THE WORKS OF ARCHIMEDES

Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams

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Volume I
The Two Books *On the Sphere and the Cylinder*

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ON THE SPHERE AND THE CYLINDER, BOOK I

/Introduction: general/

Archimedes to Dositheus:[1] greetings.

Earlier, I have sent you some of what we had already investigated then, writing it with a proof: that every segment contained by a straight line and by a section of the right-angled cone[2] is a third again as much as a triangle having the same base as the segment and an equal height.[3] Later, theorems worthy of mention suggested themselves to us, and we took the trouble of preparing their proofs. They are these: first, that the surface of every sphere is four times the greatest circle of the <circles> in it.[4] Further, that the surface of every segment of a sphere is equal to a circle whose radius is equal to the line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment.[5] Next to these, that, in every sphere, the cylinder having a

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[1] The later reference is to *QP*, so this work – *SC* I – turns out to be the second in the Archimedes–Dositheus correspondence. Our knowledge of Dositheus derives mostly from introductions by Archimedes such as this one (he is also the addressee of *SC* II, *CS*, *SL*, besides of course *QP*): he seems to have been a scientist, though perhaps not much of one by Archimedes' own standards (more on this below). See Netz (1998) for further references and for the curious fact that, judging from his name, Dositheus probably was Jewish.

[2] "Section of the right-angled cone:" what we call today a "parabola." The development of the Greek terminology for conic sections was discussed by both ancient and modern scholars: for recent discussions referring to much of the ancient evidence, see Toomer (1976) 9–15, Jones (1986) 400.

[3] A reference to the contents of *QP* 17, 24. [4] *SC* I.33.

[5] Greek: "that to the surface . . . is equal a circle . . ." The reference is to *SC* I.42–3.

base equal to the greatest circle of the <circles> in the sphere, and a height equal to the diameter of the sphere, is, itself,[6] half as large again as the sphere; and its surface is <half as large again> as the surface of the sphere.[7]

In nature, these properties always held for the figures mentioned above. But these <properties> were unknown to those who have engaged in geometry before us – none of them realizing that there is a common measure to those figures. Therefore I would not hesitate to compare them to the properties investigated by any other geometer, indeed to those which are considered to be by far the best among Eudoxus' investigations concerning solids: that every pyramid is a third part of a prism having the same base as the pyramid and an equal height,[8] and that every cone is a third part of the cylinder having the base the same as the cylinder and an equal height.[9] For even though these properties, too, always held, naturally, for those figures, and even though there were many geometers worthy of mention before Eudoxus, they all did not know it; none perceived it.

But now it shall become possible – for those who will be able – to examine those <theorems>.

They should have come out while Conon was still alive.[10] For we suppose that he was probably the one most able to understand them and to pass the appropriate judgment. But we think it is the right thing, to share with those who are friendly towards mathematics, and so, having composed the proofs, we send them to you, and it shall be possible – for those who are engaged in mathematics – to examine them. Farewell.

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[6] The word "itself" distinguishes this clause, on the relation between the volumes, from the next one, on the relation between the surfaces. In other words, the cylinder "itself" is what we call "the volume of the cylinder." This is worth stressing straight away, since it is an example of an important feature of Greek mathematics: relations are primarily between geometrical objects, not between quantitative functions on objects. It is not as if there is a cylinder and two quantitative functions: "volume" and "surface." Instead, there are two geometrical objects discussed directly: a cylinder, and its surface.

[7] *SC* I.34.

[8] *Elements* XII.7 Cor. Eudoxus was certainly a great mathematician, active probably in the first half of the fourth century. The most important piece of evidence is this passage (together with a cognate one in Archimedes' *Method*: see general comments). Aside for this, there are many testimonies on Eudoxus, but almost all of them are very late or have little real information on his mathematics, and most are also very unreliable. Thus the real historical figure of Eudoxus is practically unknown. For indications of the evidence on Eudoxus, see Lasserre (1966), Merlan (1960).

[9] *Elements* XII.10. [10] See general comments.

