc *ABC* is continually bisected and tangents to the circle are continually drawn until the portions between the tangents and the circumference are less than space *Z* by which cylindrical surface *ABC* is posited as exceeding plane [surface] *AFC*, then *a fortiori* the same cylindrical surface *ABC* will be greater than the plane surface erected on *AK*, *KL*, *LM*, *MN*, *NC* plus the remaining portions; this is against the fourth postulate. Therefore, plane surface *AFC* is not less than cylindrical surface *ABC*. It will be demonstrated in the same way that it is not equal [to it]. Therefore, it will be greater, which has been proposed.

Proposition IV.

A PYRAMIDAL SURFACE IS GREATER THAN AN INSCRIBED OR INCLUDED CONICAL SURFACE.

This will be demonstrated by a procedure no different than [that used for] the preceding proposition. Whence, if a columnar surface exceeds both the inscribed cylindrical and pyramidal surfaces, even more will it exceed an included surface [which it does not touch], inasmuch as the inscribed surface exceeds the surface included in it.

Moreover, it ought to be noted that in a cylindrical surface the above-mentioned remaining portions [or segments] in the preceding as well as in the first [proposition] of this [work] are to be taken on each base where the circumscribing surface has the same termini as the inscribed surface, as the fourth postulate supposes. Therefore the proposition is evident.

Proposition V.

THERE EXISTS SOME LINE TO WHICH A GIVEN LINE HAS A GIVEN RATIO.[1]

Let the given ratio be that of magnitude A to magnitude B [see Fig. III.5A.3], and let the given line be C; and so by the first postulate there will exist lines proportional to magnitudes A and B, which lines we let be D and E, so that $A/B = D/E$. By VI.10 of Euclid[2] let $D/E = C/F$. Hence, $C/F = A/B$. Q.E.D.

Proposition VI.

THERE EXISTS SOME MEAN PROPORTIONAL BETWEEN ANY TWO LINES.

So that if we let the given straight lines AB and BC [see Fig. III.5A.4] be joined as a straight line and if we describe on the whole AC a semicircle and erect perpendicular BD up to the circumference, then by VI.9 of Euclid[1] BD will be the mean proportional between these [lines] AB and BC. Therefore the proposition is evident.

Proposition VII.

THERE EXISTS SOME SQUARE TO WHICH A GIVEN SQUARE MAY HAVE A GIVEN RATIO.

Let the given ratio be that of magnitude A to magnitude B [see Fig. III.5A.5], and let the given square be that of line C. Now by the fifth proposition C will to some line, say D, as A is to B. So, by the preceding [proposition] there will be a mean proportional E between lines C and D. And so $C/D = A/B = C^2/E^2$, inasmuch as $C/D = (C/E)^2$. Therefore the proposition is evident.

Proposition VIII.

THE CIRCUMFERENCES OF CIRCLES ARE PROPORTIONAL TO THEIR DIAMETERS.[1]

Let the circles be AIB and CKD with diameters [respectively] AB and CD [see Fig. III.5A.6]. And let circum. AIB/circum. CKD = diam. AB/line EF, by [Proposition] V of this [work].

Prop. V

[1] See Com., Prop. V, lines 2–3.

[2] *Ibid.*, line 7.

Prop. VI

[1] See Com., Prop. VI, line 6.

Prop. VIII

[1] See Com., Prop. VIII, lines 2–3.

Now it is to be demonstrated that line *EF* will be equal to line *CD*.

For if line *EF* > line *CD*, let there be imagined circle *ELF* concentric with circle *CKD*, and let there be inscribed in circle *ELF* a regular polygon *ENF* that does not at all touch circle *CKD*, by XII.13 of Euclid's *Elements*.[2] And let a similar figure *AMB* be inscribed in circle *AIB*. And line *AB*/line *EF* = perim. fig. *AMB*/perim. fig. *ENF* = circum. *AIB*/circum. *CKD*. Therefore, alternately, perim. *AMB*/circum. *AIB* = perim. *ENF*/circum. *CKD*. But by the second postulate circum. *AIB* > perim. *AMB*. Therefore circum. *CKD* > perim. *ENF*, i.e., the included is greater than the including, which is contrary to the said postulate. Therefore line *EF* $\ngtr$ line *CD*.

But if it is less, then, inversely, circum. *CKD*/circum. *AIB* = line *EF*/line *AB*. And so let line *EF*/line *AB* = line *CD*/line *GH*, by [Proposition] V of this [work]. And line *CD*/line *GH* = circum. *CKD*/circum. *AIB*. And, by V.14 [of Euclid], since *CD* > *EF*, so also *GH* > *AB*. Whence the same impossibility follows as before, namely that the ratio of the first circumference to the second circumference is as the first diameter to a line greater than the second diameter. Therefore line *EF* $\nless$ line *CD*. But neither was it greater. Therefore it will be equal. Q.E.D.

Proposition IX.

SIMILAR ARCS OF CIRCLES ARE PROPORTIONAL TO THEIR CHORDS AND ALSO TO THEIR DIAMETERS.

For by definition similar arcs assume equal angles both at the center and in the arcs. And hence by the last [proposition] of [Book] VI of Euclid they are proportional to the whole circumferences of the circles. But by the preceding [proposition] the circumferences are proportional to the diameters; therefore similar arcs are also in the same way proportional to their diameters. Hence by the similarity of triangles, they are proportional to their chords. Q.E.D.

Proposition X.

CIRCLES ARE PROPORTIONAL TO THE SQUARES OF THEIR DIAMETERS.[1]

Let the circles be *AIB* and *CKD* with diameters *AB* and *CD* [see Fig. III.5A.7]; and let circle *AIB*/circle *CKD* = AB^2/EF^2, by [Proposition] VII of this [work].

Now it will have to be demonstrated that line *EF* is equal to line *CD*.

For otherwise it is either greater or less. If line *EF* > line *CD*, then let the same construction be made as in Proposition VIII. And AB^2/EF^2 = fig. *AMB*/fig. *ENF* = circle *AIB*/circle *CKD*. And therefore, alter-

[2] *Ibid.*, line 10.

Prop. X

[1] See Com., Prop. X, lines 1–25.

nately, fig. AMB/circle AIB = fig. ENF/circle CKD. But by the third postulate circle AIB > fig. AMB. Therefore, circle CKD > fig. ENF, which is impossible by the said postulate. Therefore line $EF \not> $ line CD.

But if it is less, then, inversely, circle CKD/circle AIB = EF^2/AB^2. Therefore, by [Proposition] VII let $EF^2/AB^2 = CD^2/GH^2$, and CD^2/GH^2 = circle CKD/circle AIB. And since $CD > EF$, so by V.14 [of Euclid] $GH > AB$. Whence follows the same impossibility,[2] namely that the ratio of the first circle to the second circle is as the square of the first diameter to the square of a line greater than the second diameter. Therefore line $EF \not< $ line CD. But neither was it more; therefore it is equal. Q.E.D.

Proposition XI.

SIMILAR SECTORS AS WELL AS SIMILAR SEGMENTS OF CIRCLES ARE PROPORTIONAL TO THE SQUARES OF THEIR DIAMETERS.

By the last [proposition] of [Book] VI of Euclid, similar sectors are proportional to their whole circles. Wherefore by the preceding [proposition] they are also proportional to the squares of their diameters. Also, the triangles in sectors of this sort that are on the chords of the segments and with vertices at the center are also proportional to the said squares. Hence, since the magnitudes which are subtracted, namely the triangles, are proportional to the wholes, namely the sectors, therefore by V.19 of the *Elements* the remainders, namely the segments, will also be proportional to the wholes [and hence to the squares of the diameters]. Q.E.D.

Proposition XII.

THE SURFACES OF SIMILAR CYLINDERS ARE PROPORTIONAL TO THE SQUARES OF THE DIAMETERS OF THEIR BASES.[1]

Let the construction be as in [Proposition] X above, and let similar cylinders be imagined as erected on circles AB, CD, EF and GH [see Fig. III.5A.8] and columns be imagined as erected on regular polygons AB, CD, EF and GH and of the same altitudes as the cylinders. And in place of the circles [used in the demonstration of Proposition X], let the cylindrical surfaces whose bases are the circles be assumed [here]; in place of the polygons inscribed in the circles [in Proposition X], let the lateral surfaces of the columns whose bases are the polygons be assumed [here]; and in place of the third postulate [used in Proposition X], let [Proposition] I of this [work] be cited [here]. And [with these substitutions made] let the same procedure be retained in this demonstration [as in Proposition X].

[2] *Ibid.*, lines 21–23.

Prop. XII

[1] See Com., Prop. XII, lines 2–11.

Proposition XIII.

THE SURFACES OF SIMILAR CONES ARE PROPORTIONAL TO THE SQUARES OF THE DIAMETERS OF THEIR BASES.[1]

Let the construction be as in [Proposition] X above. And let similar cones be imagined as erected on circles *AB*, *CD*, *EF* and *GH* [see Fig. III.5A.9] and also as many pyramids inscribed in the cones and of the same altitudes as the cones. And in place of the circles [in Proposition X], let the conical surfaces whose bases are the circles be assumed [here]; in place of the polygons inscribed in the circles [in Proposition X], let the lateral surfaces of the pyramids whose bases are the polygons be assumed [here]; and for the third postulate [used in Proposition X], let [Proposition] II of this [work] be cited [here]. For the argument of this demonstration is completely the same [as in Proposition X].

Proposition XIV.

THERE EXISTS SOME CIRCLE EQUAL TO ANY GIVEN SURFACE.[1]

Let *A* be any given surface [see Fig. III.5A.10]; and let circle *BED* be described on whatever line *BD*, which circle will either be equal to surface *A* or not equal to it. If it is equal to it, then the proposition is immediately evident. But if not, then by [Proposition] VII of this [work] circle *BED*/surface $A = BD^2/CF^2$. And by [Proposition] X of this [work] BD^2/CF^2 = circle *BED*/circle *CGF*; and since circle *BED*/circle *CGF* = circle *BED*/surface *A*, therefore by V.9 [of Euclid] circle *CGF* = surface *A*. Therefore the proposition is again evident.

Proposition XV.

THERE EXISTS A CIRCUMFERENCE OF SOME CIRCLE EQUAL TO ANY GIVEN LINE.

Let *A* be any given line [see Fig. III.5A.11], and let circle *BED* be described on any line *BD*. If its circumference is equal to line *A*, then the proposition is evident. But if not, then by [Proposition] V of this [work] circum. *BED*/line *A* = diameter *BD*/line *CF*. And by [Proposition] VIII of this [work], diameter *BD*/diameter *CF* = circum. *BED*/circum. *CGF*. Therefore, since circumference *BFD* has the same ratio to circumference *CGF* as it has to line *A*, by V.9 [of Euclid] circumference *CGF* will be equal to line *A*. Again the proposition will be clear.

