

# Archimedes Measurement of the Circle: Proposition 1

February 6, 2009

I'm assuming that we have before us these versions<sup>1</sup>. (of Proposition 1):

- Heiberg's (Greek w Latin transl.)
- Heath's (English Non-literal transl.)
- Dijksterhuis's (Non-literal but-with-close-attention-paid-to-the-original formulation)
- Ver Eecke's (French literal transl.)
- Mark's (English literal transl.)

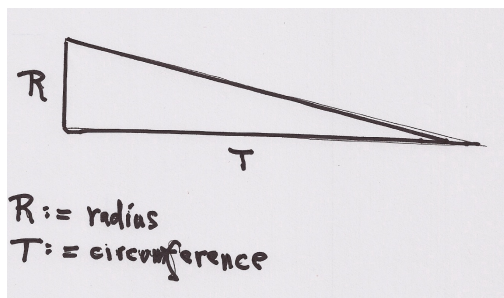
Proposition 1 is in the tradition of propositions relating the area of one figure to the area of another figure, but the important difference between this proposition and any of the ones we find in Euclid's *Elements* is that the circumference of the circle appears represented as a *linear* figure (the base of a triangle). This raises quite a number of questions, especially given the briefness of discussion in the Archimedes text.

Here is Mark's translation of the statement of the proposition:

**Proposition 1:** Every circle is equal to a right-angled triangle, whose radius  $[R]$  is equal to one of the [sides] around the right angle while the perimeter [i.e., circumference  $T$  of the circle] is equal to the base [of the triangle].

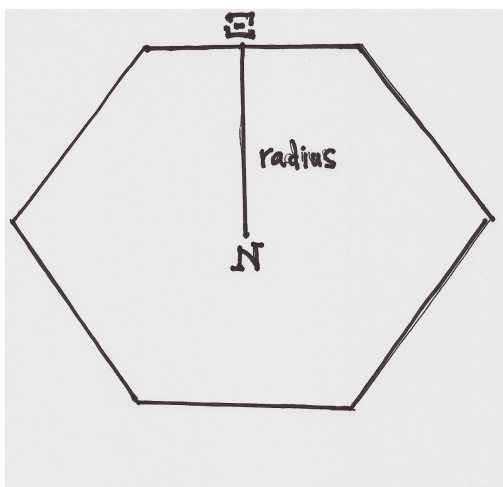
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<sup>1</sup>For a close study of the textual tradition of *Measurement of the Circle* see pages 375-594 of Wilbur Knorr's *Textual Studies in Ancient and Medieval Geometry* Birkhäuser, 1989. There Knorr traces Archimedes Theorem in Pappus' version and Theon's and their relationship to (a) the extant Greek text and (b) early Arabic translations, with some facsimiles and much discussion.



If we zoom into the architecture of Archimedes proof of this, it seems to me that there are six mathematical issues that the proof seems to call upon although they appear, if at all, very succinctly in our text.

1. **The Proposition for regular polygons.** Define the *radius* of a regular polygon to be the length of a line interval that is obtained by dropping a perpendicular to any side of the polygon from the center  $N$  of the regular polygon.



Define the *perimeter* (or *circumference*) of a polygon to be the length of its perimeter, i.e., the sum of the lengths of the sides of the polygon. If the polygon is a regular  $M$ -gon, then the circumference is  $M$  times the length of any side. Here is “my” version of Proposition 1 for regular polygons<sup>2</sup> analogous to Archimedes’ Proposition 1 circles:

The area subsumed by a regular polygon is equal to the area subsumed by a right-angled triangle for which the two right-angle sides are of lengths equal to the *radius* and the *circumference* (respectively) of the polygon.

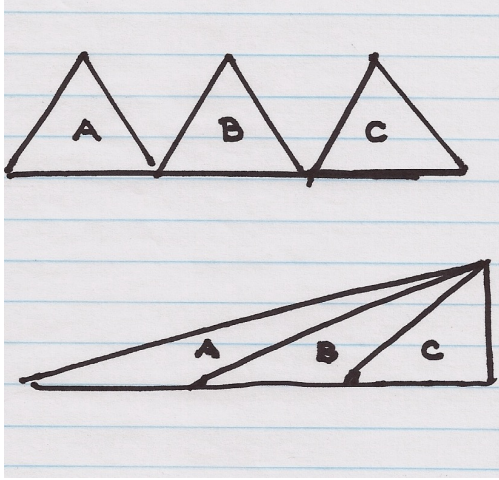
Now this “polygon-version” of the theorem is nicely within the scope of Euclidean vocabulary; its proof is within the scope of Euclid as well.

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<sup>2</sup>I say “my” version because, even though it is—in my opinion—implicitly invoked, in Archimedes’ text, it isn’t dwelt on.