TEXTUAL COMMENTS

The first page of codex A was crumbling already by 1269 (when its first extant witness, codex B, was prepared), and the page was practically lost by the fifteenth century (when the Renaissance codices began to be copied). Heiberg's first edition (1880–81), based only on A's Greek Renaissance copies, was very much a matter of guesswork as far as that page was concerned, so that this page was thoroughly revised in the second edition in light of the codices B and (the totally independent) C. I translate Heiberg's text as it stands in the second edition (1910). It is interesting that Heath (1897), based on Heiberg's first edition, was never revised: at any rate, this is the reason why my text here has to be so different from Heath's, even though this is one of the cases where Heath attempts a genuine translation rather than a paraphrase. Otherwise this general introduction is textually unproblematic.

GENERAL COMMENTS

Introduction: the genre

Introductory letters to mathematical works could conceivably have been a genre pioneered by Archimedes (of course, this is difficult to judge since we have very few mathematical works surviving from before Archimedes in their original form). At any rate, they are found in other Greek Hellenistic mathematical works, e.g. in several books of Apollonius' *Conics*, Hypsicles' *Elements* XIV, and Diocles' *On Burning Mirrors*. The main object of such introductions seems to set out the relation of the text to previous works, by the author (in this case, Archimedes relates the work to *QP*), and by others (in this case, Archimedes relates the work to that of Eudoxus). Correlated with the external setting-out – how the work relates to works external to it – is an internal setting-out – how the work is internally structured, and especially what are its main results.

For the internal setting-out, it is interesting that Archimedes orders his results as I.33, I.42–3, I.34, i.e. not the order in which they are set out in the text itself. Sequence, in fact, is not an important consideration of the work. Once the groundwork is laid, in Propositions 1–22, the second half of the work is less constrained by strong deductive relations, one result leading to the next: the main results of the second part are mainly independent of each other. Archimedes stresses then the nature of the discoveries, not their order. The main theme for those discoveries is that of the "common measure" (which is a theme of both his new results on the sphere, and his old results on the parabola). The Greek for "common measure" is *summetria*, which, translated into Latin, is a cognate of "commensurability." *Summetria* is indeed a technical term in Greek mathematics, meaning "commensurability" in the sense of the theory of irrationals (Euclid's *Elements* X Def. 1). In Greek, however, it has the overtone of "good measure," something like "harmony." What is so remarkable, then: the very fact that curvilinear and rectilinear figures have a common measure, or the fact that their ratio is so simple and pleasing? (It is even possibly relevant that, in Greek mathematical musical theory – well known to Archimedes and his audience – 4:3 and 3:2 are, respectively, the ratios of the fourth and the fifth.)

To return to the external setting-out: this is especially rich in historical detail, and should be compared with Archimedes' *Method*, 430.1–9, which is the only other sustained historical excursus made by Archimedes. The comparison is worrying in two ways. First, the *Method* passage concerns, once again, the same relation between cone and cylinder, i.e. it seems as if Archimedes kept recycling the same story. Second, the *Method* version seems to contradict this passage (*SC*: no knowledge prior to Eudoxus. *Method*: no proof prior to Eudoxus, however known already to Democritus).

Was Archimedes an old gossip then? A liar? More to the point: we see Archimedes constantly comparing himself to Eudoxus, arguing for his own superiority over him. This is the best proof we have of Eudoxus' greatness. And as for the facts, Archimedes was no historian.

Archimedes' audience: conon and dositheus

Conon keeps being dead in Archimedes' works: in the introductions to *SL* (2.2 ff.) and *QP* (262.3 ff.), also *SC* II (168.5). Born in Samos, dead well before Archimedes' own death in 212 BC, he must have been a rare person as far as Archimedes was concerned: a mathematician. That he was a mathematician, and that this was so rare, is signaled by Archimedes' shrill tone of despair: the death of Conon left him very much alone. (A little more – no more – is known of Conon from other sources, and he appears, indeed, to have been an accomplished mathematician and astronomer: the main indications are Apollonius' *Conics*, introduction to Book IV, Diocles' *On Burning Mirrors*, introduction, and Catullus' poem 66.)