From these [considerations in Propositions XIV and XV], it is manifest that a given circle is equal to some rectilinear figure and also that the circumference of a given circle is equal to some straight line.

Prop. XIII

[1] See Com., Prop. XIII, lines 2–10.

Prop. XIV

[1] See Com., Prop. XIV, lines 2–3.

For these things are demonstrated in the same way with only the assumption changed, i.e., let the circle be supposed in place of the surface [to demonstrate the converse of Proposition XIV] and the circumference of the circle in place of the given line [to demonstrate the converse of Proposition XV].

[(*Ed. 1685 in re Prop. XVI*) Maurolico calls the segment of a cone cut off by a plane parallel to the base and comprised between the aforesaid planes a truncated cone.][1]

Proposition XVI.

THE CURVED [I.E. LATERAL] SURFACES OF SIMILAR TRUNCATED CONES ARE PROPORTIONAL TO THE SQUARES OF THE DIAMETERS OF THEIR CORRESPONDING BASES.

For since truncated cones are segments of cones, similar truncated cones will be segments of similarly constructed cones. But by [Proposition] XIII of this [work], the surfaces of the cones cut off [at the top], since they are similar, are proportional to the squares of the diameters of their bases; and the surfaces of the whole similar cones are proportional to the squares of the diameters of their bases and hence are proportional to the surfaces of the cones cut off [at the top], inasmuch as the diameters of the bases [of the whole cones] are proportional to the diameters of the bases [of the cut-off cones]. Therefore by V.19 [of Euclid] the surfaces of the truncated cones which remain [after the top cones are cut off from the whole cones] will be proportional to the squares of the said corresponding diameters. Q.E.D.

Proposition XVII.

THE SURFACES OF THE SIMILAR SOLIDS DESCRIBED BY THE COMPLETE ROTATION OF SIMILAR SEMIPOLYGONS ABOUT THEIR FIXED DIAMETERS ARE PROPORTIONAL TO THE SQUARES OF THEIR DIAMETERS.

For the surfaces of such solids are composed out of an equivalent number of similar conical surfaces. Among these the pairs at the ends of the diameters are of whole cones, while the others are of truncated cones, [except that if the rotated semipolygons have an odd number of sides] a pair of the latter surfaces will be cylindrical.[1] But by [Propositions] XII, XIII and XVI above such corresponding surfaces correspondingly compared one to the other are as the squares of the diameters of their corresponding bases. But these latter squares are proportional to the squares of the diameters about which the semipolygons are rotated.

Prop. XV
[1] See Com., Prop. XV, lines 17–18.
Prop. XVII
[1] See Com., Prop. XVII, line 9.

Therefore by conjunct proportion used as often as necessary the whole surfaces of the solids will be proportional to the squares of the said diameters. Q.E.D.

Proposition XVIII.

THE SURFACES OF SPHERES ARE PROPORTIONAL TO THE SQUARES OF THEIR DIAMETERS.[1]

Let *AIB* and *CKD* be two spheres with diameters *AB* and *CD* [see Fig. III.5A.12]. And by [Proposition] VII of this [work] let spher. surf. *AIB*/spher. surf. $CKD = AB^2/EF^2$.

And it will have to be demonstrated that line $EF =$ line *CD*.

For, if it is possible, let line *EF* be greater than line *CD*. And about diameter *EF* let sphere *ELF* be imagined concentric with sphere *CKD*, and within sphere *ELF* let there be a solid of rotation *ENF* composed of conical surfaces that do not touch sphere *CKD* at all, and a solid *AMB* similar to this in sphere *AIB*. By the preceding [proposition] surf. solid *AMB*/surf. solid $ENF = AB^2/EF^2$. Therefore, surf. solid *AMB*/surf. solid $ENF =$ surf. sphere *AIB*/surf. sphere *CKD*. Alternately, surf. solid *AMB*/surf. sphere $AIB =$ surf. solid *ENF*/surf. sphere *CKD*. But by the fourth postulate, surf. sphere $AIB >$ surf. solid *AMB*. Therefore, surf. sphere $CKD >$ surf. solid *ENF*, which is impossible by the said postulate. Therefore line $EF \ngtr$ line *CD*.

But if it is less, then, inversely, surf. sphere *CKD*/surf. sphere *AIB* $= EF^2/AB^2$. Hence by [Proposition] VII of this [work] $EF^2/AB^2 = CD^2/GH^2$. And so surf. sphere *CKD*/surf. sphere $AIB = CD^2/GH^2$. And since $CD > EF$, so by V.14 [of Euclid] $GH > AB$. Whence the same impossibility follows as before, namely that the ratio of the first spherical surface to the second spherical surface is as the ratio of the square of the first diameter to the square of a diameter greater than the second diameter.[2] Therefore line $EF \nless CD$. But neither was it greater; therefore it will be equal to it. Q.E.D.

Proposition XIX.

THE SURFACES OF SIMILAR SPHERICAL SEGMENTS ARE PROPORTIONAL TO THE SQUARES OF THEIR DIAMETERS.

The same demonstration will serve for this proposition as for the preceding one, with this exception: in place of the surfaces of the whole sphere [used in Proposition XVIII] let the surfaces of the segments be taken [here], and instead of the surfaces of the whole solids of rotation [used in Proposition XVIII] let the partial surfaces of those solids which having been inscribed in similar segments of spheres are similar be taken

Prop. XVIII

[1] See Com., Prop. XVIII, lines 2–33.

[2] *Ibid.*, lines 28–31.

[here]. Hence the proposition will be concluded, namely that the surfaces of such segments will be proportional to the squares of the diameters of their spheres and also to the squares of the diameters of the circles on which the segments stand, inasmuch as the diameters of such circles are proportional to the diameters of the spheres because of the similarity [of the segments].[1]

Proposition XX.

THERE EXISTS A SURFACE OF SOME SPHERE EQUAL TO ANY GIVEN SURFACE.[1]

Let *A* be any given surface [see Fig. III.5A.13]; and let sphere *BED* be described on any line *BD* whose surface is either equal to surface *A* or is not. If it is equal, the proposition is evident. But if otherwise, then by [Proposition] VII of this [work],[2] let surf. sphere BED/surf. $A = BD^2/CF^2$. And by [Proposition] XVIII above, BD^2/CF^2 = surf. sphere BED/surf. sphere *CGF*. Therefore spherical surface *BED* has the same ratio to spherical surface *CGF* as it has to surface *A*. Therefore, by V.9 of the *Elements*, spherical surface *CGF* = surface *A*. Hence again the proposition is evident. [(*Ed. 1685 in re Fig. III.5A.13*)] Notice that the spherical figures ought to be drawn about diameters *BD* and *CF*.][3]

AND SIMILARLY IT WILL BE EVIDENT THAT THE SURFACE OF ANY GIVEN SPHERE IS EQUAL TO SOME RECTILINEAR FIGURE.

That is, with the sphere first given, [proceed] then to the rectilinear figure described.

Proposition XXI.

WITH ANY TWO SURFACES GIVEN, THERE EXISTS SOME SURFACE SIMILAR TO ONE OF THEM AND EQUAL TO THE OTHER.[1]

This proposition is that much more general than [Propositions] XIV and XX of this [work] as the surface [to which a given surface is to be equal] is more general in kind than a circular or spherical surface. Therefore, e.g., let *A* be any surface and let any segment of circle *BED* [be constructed] on chord *BD* [see Fig. III.5A.14].

I shall show that there is some segment of a circle that is equal to surface *A* and similar to segment *BED*.

Prop. XIX
[1] See Com., Prop. XIX, lines 2–12.
Prop. XX
[1] See Com., Prop. XX, lines 2–3.
[2] *Ibid.*, line 8.
[3] *Ibid.*, lines 14–15.
Prop. XXI
[1] See Com., Prop. XXI, lines 2–4.

For by [Proposition] VII of this [work], let seg. *BED*/surf. *A* = (chord $BD)^2$/(line $CF)^2$. And by III.31 of the *Elements* a segment of circle *CGF* may be constructed on line *CF* which is similar to segment *BED*. And since the chords of similar segments are proportional to the diameters of the circles, hence the squares are proportional to the squares. Then by [Proposition] XI of this [work] similar segments of the circles are proportional to the squares of their diameters. Therefore, BD^2/CF^2 = seg. *BED*/seg. *CGF*. And so since segment *BED* has the same ratio to segment *CGF* and to surface *A*, by V.9 of the *Elements* segment *CGF* will be equal to surface *A*. But it was also similar to segment *BED*. Therefore the proposition is evident.

In the case of any given figure taken together with the surface of any other figure like a cylindrical or conical figure, we shall demonstrate in the same way that there is some surface similar to the proposed figure and equal to the given surface. But in place of [Proposition] XI, we shall use [Propositions] XII, XIII or XVI of this [work].

Also in the case of a given surface and a solid of rotation, we shall prove in the same way by [Proposition] XVII that there is some solid of rotation whose surface is equal to the given surface and similar to the surface of the proposed solid of rotation.

Proposition XXII.

THERE EXISTS SOME CYLINDRICAL SURFACE ABOUT A GIVEN AXIS WHICH IS EQUAL TO ANY GIVEN SURFACE.[1]

Let *A* be the given surface [see Fig. III.5A.15], *BC* the given axis perpendicular to the plane in which indefinite [straight] line *DE* lies. Now since about axis *BC* there can be constructed an infinitude of prismatic columns whose individual lateral surfaces are less than surface *A*—and such lateral surfaces are greater than the cylindrical surfaces inscribed in them about the same axis by [Proposition] III of this [work], therefore an infinitude of cylindrical surfaces can be constructed about axis *BC* whose individual lateral surfaces are less than the given surface *A*. Also, since about axis *BC* there can be constructed an infinitude of prismatic columns whose individual lateral surfaces are greater than surface *A*—and such surfaces are less than the cylindrical surfaces in which they are inscribed by [Proposition] I of this [work], therefore there is also an infinitude of cylindrical surfaces which can be located about axis *BC* and whose individual lateral surfaces are greater than surface *A*. And so there will be some terminus in line *DE* inside of which the cylindrical surfaces are individually less than surface *A* and outside of which they are greater [than it]. Let such a terminus be point *D*; and by [Propositions] I and III, whichever of the said cylindrical surfaces is inside terminus *D* is less than

Prop. XXII

[1] See Com., Prop. XXII, lines 2–3.

the cylindrical surface whose base radius is *CD* and whichever is outside of terminus *D* is greater [than it]. And so since the given surface *A* and the cylindrical surface with base radius *CD* and axis *BC* when compared with every other surface inside and outside of point *D* are each at the same time greater than the ones or at the same time less than the others, they will be equal to each other by the last postulate.[2] Therefore the proposition is evident.