**Comments:**

- (a) Both this “polygon-version” and Archimedes Proposition 1 deal with a *right-angle* triangle (with base the circumference and altitude (synonymously: *height*) the radius of the figure this triangle is being compared to). One could rephrase these propositions by omitting the requirement that the triangle be a right-angle triangle.
- (b) In the seminar discussion Jim Carlson suggested that a visual proof of this polygonal proposition can be effected by simply cutting and “straightening out to a line” the perimeter of the polygon, and then arguing that this paper-doll figure has the same area as the triangle displayed below. (In the figure below we illustrate this with a 3-gon, alias a triangle, which when cut-and straightened-out produce the three triangles in a line labeled A,B,C; these each have the same area as the three triangles that make up the large triangle below having as base the perimeter and as height the radius.) All this uses is that the area of a triangle depends *only* on its base and height<sup>3</sup>.



This proposition, or something essentially equivalent is clearly being called forth somewhere in the Archimedean text but it is difficult to pinpoint where. (Where exactly?)

2. **The construction of regular polygons ( $2^n$ -gons, in fact) inscribed and circumscribed about a circle.** This is quite elegantly done in Euclid’s Proposition 2 of Book XII, and Archimedes surely takes that as his cue. More germane than the mere construction, though, is the recursive form of the proposition, where one starts with a square (a “regular 4-gon”) and puffs it out to an octagon, and thence to a decahexagon, etc. showing that in each successive step, we are *at least halving the discrepancy between the areas of the circle and the inscribed polygon* in the sense that (putting it in modern terms) if

$$d_n = \text{area of the circle} - \text{area of the } 2^n - \text{gon},$$

is the discrepancy, i.e., the area of the part of the circle outside the constructed inscribed regular  $2^n$ -gon, then

$$d_{n+1} < \frac{1}{2}d_n.$$

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<sup>3</sup>All that we will ever need in this general discussion is that the area of a triangle is some fixed constant times the base times the height; that the “fixed constant” is  $\frac{1}{2}$  is evident, but irrelevant, except perhaps for paraphrases of Archimedes Prop 1 as given by Hero (*Metrica* I.26, I.37) as in Mark’s comments.

And ditto for  $D_n$ , the area of the circumscribed  $2^n$ -gon outside the circle.

**Note:** I will go through this slowly in my presentation.

### 3. The relationship between the Proposition for regular polygons and Archimedes

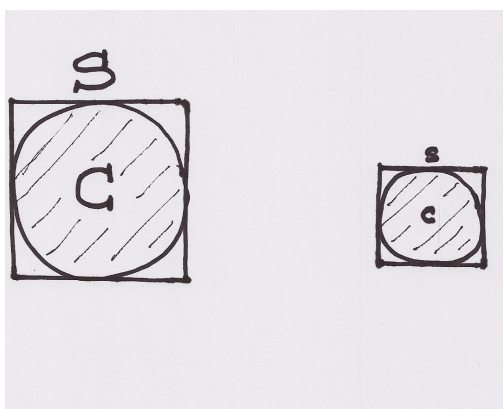
**Proposition 1.** One thing that is relevant is that the relationship between these two propositions is very analogous to the relationship between XII.1 and XII.2 (also the relationship between XII.7 and XII.10). Let's pictorialize this as

$$\frac{\text{Archimedes Prop. 1}}{\text{Regular polygon Prop.}} = \frac{XII.2}{XII.1}.$$

But this neat analogy hides some important difficulties that the “left-hand-side” must face and the “right-hand-side” need not face:

4. **The exhaustion:** Here Euclid's X.1 is the ur-model (successive halving of quantities gets you as small as you want) and it fits particularly snugly with the successive approximations described above, each (at least) halving the error of the former.

In general terms here is a sketch of the format for passing from XII.1 to XII.2 (recall the statements of these). Suppose that there is a discrepancy between the ratios of the areas of two circles,  $\frac{C}{c}$ , and ratios of the areas of the corresponding circumscribed squares,  $\frac{S}{s}$ .



Suppose, for example, that

$$\frac{S}{s} - \frac{C}{c} = e$$

with  $e > 0$ . Choose  $n$  sufficiently large so that the  $2^n$ -gons  $P_n$  and  $p_n$  inscribed in each circle have the property that the ratio of their areas  $\frac{P_n}{p_n}$  is closer to  $\frac{C}{c}$  than  $e$  (which we can do by the discussion during their construction and X.1, and—it seems to me—a bit of arithmetic that goes unmentioned). But by XII.1,  $\frac{S}{s} = \frac{P_n}{p_n}$  giving a counter-example. If  $e$  is negative do the analogous thing with circumscribed  $2^n$ -gons.

To deduce Archimedes Prop. 1 from the Regular Polygon Proposition first make these definitions.