Archimedes shows less admiration towards Dositheus. The letter is curt, somewhat arrogant, almost dismissive – though note that the first person plural would be normal and therefore less jarring for the ancient reader. The concluding words, with the refrain "but now it shall become possible – for those who will be able – to examine those <theorems>," ". . . and it shall be possible – for those who are engaged in mathematics – to examine them" stress that only one readership may *examine* the results – "those who are engaged in mathematics." There is another, much more peripheral readership: ". . . those who are friendly towards mathematics," and it is with them that Archimedes says that he had decided to "share." In other words, Dositheus is one of the "friends." He is no mathematician according to Archimedes' standards. Archimedes' hope is that, through Dositheus, the work will become public and may reach some genuine mathematicians (the one he had known – Conon – being dead).

It seems, to judge by the remaining introductions to his works, that Archimedes never did find another mathematician.

/"Axiomatic" introduction/

First are written the principles and assumptions required for the proofs of those properties.

/Definitions/

Eut. 244 /1/ There are in a plane some limited[11] curved lines, which are either wholly on the same side as the straight <lines>[12] joining their limits or have nothing on the other side.[13] /2/ So[14] I call "concave in the same
Eut. 245 direction" such a line, in which, if any two points whatever being taken, the straight <lines> between the <two> points either all fall on the same side of the line, or some fall on the same side, and some on the line itself, but none on the other side. /3/ Next, similarly, there are also some limited surfaces, which, while not themselves in a plane, do have the limits in a plane; and they shall either be wholly on the same side of the plane in which they have the limits, or have nothing on the other side. /4/ So I call "concave in the same direction" such surfaces, in which, suppose two points being taken, the straight <lines> between the points either all fall on the same side of the surface, or some on the same side, and some on <the surface> itself, but none on the other side.

/5/ And, when a cone cuts a sphere, having a vertex at the center of the sphere, I call the figure internally contained by the surface of the cone, and by the surface of the sphere inside the cone, a "solid sector." /6/ And when two cones having the same base have the vertices on each of the sides of the plane of the base, so that their axes lie on a line, I call the solid figure composed of both cones a "solid rhombus."

And I assume these:

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11 The adjective "limited," throughout, is meant to exclude not only infinitely long lines (which may not be envisaged at all), but also closed lines (e.g. the circumference of a circle), which do not have "limits."

12 The words "straight <line>" represent precisely the Greek text, *eutheia*: "straight" is written and "line" is left to be completed. This is the opposite of modern practice, where often the word "line" is used as an abbreviation of "straight line." Outside this axiomatic introduction, whenever the sense will be clear, I shall translate *eutheia* (literally meaning "straight") by "line."

13 See Eutocius for the important observation that "curved lines" include, effectively, any one-dimensional, non-straight objects, such as "zigzag" lines. See also general comments on Postulate 2.

14 Here and later in the book I translate the Greek particle δή with the English word 'so'. The Greek particle has in general an emphatic sense underlining the significance of the words it follows. In the mathematical context, it most often serves to underline the significance of a transitional moment in an argument. It serves to emphasize that, a conclusion having been reached, a new statement can finally be made or added. The English word "so" is a mere approximation to that meaning.

/Postulates/

Eut. 245 /1/ That among lines which have the same limits, the straight <line> Eut. 246 is the smallest. /2/ And, among the other lines (if, being in a plane, they have the same limits): that such <lines> are unequal, when they are both concave in the same direction and either one of them is wholly contained by the other and by the straight <line> having the same limits as itself, or some is contained, and some it has <as> common; and the contained is smaller.

/3/ And similarly, that among surfaces, too, which have the same limits (if they have the limits in a plane) the plane is the smallest. /4/ And that among the other surfaces that also have the same limits (if the limits are in a plane): such <surfaces> are unequal, when they are both concave in the same direction, and either one is wholly contained by the other surface and by the plane which has the same limits as itself, or some is contained, and some it has <as> common; and the contained is smaller.