Proposition XXIII.

THERE EXISTS SOME CONICAL SURFACE ABOUT A GIVEN AXIS THAT IS EQUAL TO ANY GIVEN SURFACE.[1]

Let *A* be the given surface [see Fig. III.5A.16] and *BC* the given axis perpendicular to the plane in which indefinite straight line *DE* lies. Now since about axis *BC* there can be placed an infinitude of pyramids whose lateral surfaces are individually less than surface *A* and such lateral surfaces are greater than the conical surfaces about the same axis inscribed in them by [Proposition] IV of this [work], therefore an infinitude of conical surfaces can be erected about axis *BC* whose individual lateral surfaces are less than given surface *A*. Also, since about axis *BC* there is an infinitude of pyramids whose individual lateral surfaces are greater than surface *A*—and such surfaces are less than the conical surfaces in which they are inscribed by [Proposition] II of this [work], therefore there is also an infinitude of conical surfaces around axis *BC* that are individually greater than surface *A*. And so there will be some terminus in line *DE* inside of which the conical surfaces are individually less than surface *A* and outside of which they are greater [than it]. Let such a terminus be point *D*; and by [Propositions] II and IV whichever of the said conical surfaces is inside of point *D* is less than the conical surface whose base radius is *CD* and whichever is outside of point *D* is greater than it. And so since the given surface *A* and the conical surface with base radius *CD* and axis *BC* when compared with every other surface assumed inside and outside of point *D* are each at the same time greater than the former or at the same time less than the latter they are demonstrated to be equal to each other by the last postulate. Therefore the proposition is evident.

Proposition XXIV.

IN THE CASE OF TWO SPHERICAL SEGMENTS STANDING ON ONE PLANE, THE GREATER SURFACE IS THAT OF THE EXTERIOR SEGMENT.

So that if spherical segments *ABC*, *DEF* [see Fig. III.5A.17] stand on a plane in which the straight line *ADFC* lies, with segment *ABC* the

[2] *Ibid.*, line 24.
Prop. XXIII
[1] See Com., Prop. XXIII, lines 2–3.

exterior surface, the surface of *ABC* will be the greater. For if in the other direction on the same plane we let spherical segments *AGC* and *DHF* stand and each is equal and similar to the segment coterminal with it, then by the fourth postulate the total surface *ABCG* will be greater than the total surface *DEFH*, i.e. the including surface than the included. Therefore the half is greater than the half, i.e. surface *ABC* than surface *DEF*. Q.E.D.

Proposition XXV.

THERE EXISTS SOME SURFACE OF SOME SPHERICAL SEGMENT DESCRIBED ON A GIVEN CENTER, THE SEGMENT HAVING BEEN CUT OFF BY A GIVEN PLANE, EQUAL TO ANY GIVEN SURFACE.[1]

Let *A* be the given surface [see Fig. III.5A.18], *B* the given center, and the given plane the one in which straight line *CD* lies so that *BC* is perpendicular to this indefinite plane. Now, as in [Propositions] XXII and XXIII of this [work], there is an infinitude of cylindrical or conical surfaces having an axis in line *BC* that are individually less than and greater than the given surface *A*; and since by the fourth postulate a spherical or conical surface is greater than the surface of the spherical segment inscribed in it and less than the surface of the spherical segment in which it is inscribed, therefore there will be an infinitude of surfaces of spherical segments about center *B* and cut off by the plane *CD* that are less than surface *A* and also that are greater than it. And so there will be some terminus in line *CD* inside of which the surfaces of the spherical segments will be [individually] less than surface *A* and outside of which they will be greater [than it]. Let point *D* be such a terminus, and by the preceding [proposition] whichever surface of a spherical segment is inside of point *D* is less than the surface of the spherical segment with base radius *CD* and whichever is outside of point *D* is greater than it. And so since the given surface *A* and the surface of the spherical segment whose base radius is *CD* when compared [individually] to every other surface assumed inside and outside of point *D* are each at the same time greater than the former or at the same time less than the latter, therefore, by the last postulate, they will be equal [to one another]. Q.E.D.

Proposition XXVI.

THERE EXIST TWO MEAN PROPORTIONALS TO ANY TWO GIVEN LINES.

Let *AB* and *BC* be any two given straight lines [see Fig. III.5A.19], and let the rectangle *ABCD* be completed, and let *BC* and *BA* be extended indefinitely, and let the center of the rectangle be *E*, in which the two diagonals *AC* and *BD* intersect. Now with *BA* and *BC* produced, let

Prop. XXV

[1] See Com., Prop. XXV, lines 2–4.

straight line FDG be applied through D in such a way that the two lines EF and EG are equal.

I say then that CF and AG lie as mean proportionals between AB and BC.

For let EH and EK be drawn perpendicularly to AB and BC. And by II.6 of Euclid $(BF \cdot FC) + KC^2 = KF^2$. And let the common term EK^2 be added [to both sides]. So $(BF \cdot FC) + KC^2 + KE^2 = KF^2 + KE^2$ or $(BF \cdot FC) + EC^2 = EF^2$. In entirely the same way we shall demonstrate that $(BG \cdot GA) + EA^2 = EG^2$. $EF^2 = EG^2$ by hypothesis. Therefore, $(BF \cdot FC) + EC^2 = (BG \cdot GA) + EA^2$. Further, $EC^2 = EA^2$, since EC and EA are each half of the same line AC. Therefore, $(BF \cdot FC) = (BG \cdot GA)$. Wherefore by VI.15 of Euclid[1] $BG/BF = FC/GA$. And, because of the similarity of triangles GBF and DCF, DC (or AB)$/CF = FC/GA$. Also, because of the similarity of triangles GAD and DCF, $DC/CF = GA/AD$ (or BC). And so the four lines AB, CF, GA and BC are continually proportional. And this was proposed for demonstration.

And this is the method of Hero, who wrote a *Mechanica*.[2] Now let us present the traditions of other philosophers on the problem, which we have taken from Eutocius.

Proposition XXVII.

TO DEMONSTRATE THE SAME THING IN ANOTHER WAY.

Let AB and BC be the given lines [see Fig. III.5A.20], and let rectangle $ABCD$ be completed, around which we circumscribe a circle by IV.9 of Euclid. Then between BA and BC extended indefinitely let straight line $FDHG$ be drawn in such a way that FD and HG are equal to one another, for then CF and AG will be the mean proportionals between AB and BC. For by III.35 of Euclid,[1] $(AG \cdot GB) = (HG \cdot GD)$ and $(CF \cdot FB) = (DF \cdot FH)$. But $(DF \cdot FH) = (HG \cdot GD)$, inasmuch as lines DF and GH are posited as equal. Therefore $(CF \cdot FB) = (AG \cdot GB)$. Wherefore, by VI.15 [of Euclid] the situation is no different from that of the preceding [proposition]: from the similarity of the triangles we shall demonstrate that AB, CF, GA and BC are continually proportional.

And this is the method of Apollonius and Philo of Byzantium, as Johannes Philoponus the Alexandrian attests.[2] And this method is almost the same as that of Hero, although the demonstrations differ somewhat.

Proposition XXVIII.

TO SHOW THE VERY SAME THING IN ANOTHER WAY.

Let AB and BG be the two given lines [see Fig. III.5A.21] joined at a

Prop. XXVI

[1] See Com., Prop. XXVI, line 21.

[2] *Ibid.*, line 27.

Prop. XXVII

[1] See Com., Prop. XXVII, line 7.

[2] *Ibid.*, lines 15–16.

right angle and let them be extended, and let *BE* and *BD* each be equal to *AB*, and let the semicircle *ADE* be described on center *B* with *AB* as a radius, and with *AG* drawn let it be extended to the circumference in point *Z*, and about *E* let rule *EH* be moved until the segment *TK* of the rule intercepted between the circumference and *AZ* is bisected by *BD* in point *L*. And through point *K* let line *MN* be drawn parallel to *BD*, and *MA*/*AB* = *EM*/*MX* = *MN*/*MO* [*X* and *O* having been taken so as to preserve the proportion].

And so I say that lines *BG*, *MX*, *MO* and *AB* are continuous proportionals.

For let *TP* be drawn parallel to *BD*, and *BP* and *BM* are equal inasmuch as *TL* and *LK* (placed between the same parallels) are equal to each other. Therefore, *KM*/*ME* = *TP*/*PE* = *MN*/*MA* = *ME*/*MN*, inasmuch as *MN* is the mean proportional between *AM* and *ME*. Therefore *ME* is also a mean proportional between *KM* and *MN*. Therefore, there is no doubt but that *KM*, *ME*, *MN* and *MA* are in continued proportion. But *BG*/*KM* = *AB*/*MA* because of the similarity of triangles. And so by hypothesis, *MX*/*ME* = *MO*/*MN*. Therefore by alternate proportionality it follows that lines *BG*, *MX*, *MO* and *AB* are in the same continued proportion. Q.E.D.

This is the tradition of Pappus in the *Mechanics*, and Diocles and Porus [i.e. Sporus] seem to use the same method.

Proposition XXIX.

TO DEMONSTRATE THE VERY SAME THING IN ANOTHER WAY.

Let *AB* and *BC* be the given straight lines [see Fig. III.5A.22] joined at right angles and produced indefinitely, and let these extended lines be *ABD* and *CBE*, and let a right angle [i.e. a carpenter's square] *FGH* be constructed; and in one arm, e.g. *FG*, let a rule *KL* be moved in a channel which is *FG*, so that rule *KL* is always parallel to *GH*. Let another rule be inserted in arm *GH*, namely *HM*, and let *HM* be made parallel to *FG*, so that rule *KL* fitted to the channels of *FG* and *HM* may be moved back and forth [remaining] always parallel to *GH*. Then angle *ABC* is to be accommodated to this structure in such a way, I say, that lines *ABD* and *CBE* are interposed between the rules with point *A* touching rule *KL* and point *C* touching rule *GH* and with [line] *ABD* passing through angle *G* and line *CBE* through angle *K*. For with this done, surely angles *AED* and *EDC* will be right angles. Wherefore, by VI.7 of the *Elements* twice assumed, the lines *AB*, *BE*, *BD* and *BC* are continuous proportionals. And so between the two given [lines] *AB* and *BC* there are just as many proportional lines, and this was to be shown.