Let  $C$  be a circle, and let  $r_n, t_n$  be the radius and the circumference (respectively) of the *in*-scribed regular  $2^n$ -gon, and  $R_n, T_n$  be the radius and the circumference (respectively) of the *circum*-scribed regular  $2^n$ -gon, noting that  $R_n = R$ , where  $R$  is the radius of the circle about which the polygon is circumscribed. Euclid's XII.2 already gives us a way of seeing

that the discrepancy of areas (i.e.,  $D_n$  and  $d_n$ ) goes to zero as  $n$  goes to infinity<sup>4</sup> But we must also know some further things:

- (a) **About the radii:** The radii of the inscribed regular polygons are increasing (as  $n$  increases) and converging to the radius  $R$  of the circle.
- (b) **About the circumferences:** The circumferences of the inscribed regular polygons are increasing (as  $n$  increases) and the circumferences of the inscribed regular polygons are decreasing and these circumferences are “nesting” and converging in the sense that
  - we have the sandwich<sup>5</sup> an of inequalities:

$$\dots t_n < t_{n+1} < \dots \quad \dots < T_{n+1} < T_n < \dots$$

and

- $T_n - t_n$  can be made smaller than any positive quantity by choosing  $n$  large enough.

All this, so far—despite the fact that it is given in “modern-speak”—is doable with standard Euclidean methods (even if it might be necessary to rephrase it somewhat), and—we might look for a mini-argument of this type hinted at in the way Archimedes sets up his proof.

5. **The question of definition versus approximation-with-arbitrary-accuracy-from-above-and-from-below:** We have not yet faced the main issue of interest in this proposition, namely the *circumference of the circle itself!* This circumference (or, for that matter, the arc-length of *any* curvilinear figure) has no direct counterpart in Euclidean vocabulary. Nevertheless, the sandwich nest in the two bullets above would allow, say, Eudoxus, to refer to *the* circumference of the circle as the (unique) length  $T$  that fits in the sandwich:

$$\dots t_n < t_{n+1} < \dots < T < \dots < T_{n+1} < T_n < \dots$$

A question: In which of the two possible ways should our text be read?

- Are we to think of  $T$  as a length to be measured, and the vise (or “sandwich”) above as simply getting closer and closer in measurement to  $T$ ,
- or are we to give it the more modern interpretation and view it as having the ontological consequence that  $T$  exists?

It is interesting to compare this with Casselman’s remarks<sup>6</sup>.

Once we accept this, the rest is clear, I believe. For by the Regular Polygon Proposition, the area  $p_n$  of the  $n$ -th regular inscribed polygon is  $\frac{1}{2}r_n t_n$ , and the area  $P_n$  of the  $n$ -th regular circumscribed polygon is  $\frac{1}{2}R_n T_n$ . So if  $C, R, T$  are the areas subsumed by the circle, the radius of the circle, and its circumference (following the above definition) we have equal nests:

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<sup>4</sup>It is interesting that Archimedes’ proof discusses explicitly *only* the circumscribed polygonal construction, while Euclid discusses the inscribed.

<sup>5</sup>I also like to think of this nest of inequalities as a *vise* closing in on one length

<sup>6</sup>The above discussion merely gives a linear length ( $T$ ) to the circumference of the circle (“linearizes the circle,” so to speak) by the sandwich described. This, if you are to go on to do measurements, etc. would be sufficient. Casselman suggests something more; i.e., that Archimedes envisions (or perhaps I should say “also envisions”) a more conceptual approach to the issue by contemplating a general axiom about lengths of convex arcs, following ideas related to those found in *The sphere and cylinder*, and this would give a conceptual formulation to what has been achieved in the sandwich described above.

$$\dots \frac{1}{2}r_n t_n < \frac{1}{2}r_{n+1} t_{n+1} < \dots < \frac{1}{2}RT < \dots < \frac{1}{2}RT_{n+1} < \frac{1}{2}RT_n < \dots$$

and

$$\dots p_n < p_{n+1} < \dots < C < \dots < P_{n+1} < P_n < \dots$$

## 6. Reductio versus Limit

A modern might simply use modern vocabulary and simply *pass to a limit* in the above vise to conclude that the areas  $\frac{1}{2}RT$  and  $C$ —are equal; hence Proposition 1. But what is envisioned in the Archimedes text is the alternate route (which I'll phrase in relatively modern language) namely

Assume that the proposition is false, and therefore that there is a discrepancy  $d$  between the areas of the two figures that are being compared, and *then* find an  $n$  large enough so that the differences between the areas of inscribed, circumscribed  $2^n$ -gons, and circle are so small that the vise above shows that the discrepancy can not be as large as  $d$ .

A query for Archimedes: An immediate Corollary of Proposition 1 is the statement analogous to XII.2. Namely, the ratio of the circumferences of any two circles is equal to the ratio of their diameters (or their radii); and the equivalent way of stating this is that the ratio of circumference to diameter of any circle is independent of the (size of the) circle. In brief, as with Euclid's XII.2 which may be taken as (implicitly) giving us a definition of  $\pi$  as a ratio of areas, Archimedes Proposition 1 can be taken to simply be giving us another *definition* of  $\pi$  as a ratio of lengths. All this is implicit, since Propositions 2,3 are already taking off numerically on that theme. How is it, Archimedes, that you don't say a word about this and simply assume that your audience is on board with it?