Now this was the discovery of Plato when Apollo, having been consulted by the Delians who were suffering under a pestilence, responded that the plague would cease if his altar were to be doubled. Since the altar had the shape of a cube and a cube could only be duplicated by the interposition of two mean proportionals, the problem of this sort was proposed.[1]

Proposition XXX.

TO DEMONSTRATE THE SAME THING IN ANOTHER WAY.

Let *AB* and *CD* be the given straight lines [see Fig. III.5A.23]. And insert between them [three] equal and similar rectangles whose longer arms are each equal to *AB*, the greater of the given lines. And these rectangles are to be disposed in the same plane so that their shorter sides lie on the same straight line *BK* and their diagonals are mutually parallel. Then with the middle rectangle remaining stationary, let one of the others be pushed along above it and the other pushed along below it but in the same plane [and toward each other] until the extremes of the diagonals, namely points *A*, *E* and *H*, are in a single line *AK* which furthermore intercepts with straight line *BK* a segment *DC* on the side of the last rectangle, the segment being equal to the lesser of the [two] given [lines]. With this done, lines *EF* and *GH* will be the mean proportionals between *AB* and *CD*. For because of the similarity of triangles, $AB/EF = AK/KE$, and $AK/KE = FK/KG$. But $FK/KG = EF/HG$. Therefore, $EF/HG = AB/EF$. Again, $EF/HG = EK/KH$. Now $EK/KH = GK/KC$. But $GK/KC = HG/CD$. Therefore, $EF/HG = HG/CD$. Therefore it is evident that *AB*, *EF*, *HG* and *CD* are in continued proportion, as was to be demonstrated. And this is the tradition of Eratosthenes.

Proposition XXXI.

TO DEMONSTRATE THE SAME THING IN ANOTHER WAY.

Let *AB* and *BG* (! *AG*) be the given lines [see Fig. III.5A.24] and let them be placed in a straight line [i.e., *ABG*], and let parabola *ADH* be described about axis *AG*, and let straight line *BD* perpendicular to *AG* cut the parabola in point *D*. Then through point *D* let a straight line *EDZ* be drawn parallel to *AG*. Then from points *A* and *G* let lines *AE* and *GZ* be drawn to points *E* and *Z* and be parallel to line *BD*. Also let hyperbola *ZH* be described through point *Z* within asymptotes *EA* and *AG*. Let the hyperbola cut the parabola in point *H*, from which let fall perpendiculars *HT* and *HK* to lines *AE* and *AG*. And by [Proposition] VI of this [work] let *AL* be the mean proportional between *AB* and *AT*.

And so I say that *AL* and *AT* are the mean proportionals between *AB* and *AG*.

For since *HZ* is a hyperbola with *KA* and *AG* its asymptotes, so by II.12 of the *Conical Elements* [of Apollonius][1] rectangle *AH* will be equal

Prop. XXIX

[1] See Com., Prop. XXIX, lines 18–22.

Prop. XXXI

[1] See Com., Prop. XXXI, lines 13–14.

to rectangle AZ. Therefore by VI.15 of the *Elements* of Euclid, $AT/AG = GZ$ (or BD)$/TH$. But by I.20 of the *Conical Elements*,[2] $AB/AT = (BD/TH)^2$, inasmuch as ADH is a parabola. Therefore $AB/AT = (AT/AG)^2$. Therefore AB to AL, or AL to AT (for the ratios are the same by hypothesis), will be as AT to AG. Therefore, AB, AL, AT and AG are continuous proportionals. Q.E.D.

And this is the discovery of Menaechmus.[3] It is to be more commended than the other methods since it uses geometrical and more complete precepts while the others seem to achieve their objective by a certain fortuitous procedure. But add to this [demonstration], that if AB and BD are posited as being equal, and then the parabola ADH is described through points A and D, and the rest as before, then by the above demonstration BD (or AB)$/TH = AT/AG$. But by I.20 of the *Conics* [of Apollonius], $AB/AT = (BD/TH)^2$ or $AB/AT = (AB/TH)^2$ [since $AB = BD$]. Therefore, by this way, AB, TH, AT and AG are continually proportional. Thus the interposition of AL is not necessary, as Menaechmus teaches.

Proposition XXXII.

TO DEMONSTRATE THIS VERY SAME THING IN STILL ANOTHER WAY.

Let AB and BG be the two given straight lines [see Fig. III.5A.25] with AB the greater, and let circle BGA be described on diameter AB, within which circle the line BG is applied by IV.1 of Euclid,[1] and let BG extended meet, in point D, line AD tangent to the circle. Also let [chord] EG cut diameter AB at right angles in point Z. And on diameter EG let a semicircle EHG be drawn perpendicular to circle BGA. Then upon semicircle BGA let a right semicylinder be erected and let a semicircle be described on the diameter AB in the rectangle that passes through the axis of the semicylinder. Let this semicircle be moved on the plane of circle BGA so that it always remains perpendicular to that plane, the motion of AB being towards G and the terminus B remaining unmoved. And let the semicircle, after it has been moved, be TKB on diameter BT and let it cut the circumference of BGA in point L. With this motion circumference TKB will describe on the cylindrical surface a certain curved line. Then let $\triangle BDA$ be rotated on axis AB. With this motion point G will be carried in circumference EHG, and line BD thus rotated describes a conical surface, and it will cut the curved line described on the cylindrical surface in a certain point K in which the circumference TKB will cut the side BD in motion. Therefore let $\triangle BDA$ after its translation to this position be $\triangle BMA$ with its side BM cutting the circumference TKB in point

[2] *Ibid.*, line 16.

[3] *Ibid.*, lines 22–30.

Prop. XXXII

[1] See Com., Prop. XXXII, lines 4–5.

K in the cylindrical surface. Let straight line *KL* be drawn. This will be the common section of plane *TKB* and the cylindrical surface, since the cylinder is a right cylinder; and therefore the plane *TKB*, being perpendicular to the base of the cylinder, is parallel to the axis of the cylinder; and therefore *KL*, parallel to that same axis, will be perpendicular to circle *BGA* and hence to straight line *TB*. Furthermore, line *HN*, which by XI.19 of Euclid will be perpendicular to circle *BAG* and hence to straight line *TB*, will be the common section of circle *TKB* and circle *EHG*. And let *TK* and *LH* be drawn.

And so I say that *BL* and *BK* will lie as mean proportionals between *BG* and *AB*, which is demonstrated as follows.

For by VI.8 of the *Elements* of Euclid,[2] $HN^2 = EN \cdot NG$; and with III.34 [of Euclid][3] adduced, $HN^2 = BN \cdot NL$. From this it follows that $\angle BHL$ is a right angle. Since angles *BLK* and *BKT* are right angles, hence triangles *BHL*, *BLK* and *BKT* are similar, inasmuch as they are equiangular. Whence it follows that *BH*, *BL*, *BK* and *BT* are continually proportional. But $BH = BG$ because they are elements [of the surface] of the right cone whose axis is *BZ* and whose apex is *B*. Also $BT = BA$ by hypothesis. Therefore, *BG*, *BL*, *BK* and *BA* will also be continually proportional. Q.E.D.

This is the discovery of Archytas of Taras, as Eudemus and Eutocius report.[4] It is indeed an ingenious theoretical device worthy of such a man, that despite its difficult construction it is nevertheless quite easy to demonstrate.

Proposition XXXIII.

THERE EXISTS SOME CUBE WHICH HAS A GIVEN RATIO TO A GIVEN CUBE.[1]

Let the given ratio be that of magnitude *A* to magnitude *B* [see Fig. III.5A.26], and let the given cube be that of line *C*. Now by [Proposition] V of this [work], *A* is to *B* as line *C* is to some line *D*. And so by any of the seven preceding [propositions],[2] two mean proportionals will lie between *C* and *D*; let these be *E* and *F*. And by XI.36 of Euclid,[3] $C/D = C^3/E^3$, or $A/B = C^3/E^3$. Therefore the given cube of *C* has to cube of *E* the given ratio of *A* to *B*. Q.E.D.

[2] *Ibid.*, line 31.
[3] *Ibid.*, line 32.
[4] *Ibid.*, line 40.

Prop. XXXIII
[1] See Com., Prop. XXXIII, lines 2–3.
[2] *Ibid.*, line 7.
[3] *Ibid.*, line 8.

Proposition XXXIV.

SIMILAR CYLINDERS ARE PROPORTIONAL TO THE CUBES OF THEIR BASE DIAMETERS.[1]

Let *XAIB* and *ZCKD* be the similar cylinders [see Fig. III.5A.27] with base diameters [respectively] of *AB* and *CD*. And let cylinder *XAIB*/cylinder *ZCKD* = AB^3/EF^3 by the preceding [proposition].

And it will have to be demonstrated that line *EF* = line *CD*.

For otherwise it will be greater or less [than it]. If line *EF* > line *CD*, let there be imagined a circle *ELF* concentric with circle *CKD*; and let there be inscribed in circle *ELF* a polygon *ENF* that does not touch circle *CKD* at all, by XII.13 of Euclid.[2] And let a polygon *AMB* similar to it be inscribed in circle *AIB*. And on the [polygonal] figures let [prismatic] columns *QENF* and *XAMB* be erected, which [by their erections] become inscribed in similar cylinders *QELF* and *XAIB*. And by XII.8 of Euclid,[3] AB^3/EF^3 = col. *XAMB*/col. *QENF*, inasmuch as the columns as well as the cylinders are similar. Therefore col. *XAMB*/col. *QENF* = cyl. *XAIB*/cyl. *ZCKD*. And, alternately, col. *XAMB*/cyl. *XAIB* = col. *QENF*/cyl. *ZCKD*. But by the fifth postulate cyl. *XAIB* > col. *XAMB*, since the column is inscribed in the cylinder. Therefore, cyl. *ZCKD* > col. *QENF*, which is impossible by the said postulate. Therefore, line *EF* $\ngtr$ line *CD*.

Now if it is less, then inversely, cyl. *ZCKD*/cyl. *XAIB* = EF^3/AB^3. Therefore $EF^3/AB^3 = CD^3/GH^3$, by the preceding [proposition]. And CD^3/GH^3 = cyl. *ZCKD*/cyl. *XAIB*. And since *CD* > *EF*, therefore by V.8, V.12 and V.14 [of Euclid], *GH* > *AB*. From this follows the same impossibility,[4] namely that the ratio of the first cylinder to the second cylinder is as the ratio of the column [whose base diameter] is the diameter of the first base to the column [whose base diameter is] a line greater than the diameter of the second base. Therefore line *EF* $\nless$ line *CD*. But neither was it more; therefore it is equal. Q.E.D.

Proposition XXXV.

SIMILAR CONES ARE PROPORTIONAL TO THE CUBES OF THEIR BASE DIAMETERS.

The demonstration is entirely the same [as that of the preceding proposition] if instead of [prismatic] columns [included in cylinders] you include pyramids in cones and you use the same arguments.

Prop. XXXIV

[1] See Com., Prop. XXXIV, lines 2–32.
[2] See Com., Prop. VIII, line 10.
[3] See Com., Prop. XXXIV, lines 13–14.
[4] *Ibid.*, lines 28–30.

Proposition XXXVI.

SIMILAR TRUNCATED CONES ARE PROPORTIONAL TO THE CUBES OF THEIR CORRESPONDING BASE DIAMETERS.

For similar truncated cones are segments of similarly constructed cones. But by the preceding [proposition] the whole cones are proportional to the cubes of their base diameters and the cones cut off [at the top] become proportional to the cubes of the diameters of their cutting [circles]. But this set of cubes is proportional to the other, since the diameters of the bases of the one set are proportional to the diameters of the bases of the other set because of the similarity of the segments. Therefore the cones cut off [at the top] are proportional to the whole cones. Therefore V.19 [of Euclid], the remaining segments, which are the truncated cones, are as the cut-off cones and as the whole cones, and therefore are proportional to the cubes of their corresponding bases. Q.E.D.

Proposition XXXVII.

SIMILAR SOLIDS OF ROTATION, I.E. THOSE OF SIMILAR PLANE SEMIPOLYGONS AROUND THEIR FIXED DIAMETERS, ARE PROPORTIONAL TO THE CUBES OF THESE DIAMETERS.

For solids of this sort are made up of the same number of similar cones, among which the pairs at the ends of the diameters are complete cones, while the others are truncated cones, except that sometimes [i.e. when the number of sides of the semipolygons is an odd number] there is a pair of cylinders. Now by [Propositions] XXXIV, XXXV and XXXVI above, such cones are proportional to the cubes of their corresponding base diameters (when we compare the corresponding cones). But these cubes are proportional to the cubes of the diameters around which the [similar] semipolygons are rotated, inasmuch as the diameters are proportional to the diameters because of the similarity of the figures. Therefore by V.13 of the *Elements*[1] the aggregates of all the cones, i.e., the whole solids of rotation, will be proportional to the cubes of their diameters. Q.E.D.

Proposition XXXVIII.

SPHERES ARE PROPORTIONAL TO THE CUBES OF THEIR DIAMETERS.[1]

Let *AIB* and *CKD* be two spheres with diameters *AB* and *CD* [see Fig. III.5A.28]. By [Proposition] XXXIII of this [work] sphere *AIB*/sphere *CKD* = AB^3/EF^3, where *EF* is some line.

It will have to be demonstrated that line *EF* = line *CD*.

Prop. XXXVII
[1] See Com., Prop. XXXVII, line 14.
Prop. XXXVIII
[1] See Com., Prop. XXXVIII, lines 2–27.

For otherwise it will be greater or less. If $EF > CD$, let there be imagined sphere *ELF* concentric with sphere *CKD*. And by XII.13 [of Euclid] let a solid of rotation *ENF* [described by a polygon] be inscribed in sphere *ELF* so that it does not touch sphere *CKD* at all. And let a similar solid *AMB* be inscribed in sphere *AIB*. By the preceding [proposition] AB^3/EF^3 = solid *AMB*/solid *ENF*. Hence solid *AMB*/solid *ENF* = sphere *AIB*/sphere *CKD*. And, alternately, solid *AMB*/sphere *AIB* = solid *ENF*/sphere *CKD*. But sphere *AIB* > solid *AMB* by the fifth postulate. Therefore sphere *CKD* > solid *ENF*, which is impossible by the said postulate. Therefore line $EF \not> $ diameter *CD*.

But if it is less, then, inversely, sphere *CKD*/sphere $AIB = EF^3/AB^3$. Hence by [Proposition] XXXIII of this [work] $EF^3/AB^3 = CD^3/GH^3$, and CD^3/GH^3 = sphere *CKD*/sphere *AIB*. And since $CD > EF$, by what has been premised and by V.14 [of Euclid] $GH > AB$. Whence the same impossibility follows, namely that the ratio of the first sphere to the second sphere is the same as the ratio of the first inscribed solid to a solid [described around] a line greater than the second diameter. Therefore line $EF \not< $ diameter *CD*. But neither was it greater; therefore it is equal. Q.E.D.

Proposition XXXIX.

SIMILAR SPHERICAL WEDGES [I.E. SECTORS] ARE PROPORTIONAL TO THE CUBES OF THEIR DIAMETERS.

I understand by a spherical wedge [i.e. sector] a body with a conical [lateral] surface having its apex in the center of the sphere and contained [at its base] by an assumed spherical surface. It is indeed composed of a cone and a spherical segment that have as their common base the circle which cuts the sphere into two unequal parts. This is demonstrated in a way not different from that of the preceding [proposition], with this exception, that instead of the total spheres similar spherical wedges are taken; and instead of [complete] solids of rotation similar wedges of rotation are taken. Now a solid wedge of rotation is composed of the aforesaid cone of the spherical wedge and the segment of the solid of revolution that falls within the spherical segment. For solid wedges of rotation of this sort are similar, inasmuch as they fall within similar spherical wedges. And therefore by the next to the last proposition [above] they are proportional to the cubes of the diameters of the spheres and also to the cubes of the diameters of the said bases since these latter diameters are proportional to the diameters of the spheres because of the similarity of the segments.

Proposition XL.

SIMILAR SPHERICAL SEGMENTS ARE PROPORTIONAL TO THE CUBES OF THEIR DIAMETERS.

For since the spherical segment (assigned by definition in the preceding [proposition]) is that which is left after cutting off the cone of the spherical wedge [or sector], and by the preceding [proposition] such wedges, or by [Proposition] XXXV such cones (for they are similar), are proportional to the cubes of the diameters of their bases and also to the cubes of the diameters of the spheres; that is, since the whole [sectors] are proportional to the [cones] cut off, hence the parts which remain will be proportional to the wholes, i.e., the spherical segments will be proportional to the cubes of the said diameters. Q.E.D.

Therefore, since similar spherical segments less than hemispheres are proportional to the cubes of the diameters, so also will the remaining spherical segments greater than hemispheres be proportional to the same cubes.

Proposition XLI.

THERE EXISTS SOME SPHERE EQUAL TO ANY GIVEN SOLID.[1]

Let *A* be any given solid [see Fig. III.5A.29]; and let any line *BD* be drawn, on which as a diameter let sphere *BED* be imagined. Therefore, either sphere *BED* will be equal to solid *A* or not at all. If it is equal, the proposition is evident. But if not, the sphere *BED* is to solid *A* as the cube of line *BD* is to the cube of [some] line *CF*, by [Proposition] XXXIII of this [work]. By [Proposition] XXXVIII of this [work], BD^3/CF^3 = sphere *BED*/sphere *CGF*; and so, since sphere *BED* has the same ratio to solid *A* as it has to sphere *CGF*, therefore sphere *CGF* = solid *A*. Again, therefore, the proposition is evident.

Proposition XLII.

WITH ANY TWO SOLIDS GIVEN, THERE EXISTS SOME SOLID SIMILAR TO ONE OF THEM AND EQUAL TO THE OTHER.[1]

For example, let *A* be any solid and let *BED* be a cone on a base whose diameter is *BD* [see Fig. III.5A.30].

Now I shall demonstrate that there is a cone similar to cone *BED* and equal to given solid *A*.

For by [Proposition] XXXIII of this [work], cone *BED* is to solid *A* as the cube of line *BD* is the cube of some line *CF*; and on circle *CF* let cone *CGF* similar to cone *BEF* be erected. Then by [Proposition] XXXV of this [work], cone *BED*/cone *CGF* = BD^3/CF^3; and therefore cone *BED*/cone *CGF* = cone *BED*/solid *A*. Hence since cone *BED* has the same ratio to solid *A* as to cone *CGF*, cone *CGF* = solid *A*. Therefore the proposition is evident.

But if on the base whose diameter is *BD* a cylinder is supposed, then

Prop. XLI

[1] See Com., Prop. XLI, lines 2–3.

Prop. XLII

[1] See Com., Prop. XLII, lines 2–3.

by [Propositions] XXXIII and XXXIV of this [work] we shall demonstrate in the same way [as above] that cylinder *GCF* is similar to cylinder *EBD* and equal to solid *A*. We shall be able to effect this same thing for truncated cones by [Propositions] XXXIII and XXXVI, for solids of rotation by [Proposition] XXXVII, for spherical wedges by [Proposition] XXXIX, for spherical segments by [Proposition] XL. Nor will the proposition be evident in any other way for cubes, or prisms, or pyramids, or parallelepipeds, or regular polyhedra, or irregular solids, inasmuch as it has been demonstrated in Book XII of the *Elements* that similar solids are always proportional to the cubes of their corresponding sides.

Proposition XLIII.

A CIRCLE IS EQUAL TO THE PRODUCT OF ITS RADIUS AND ITS SEMICIRCUMFERENCE.[1]

Let *AGH* be a circle of radius *AB* [see Fig. III.5A.31] and let *C* be the surface measured by the product of radius *AB* and the semicircumference of circle *AHG*.

It is to be demonstrated that circle *AHG* = surface *C*.

For by [Proposition] XIV of this [work] there will exist some circle equal to surface *C*. Therefore let circle *DIK* with radius *BD* be equal to surface *C*, and it will have to be shown that line $BD = BA$. For otherwise either one of the two will be greater. And let circle *DIK* be described concentric with circle *AGH*, and by XII.12 of Euclid within the greater of these circles let there be described a polygon that does not touch the lesser circle at all. Then by VI.1 of Euclid the surface of such an inscribed figure is equal to the product of *BE* (which is some line between *AB* and *BD* in length) and *ELM* (which is the semiperimeter of the figure). Let this product be *F*. And so if $AB > BD$, then $BE <$ line *AB* and hence surface $C >$ surface *F*. Therefore circle *DIK* (which was posited as equal to figure *C*) is greater than figure *F* or than *ELM*, which is impossible by the third postulate.

But if $AB < BD$, then $BE >$ line *AB*. Hence surface $F >$ surface *C*, and therefore figure *F* or *ELM* will be greater than space *C* or circle *DIK* within which it is inscribed, which is impossible by the said postulate. Therefore since line $BD \not< $ line *AB*, and line $BD \not>$ line *AB*, hence line $BD =$ line *AB*. Q.E.D.

Moreover we shall show in the same way that a cylindrical surface is equal to the product of an element of the cylindrical surface and the circumference of its base. This [is done] by means of a [prismatic] column inscribed in the greater cylinder so that it does not touch the lesser. And also [we shall demonstrate similarly] that the surface of a cone

Prop. XLIII

[1] See Com., Prop. XLIII, lines 2–25.

is equal to the product of an element of the conical surface [i.e., its slant height] and the semicircumference of the base. This [is done] by means of a pyramid inscribed in the greater cone so that it does not touch a lesser similar cone or a cone under the same axis.[2]

Proposition XLIV.

IN TWO UNEQUAL CIRCLES THE PRODUCT OF THE GREATER DIAMETER AND THE LESSER CIRCUMFERENCE IS EQUAL TO THE PRODUCT OF THE LESSER DIAMETER AND THE GREATER CIRCUMFERENCE. THE SAME THING IS TO BE SAID FOR THEIR RADII AND SEMICIRCUMFERENCES.[1]

Let *AGE* and *BHF* be two unequal circles with diameters *AE* and *BF* [see Fig. III.5A.32]. And let their circumferences be equal to lines *C* and *D*, i.e., by [Proposition] XV of this [work] let line *C* be equal to the circumference of circle *AGE* and line *D* to the circumference of circle *BHF*.

Therefore it is to be demonstrated that rectangle $AE \cdot D$ = rectangle $BF \cdot C$.

For since by [Proposition] VIII of this [work] $AE/BF = C/D$, so by VI.8 of the *Elements* of Euclid the first part of the proposition is immediately evident. But if *AI* and *BL* are posited as the radii, then by the same [Proposition] VIII [of this work] lines *C* and *D* will be the semicircumferences. Hence what remained to be demonstrated will be evident.

Proposition XLV.

IN THREE CIRCLES WHOSE DIAMETERS ARE IN CONTINUED PROPORTION THE PRODUCT OF THE DIAMETER OF THE FIRST AND THE CIRCUMFERENCE OF THE LAST IS EQUAL TO THE PRODUCT OF THE DIAMETER OF THE MIDDLE CIRCLE AND ITS CIRCUMFERENCE. THE SAME THING IS TO BE SAID FOR THEIR RADII AND SEMICIRCUMFERENCES.[1]

For by [Proposition] VI of this [work] let line *EI* be the mean proportional between diameters *AG* and *BL* [see Fig. III.5A.33]. Then by [Proposition] VIII of this [work] *F* the circumference of the circle with diameter *EI* will be the mean proportional between circumferences *C* and *D* inasmuch as the circumferences are proportional to the diameters.

And it will have to be demonstrated that rectangle $AG \cdot D$ = rectangle $EI \cdot F$ or that rectangle $BL \cdot C$ = rectangle $EI \cdot F$ (for rectangle $AG \cdot D$ = rectangle $BL \cdot C$ by the preceding [proposition]).

For since $AG/EI = F/D$, by VI.15 of Euclid the first part of the

[2] *Ibid.*, lines 26–31.

Prop. XLIV

[1] Cf. Text C, Prop. VI, Cor. III.

Prop. XLV

[1] Cf. Text C, Prop. VI, Cor. IV.

proposition is evident. And the remainder [of the proposition follows] in the same way. For if lines *AN*, *EO* and *BR* are posited as radii, then lines *C*, *F* and *D* will be the semicircumferences by [Proposition] VIII of this [work]. Therefore the proposition is evident.

Proposition XLVI.

IN THREE CIRCLES WHOSE RADII ARE CONTINUALLY PROPORTIONAL, THE PRODUCT OF THE RADIUS OF THE FIRST AND THE SEMICIRCUMFERENCE OF THE LAST IS EQUAL TO THE MIDDLE CIRCLE. THE SAME THING IS TO BE SAID IF THESE LINES IN THE FIRST AND THE LAST ARE DESIGNATED AS WHOLE DIAMETERS AND WHOLE CIRCUMFERENCES.[1]

For since by the preceding [proposition], and following the same figure, each of the rectangles $AN \cdot D$ and $BR \cdot C$ is equal to the rectangle $EO \cdot F$, and that last rectangle is equal to the product of the radius *EO* and its semicircumference *F*, so by [Proposition] XLIII above that product is equal to each of the rectangles $AN \cdot D$ and $BR \cdot C$, each of which is the product of the radius of one of the extreme [circles] and the semicircumference of the other, or even of the whole diameter and the whole circumference by supposition. Therefore, that which is proposed for demonstration is evident.

Completed at Termini in the first hour of the night of Thursday beginning Lent, which was the thirteenth of the month of February of the eighth indiction, 1550.

Commentary to Text A

Postulates

1 "Postulata." These postulates play an important role in the treatise. The first one, which asserts that the ratio of any magnitudes of the same kind can be represented by some ratio of two [straight] lines, is the assumption that makes Book V of Euclid's *Elements* such a powerful tool for the manipulation of homogeneous magnitudes. Campanus stresses this in his comment to Book V, Def. 3 (*Elementa*, Basel, 1546, p. 104): "Sed quaecunque proportio reperitur in uno genere continuorum, eadem reperitur in omnibus aliis. Nam qualitercunque se habet aliqua linea ad quamlibet aliam: sic se habet quaelibet superficies ad aliquam aliam, et quodlibet corpus ad aliquod aliud, similiter et tempus." Postulates 2–5, the "inclusion postulates," assume that a circumscribing or including line, surface or solid is greater than any line, surface or solid

Prop. XLVI

[1] Cf. Text C, Prop. VI, Cor. IV. See also Com., Prop. XLVI, lines 5–7.

it circumscribes or includes. Postulate 6 is important for Propositions XXII, XXIII and XXV. It resembles in form Definition 5 of Book V of Euclid's *Elements*. Where the latter limits itself to a definition of the equality of ratios, Maurolico's postulate embraces the equality of any two comparable magnitudes. The postulate was constructed to allow the possibility of equality between rectiplanar and curved surfaces where ordinary tests of congruence or rectilinear transformation cannot apply. This postulate in effect asserts that magnitude *A* is said to be equal to *B* if it can be shown to occupy the same relative position as *B* in a whole, compact infinite set of continually increasing magnitudes of the same kind, i.e. to say, *A* and *B* are equal to each other if each is the limit of the same sets of greater and lesser magnitudes. Hence, if *A* and *B* are always simultaneously compared to any magnitude *C* in its infinite set they will on any such simultaneous comparison be at the same time greater or at the same time less than magnitude *C* to which they are each compared. Alexander of Aphrodisias, according to Philoponus, used such a postulate of equality for explicating Bryson's method of quadrature (see Volume One, p. 427).

Proposition I

2–3 "circumscripta" Perhaps this would be less confusing if he had said *inscripta* here (and also in Postulates 2, 3 and 5) instead of *circumscripta*, as in fact he did in Postulate 4 and Proposition II. In any case what is intended is the figure that has been "circumscribed" by the circumscribing figure, i.e., it is the "inscribed" figure itself. Incidentally, we may call the first four propositions the "inclusion" propositions just as we called Postulates 2–5 the "inclusion postulates." Their purpose is to show that a circumscribing or including cylinder (or cone) is greater than an inscribed or included prismatic column (or pyramid) and that a circumscribing or including prismatic column (or pyramid) is greater than an inscribed or included cylinder (or cone). We shall see later that the kind of included figure often intended is one that is less than an inscribed figure and is in fact one that the inscribed figure does not touch.

Proposition V

2–3 "Esse . . . rationem." This is the first of three important existence theorems. See the similar theorems for a square (Proposition VII) and for a cube (Proposition XXXIII).

7 "per decimam sexti" The appeal is to the *additio of* Campanus (see *Elementa*, Basel, 1546, p. 146) where Campanus describes the finding of a fourth proportional. This is a separate proposition (VI.12) in the Greek text and Zamberti's translation (*ibid.*, p. 147). Hence in his use of the number VI.10 we have the

first of a whole host of citations that indicate that Maurolico was using Campanus' version of the *Elements*. I shall note in these commentaries only those propositions where the citation is clearly from the version of Campanus. Where the citation is identical in the Greek text and the version of Campanus, I offer no comment.

Proposition VI

6 "per nonam sexti" Again it is Campanus' version that is being followed instead of the Greek text or Zamberti's translation where the appropriate proposition is VI.13 (see *Elementa*, Basel, 1546, pp. 145, 147).

Proposition VIII

2–3 "Circulorum . . . proportionales." Note the similar proposition with different enunciation and proof in the *Verba filiorum* (Volume One, pp. 260–64). Maurolico also proved this theorem twice in his *On the Sphere and the Cylinder* (Text C, Proposition VI), using his "easier way." But here in the *Praeparatio* Maurolico has given his first example of the "converse way" and it can be summarized as follows.

Let us suppose unequal circles with diameters d and d' and circumferences c and c'. Then, by Prop. V, let $c / c' = d / e$, where e is some line. Hence to prove that $c / c' = d / d'$. we must prove that $e = d'$. If $e \neq d'$, then $e > d'$ or $e < d'$.
I. Suppose, if possible that $e > d'$. Then let a circle with diameter e be circumscribed about and be concentric with the circle with diameter d'. By XII.13 of Euclid (=Gr XII.16) inscribe within a circle of diameter e a regular polygon of perimeter p' that does not touch c' and within circumference c a similar polygon of perimeter p. Hence [by XII.1 of Euclid] $d / e = p / p'$. And so, since $c / c' = d / e$ by hypothesis, $p / p' = c / c'$. And, alternately, $p / c = p' / c'$. But, by Post. 2, $c > p$. Hence $c' > p'$, i.e., the included > the including, which is against Post. 2. Therefore, $e \ngtr d'$.
II. Suppose, if possible, $e < d'$. Then invert the proportion of the initial supposition, so that $c' / c = e / d$. By Prop. V, let $e / d = d' / h$. Hence $c' / c = d' / h$. [And since, alternately $d' / e = h / d$] and $d' > e$, hence $h > d$. Therefore, with $c' / c = d' / h$, we have the same kind of impossibility disproved in the first part, namely that a first circumference (here c') is to a second (here c) as the first diameter (here d') to a line (h) greater than the second diameter (here d). Therefore, $e \nless d'$. But it was proved that $e \ngtr d'$. Hence $e = d'$ and thus $c / c' = d / d'$. Q.E.D.

10 "per decimamtertiam duodecimi" Again the citation is to the Campanus version, for in the Greek text and the Zamberti translation the proposition appears as XII.16 (*Elementa*, Basel, 1546, pp. 413–14).

Proposition X

1–25 "Circuli. . . . demonstrandum." This is Maurolico's second example of the use of his "converse way." The proof is formally identical with that of Proposition VIII and need not be summarized here. This proposition was proved in a different way, employing the more usual form of the exhaustion method, in Euclid's *Elements*, Prop. XII.2. A proof using the "easier way" was suggested by A.-M. Legendre (see below, Com. Text C, Prop. VI).

21–23 "idem . . . diametro" Here in the second part of the proof circle *CKD* becomes the first circle, circle *AIB* the second, *CD* the first diameter, *AB* the second diameter and *GH* the line greater than the second diameter. And so the impossibility is that refuted in the first part of the proof.

Proposition XII

2–11 "Cylindrorum. . . . servetur." Maurolico notes that the proof of this proposition will be the same as that of Proposition X (i.e. by what I have called the "converse way") except that the including figure is a cylinder instead of a circle and the included figure is a prismatic figure based on a regular polygon instead of a regular polygon and hence Proposition I instead of Postulate 3 is the inclusion assumption controverted.

Proposition XIII

2–10 "Conorum. . . . syllogismus." As in the preceding proposition, Maurolico declares that the proof will be the same as that of Proposition X, except that this time the including figure is a cone, the included figure a pyramid and the inclusion assumption controverted Proposition II.

Proposition XIV

2–3 "Datae . . . aequalem." This proposition is crucial for the only use of the "easier way" in the *Praeparatio*, namely that in the proof of Proposition XLIII.

Proposition XV

17–18 "Vocat . . . plana." This editorial comment added by Borelli or perhaps Cyllenius of course refers not to Proposition XV but to the term *conus-colurus* (or *colurus-conus*) used in Proposition XVI. It was used in Greek mathematical literature, e.g. in Nicomachus, *Intro. Arith.*, ed. R. Hoche (Leipzig, 1866), XIV, 5, p. 104, line 1, and in Eutocius' *Comm. in libros de sphaera et cylindro* (Archimedes, *Opera omnia*, Vol. 3, p. 42, line 27 and p. 44, line 4). And it appears in Latin in the translations of Eutocius by Moerbeke (Vol. 2, 35vY) and Cremonensis (ed. 1544, p. 11), and also in Valla's *De expetendis et fugiendis rebus* (Venice, 1501), sign. z iiii recto, where no doubt Maurolico first encountered it. Note also other citations to Hero and Pappus in C. Mugler, *Dictionnaire historique de la terminologie géométrique des Grecs* (Paris, 1958), p. 255. Maurolico was already using the expression *conus-colurus* in 1534 (see below, Text C, Prop. VII, line 2).

Proposition XVII

9 "binae cylindricae" In Archimedes' *On the Sphere and the Cylinder*, the only solid of rotation was that described by a regular polygon whose number of sides was divisible by four and where therefore the number of sides of the semipolygon was an even number. But in the *Liber de curvis superficiebus*, semipolygons of an odd number of sides were also used (see Vol. 1, p. 476) and they were consequently used by Maurolico in his version of the *De sphaera et cylindro* (see below, Text C, Prop. VIII). Needless to say, it is only where the semipolygon rotated has an odd number of sides that the middle side describes a cylindrical surface.

Proposition XVIII

2–33 "Sphaerarum. . . . demonstrandum." Once more the "converse way," already used in the proofs of Propositions VIII, X, XII and XIII, is here applied in almost exactly the same fashion.

28–31 "idem . . . diametro" In the second half of the proof the surface of the first sphere is that of *CKD*, the second that of *AIB*; the first diameter is accordingly *CD*, the second *AB*, and the line greater than the second diameter *GH*. And so the impossibility is that refuted in the first part of the proof.

Proposition XIX

2–12 "Similium. . . . proportionales." Maurolico here indicates that the same procedure as in Proposition XVIII (i.e., the "converse way") is to be used here, except in this proof the surfaces of the spherical segments are to be taken instead of the surfaces of the spheres and the surfaces of the segments of the solids of rotation inscribed in the segments of the spheres are to be taken instead of the whole surfaces of those solids.

Proposition XX

2–3 "Cuilibet . . . aequalem." This proposition justifies the procedure of Proposition X of Maurolico's *On the Sphere and the Cylinder* (see Text C), for it allows that if the rectangle composed of the diameter and the circumference of a great circle of the sphere is not equal to the surface of the given sphere, it is equal to that of some sphere and hence to the surface of one either greater or less than the given sphere.

8 "[per septimam huius]." This phrase has been bracketed as superfluous in view of the appearance of the same phrase in line 7.

14–15 "Adverte . . . CF." This is an editorial comment of either Borelli or Cyllenius. It seems to confirm that there was only one figure in the manuscript to serve for both Propositions XX and XXI and that was a figure where the segments on *BD* and *CF* were less than hemispheres. Hence the editor is simply pointing out that when the figure is used for Proposition XX, *BD* and *CF* ought to be diameters of spherical figures.

Proposition XXI

2–4 "Datis . . . aequalem." This proposition provides a kind of general justification for the hypotheses used in the "easier way" of the *Liber de curvis superficiebus* and of Maurolico's *On the Sphere and the Cylinder* (see particularly Propositions II, IV and XV in Text C below), since it allows us to posit that the given rectilinear figures of these propositions can be equal to some curved surfaces of the kinds specified in the propositions (i.e. cone and cylinders in Propositions II and IV and a spherical segment in Proposition XV). Then the succeeding propositions of the *Praeparatio* (i.e. Propositions XXII, XXIII and XXV) more specifically allow that they be equal to some particular curved surfaces of the given kind (e.g. cones and cylinders of a given axis in Propositions II and IV and a spherical segment erected on the same plane but concentric with the given segment in Prop. XV). But in fact without the three succeeding propositions that are necessary for the actual proofs of Propositions II, IV and XV of Maurolico's *On the Sphere and the Cylinder*, one could construct similar proofs on the basis of this Proposition XX alone by holding, for example in the case of the proof of Proposition II, that if the given rectilinear surface is not equal to the given conical surface, it must be equal to the surface of a similar cone that is greater or less than the given cone rather than to the surface or cone of the same axis but of greater or lesser base circumference. Indeed such an alternative proof is suggested at the end of the last paragraph of the proof of Proposition XLIII of the *Praeparatio*.

Proposition XXII

2–3 "Cuilibet . . . axem." As I have noted before, this proposition gives specific justification for the procedure of the proof of Proposition IV of Maurolico's *De sphaera et cylindro* (and thus of its equivalent Proposition II of the *Liber de curvis superficiebus*), which commences by asserting that if the rectangle composed of the axis of a cylinder and the circumference of its base is not equal to the lateral surface of the given cylinder, then it is equal to the lateral surface of some other cylinder of the same axis but of greater or lesser base circumference.

24 "ultimum postulatum" Consult the commentary to the postulates for the significance of this last postulate.

Proposition XXIII

2–3 "Cuilibet . . . axem." This proposition is crucial for the basic procedure of the proof of Proposition II of Maurolico's *De sphaera et cylindro* (and thus for its equivalent Proposition I of the *Liber de curvis superficiebus*), which begins by asserting that if the right triangle with sides about the right angle equal respectively to the slant height of a cone and the circumference of its base is not equal to the lateral surface of the given cone, then it is equal to some conical surface of a cone of the same axis but of greater or lesser base circumference.

Proposition XXV

2–4 "Cuilibet . . . superficiem." This proposition is essential for the basic procedure of the proof of Proposition XV of Maurolico's *De sphaera et cylindro*, which begins by asserting that if the product of the diameter of a sphere and the circumference of the circle whose diameter is equal to the axis of the segment of the sphere is not equal to the surface of the spherical segment, it is equal to the surface of a greater or lesser segment constructed on the same plane and concentric with the given spherical segment.

Proposition XXVI

21 "per 15^{am} sexti" Again the citation is to Campanus' version of the *Elements* rather than to the Greek text or Zamberti's translation where the number of the proposition is VI.16 (see *Elementa*, Basel, 1546, pp. 149–50).

27 "modus Heronis" Maurolico's comment perhaps implies that he drew Hero's method from the *Mechanica*, while drawing the remaining solutions of the proportional means problem (Propositions XXVII–XXXII) from Eutocius. However, Maurolico could scarcely have taken it from the *Mechanica*, I.11 (ed. of L. Nix, pp. 24–26) since the Greek text of that work had disappeared. It is not impossible that he read it in the *Belopoiika* of Hero (ed. of H. Diels and E. Schramm, pp. 52–55) since that work was available in manuscript. But if so, it is strange that Maurolico made no mention of that work. One can argue in the same way against his having seen the same proof in Pappus' *Collectio*, Book III, Sect. IX (ed. Hultsch, Vol. 1, pp. 62–65). All in all, I am reasonably certain that he took this solution from its Eutocian account where it is labeled as being from Hero's *Mechanics*. By the time he composed the *Praeparatio*, he no doubt had read the Greek text and the translations of Valla and Cremonensis of the pertinent section of Eutocius' *Commentary*. Maurolico's diagram follows the Latin lettering adopted by both Valla and Cremonensis, except that he has transposed *D* and *B*, has *K* instead of *H* and has added letter *H* where no letter appeared in the translations. Maurolico's reference to the title of

Hero's work seems compounded of the reference found in Valla's version (*De expetendis*, sign. [u vi verso]: "Ex Herone in mechanicis traditionibus") and that found in Cremonensis' (ed. Basel, 1544, p. 15: "Modus Heronis in mechanicis introductionibus et in telis fabricandis").

Proposition XXVII

7 "per 35^{am} tertii" Again Maurolico cites the version of Campanus and in fact Campanus' *additio* to the proof (*Elementa*, Basel, 1546, p. 83): "Et ex hac nota, quod puncto extra circulum signato, si ab ipso ad circulum quodlibet secantes lineae ducantur, rectangula quae continentur sub totis et earum portionibus extrinsecis adinvicem sunt aequalia." The proper proposition in the Greek text and Zamberti's translation is III.36, but no such addition as that of Campanus is found there (*ibid.*, p. 84).

15–16 "ut . . . Alexandrinus" Cf. above, Part III, Chap. 3, notes 75 and 76. There is little doubt that Maurolico's information on Philoponus' account comes from Valla's translation of Philoponus' discussion (given in note 76) rather than from the Greek text itself (cited in note 75). Maurolico mentioned Philoponus' discussion earlier in brief discussions of the mean proportionals problem written in 1533 and 1535 (see below, Chap. 6, Sect. III, n. 7).

Proposition XXIX

18–22 "Fuit . . . quaestio." This comment about Plato and the Delians was drawn from the earlier section of the account in Philoponus' *Commentary on the Posterior Analytics* (ed. of Wallies, p. 102) noted above in the comment to Proposition XXVII. Valla in his *De expetendis et fugiendis rebus opus*, sign. x ii recto, says:

> Ut Philoponus. Quo pacto duos cubos unum possis cubum facere inventum est, quod deliis (nota siquidem est historia) pestilenti lue passim laborantibus, ac pereuntibus Apollo consultus responderit eam luem sedari posse si aram duplicassent. Erat autem ea cubus hi alterum alteri aequalem cubum sibi capiendo arae imposuerunt. sed adhuc crudescente grassanteque pestilentia, respondit Apollo eos quod fuerat imparatum non fecisse, quod mandasset aram duplicandam, eos autem cubum cubo superimposuisse. Platonem adierunt consulendo, quonam pacto cubus foret duplicandus qui respondit videri sibi numen eos incessere, quod geometriam ignorarent. Cubi vero duplicationem, tum demum posse inveniri, cum binae mediae lineae proportionales essent inventae et continuo suis hanc quaestionem proposuit indagandam discipulis.

However, Maurolico's account of the so-called Platonic method itself was drawn from the section on the problem of proportional means composed by Eutocius, as he had confessed at the end of Proposition XXVI.

Proposition XXXI

13–14 "per 12^am secundi Conicorum Elementorum" Proposition II.12 in Maurolico's version of Apollonius' *Conics* (*Emendatio et restitutio Conicorum Apollonii Pergaei*, Messina, 1654, p. 58) runs: "XII. Si penes non tangentes a quodam puncto in sectione duae lineae ad ipsas non tangentes ducantur, et rursus ab alio puncto in sectione his aequidistantes similiter non tangentibus incidant: rectangulum sub his contentum aequale erit rectangulo sub illis contento."

16 "per 20^am primi Conicorum Elementorum" Propositio I.20 in Maurolico's version of Apollonius (*ed. cit.*, p. 19) runs: "XX. Si in Parabola a sectione ducantur duae lineae ad diametrum ordinate: erunt ut quadrata quae ab ipsis fiunt adinvicem, sic secatae (*!*) sub ipsis ex diametro ad summitatem." (*Punctuation slightly altered.*)

22–30 "Et haec. . . . docet." The proof described to this point was essentially drawn from the synthesis of Menaechmus' first proof. However, Maurolico changed it in one respect. He did not specify *AB* as the *latus rectum*. Hence he added the procedure of interposing a mean *AL* between *AB* and *AT*. Now in this paragraph he notes that the interposition of *AL* as a mean is not necessary if we make *DB* = *AB* and construct the parabola through points *A* and *D* with *AG* the axis. This is the same as saying that *AB* is the *latus rectum* and indeed the proof of Menechmus had so specified this (see above, Vol. 2, 37vL).

Proposition XXXII

4–5 "per secundam quarti" Surely IV.1 is the proper citation, for that proposition in all of the versions of Euclid shows how to fit in a circle a straight line equal to a given line which is not greater than the diameter of the circle, and *BG* is such a line.

31 "per octavam sexti" That is, by the corollary to VI.8.

32 "adducta 34^a tertii" Again Campanus' version of the *Elements* is being cited; the proposition is numbered III.35 in the Greek text and the translation of Zamberti (see the *Elementa*, Basel, 1546, pp. 81–82).

40 "inventio Archytae Tarentini" See my discussion of this method in Vol. 1, pp. 365–66.

Proposition XXXIII

2–3 "Esse . . . rationem." Since this proposition is immediately necessary for the proofs of Propositions XXXIV, XXXV, XXXVIII, XXXIX, XLI and XLII below (and thus indirectly necessary for Propositions XXXVI, XXXVII, and XXXIX) and since its proof requires the solution of the proportional means problem we can now see why Maurolico included so many solutions to that problem.

7 "sex (! septem)" There are seven propositions that give solutions to the problem of proportional means. But perhaps when Maurolico used "six" here, he meant to identify the first two solutions (i.e., Propositions XXVI and XXVII) as a single method.

8 "per 36^{am} undecimi" This is XI.36 in the Campanus version and XI.33 in the Greek text and Zamberti's translation (see the *Elementa*, Basel 1546, pp. 377–78).

Proposition XXXIV

2–32 "Similes. . . . demonstrandum." Maurolico's proof is again by the "converse way" already described in Section III of the chapter and in the commentary to Proposition VIII.

13–14 "per octavam 12^{i} Euclidis" Campanus' last *additio* to Proposition XII.8 (*Elementa*, Basel, 1546, p. 401) asserts: "Omnium duarum columnarum lateratarum similium est proportio alterius ad alteram, tanquam lateris ad suum relativum latus proportio triplicata." One simply has to add that the ratios of the corresponding sides of the similar prismatic columns are proportional to the diameters of the bases. Therefore, AB^3 / EF^3 = col. *XAMB* / col. *QENF*, as Maurolico states.

28–30 "Unde . . . basis." In the second half of the proof the first cylinder is now *ZCKD*, the second *XAIB*; the diameter of the first base is *CD*, that of the second is *AB*, and the line greater than the second diameter is *GH*. And so the impossibility is that refuted in the first part of the proof.

Proposition XXXVII

14 "per 13^{am} quinti" The citation is once more to Campanus' version of the *Elements*; it is Proposition V.12 in the Greek text and in Zamberti's translation (see *Elementa*, Basel, 1546, pp. 123–24).

Proposition XXXVIII

2–27 "Sphaerae. . . . demonstrandum." I have summarized this proof above in Section III of this chapter and given it as the principal example of the "converse way."

Proposition XLI

2–3 "Cuilibet . . . aequalem." This proposition justifies the basic procedure of the proof of Proposition XXV of Maurolico's *De sphaera et cylindro* (see Text C below), which commences by asserting that if the cone there described does not equal the volume of the given sphere, it equals the volume of some sphere greater or less than the given sphere.

Proposition XLII

2–3 "Datis . . . aequale." With some adaptation this proposition will serve to justify in a general way the procedure of the proof of Proposition XXX of Maurolico's *De sphaera et cylindro* (see Text C below), which commences by asserting that if the specified cone does not equal the volume of the given spherical sector, it equals the volume of one greater or less than it.

Proposition XLIII

2–25 "Circulus. . . . demonstrandum." This proof reveals the only application of the "easier way" in the *Praeparatio*. It was included, I suspect, because although he had given the proposition itself in his *On the Measurement of the Circle* (see Text B, Prop. IV), the proof there was by Archimedes' method rather than by the "easier way." Hence he used this opportunity to give its proof by the same method he had used in his *De sphaera et cylindro* of 1534. Furthermore, in the form that the proposition is given here, it provides partial authority for Propositions III, V, XI, XXV and for Corollaries to Propositions VI, X and XXXVII of his *De sphaera et cylindro* (see Text C below). A similar but somewhat more elaborate use of the "easier way" to prove this proposition was made by A.-M. Legendre, which I quote in the English translation: *Elements of Geometry and Trigonometry*, tr. of E. D. Brewster (Edinburgh, 1824), BK. IV, Prop. XII, pp. 96–97:

Proposition XII. Theorem.

The area of a circle is equal to the product of its circumference by half the radius.

Let us designate the surface of the circle whose radius is *CA* [see Fig. III.5A.34] by surf. *CA*; we shall have surf. $CA = \frac{1}{2}CA \times$ circ. *CA*.

For if $\frac{1}{2}CA \cdot$circ. *CA* is not the area of the circle whose radius is *CA*, it must be the area of a circle either greater or less. Let us first suppose it to be the area of a greater circle; and, if possible, that $\frac{1}{2}CA \cdot$circ. $CA =$ surf. *CB*.

About the circle whose radius is *CA* describe a regular polygon *DEFG* &c., such (10.IV.) that its sides shall not meet the circumference whose radius is *CB*. The surface of this polygon will be equal (7.IV.) to its perimeter $DE + EF + FG +$ &c. multiplied by $\frac{1}{2}AC$: but the perimeter of the polygon is greater than the inscribed circumference enveloped by it on all sides; hence the surface of the polygon *DEFG* &c. is greater than $\frac{1}{2}AC \cdot$circ. *AC*, which by the supposition is the measure of the circle whose radius is *CB*; thus the polygon must be greater than that circle. But in reality it is less, being contained wholly within the circumference: hence it is impossible that $\frac{1}{2}CA \cdot$circ. *AC* can be greater than surf. *CA*; in other words, it is impossible that the circumference of a circle multiplied by half its radius can be the measure of a greater circle.

In the second place, we assert it to be equally impossible that this product can be the measure of a smaller circle. To avoid the trouble of changing our figure, let us suppose that the circle in question is the one whose radius is *CB*: we are to shew that ½*CB*·circ. *CB* cannot be the measure of a smaller circle, of the circle, for instance, whose radius is *CA*. Grant it to be so; and that, if possible, ½*CB*·circ. *CB* = surf. *CA*.

Having made the same construction as before, the surface of the polygon *DEFG*, &c. will be measured by (*DE* + *EF* + *FG* + &c.) ·½*CA*; but the perimeter *DE* + *EF* + *FG* + &c. is less than circ. *CB*, being enveloped by it on all sides; hence the area of the polygon is less than ½*CA*·circ. *CB*, and still more than ½*CB*·circ. *CB*. Now by the supposition, this last quantity is the measure of the circle whose radius is *CA*: hence the polygon must be less than the inscribed circle, which is absurd; hence it is impossible that the circumference of a circle multiplied by half its radius, can be the measure of a smaller circle.

Hence, finally, the circumference of a circle multiplied by half its radius is the measure of that circle itself.

I have changed all multiplication signs from "×" to a raised dot after the first one and I have changed his expressions *surf.* and *circ.* to roman type. Further, I have italicized all letters marking the magnitudes.

26–31 "Similiter. . . . tangat." Here he points to the application of the same method (i.e., the "easier way") to the cases of the lateral surfaces of cylinders and cones. He does not give the actual proofs since they were included in *De sphaera et cylindro* (see Text C, Propositions II and IV). But in a sense his remarks here provide not only for the proofs as found in his earlier work that apply to cones or cylinders of the same altitudes and of greater or lesser base circumferences but also for proofs that could be constructed by using similar cones or cylinders.

Proposition XLVI

5–7 "Si . . . appellentur." This cannot be literally so; that is, the products of the whole diameters and the whole circumferences cannot equal the middle circle [in area]. But perhaps all that he means is that these products are equal to the middle rectangle (which is in fact four times the area of the middle circle). Still if this is so, then the supplementary statement in nowise differs from Proposition XLIV.