/5/ Further, that among unequal lines, as well as unequal surfaces and unequal solids, the greater exceeds the smaller by such <a difference> that is capable, added itself to itself, of exceeding everything set forth (of those which are in a ratio to one another).

Assuming these it is manifest that if a polygon is inscribed inside a circle, the perimeter of the inscribed polygon is smaller than the circumference of the circle; for each of the sides of the polygon is smaller than the circumference of the circle which is cut by it.

TEXTUAL COMMENTS

It is customary in modern editions to structure Greek axiomatic material by titles and numbers. These do not appear in the manuscripts. They are convenient for later reference, and so I add numbers and titles within obliques (//). Paragraphs, as well, are an editorial intervention. The structure is much less clearly defined in the original and, probably, no clear visual distinction was originally made between the introduction (in its two parts) and the following propositions. This is significant, for instance, for understanding the final sentence, which is neither a postulate nor a proposition. Archimedes does not set a series of definitions and postulates, but simply makes observations on his linguistic habits and assumptions.

GENERAL COMMENTS

Definitions 1–4

Following Archimedes, we start with Definition 1. Imagine a "curved line," and the straight line joining its two limits. For instance, let the "curved line" be the railroad from Cambridge to London as it is in reality (let this be called *real railroad*); the straight line is what you wish this railroad to be like: ideally straight (let this be called *ideal railroad*). Now, as we take the train from Cambridge to London, we compare the two railroads, the real and the ideal. Surprisingly perhaps, the two do have to coincide on at least two points (namely, the start and end points). Other than this, the real veers from the ideal. If the real sometimes coincides with the ideal, sometimes veers to the east, but never veers to the west, then it falls under this definition. If the real sometimes coincides with the ideal, sometimes veers to the west, but never veers to the east, once again it falls under this definition. But if – as I guess is the case – the real sometimes veers to the east of the ideal, sometimes to the west, then (and only then) it does not fall under this definition. In other words, this definition singles out a family of lines which, even if not always straight, are at least consistent in their direction of non-straightness, always to the same side of the straight. It is only this family which is being discussed in the following Definition 2 (a similar family, this time for planes, is singled out in Definition 3, and is discussed in Definition 4: whatever I say for Definitions 1–2 applies *mutatis mutandis* for Definitions 3–4).

Definition 2, effectively, returns to the property of Definition 1, and makes it global. That is, if Definition 1 demands that the line be consistent in its non-straightness relative to its start and end points only, Definition 2 demands that the line be consistent in its non-straightness relative to any two points taken on it (the obvious example would be the arc of a circle). It follows immediately that whatever line fulfils the property of Definition 2, must also fulfil the property of Definition 1 (the end and start points are certainly some points on the line). Thus, the lines of Definition 2 form a subset of the lines of Definition 1. This is strange, since the only function of Definition 1 is to introduce Definition 2 (indeed, since originally the definitions were not numbered or divided, we should think of them as two clauses of a single statement). But, in fact, Definition 1 adds nothing to Definition 2: Definition 2 defines the same set of points, with or without the previous addition of Definition 1. That is, to say that the property of Definition 2 is meant to apply only to the family singled out in Definition 1 is an empty claim: the property can apply to no other lines. It seems to me that the clause of Definition 1 is meant to introduce the main idea of Definition 2 with a simple case – which is what I did above. In other words, the function of Definition 1 may be pedagogic in nature.

Postulates 1–2: about what?

The wording of the translation of Postulate 1 gives rise to a question of translation of significant logical consequences. My translation has ". . . among lines which have the same limits, the straight <line> is the smallest . . ." Heiberg's Latin translation, as well as Heath's English (but not Dijksterhuis') follow Eutocius' own quotation of this postulate, and read an "all" into the text, translating as if it had "among *all* lines having the same limits . . ."

The situation is in fact somewhat confusing. To begin with, there is no unique set of "lines having the same limits," simply because there are many couples of limits in the world, each with its own lines. So, to make some sense of the postulate, we could, possibly, imagine a Platonic paradise, in it a single straight line, a sort of Adam-line; and an infinite number of curved lines produced between the two limits of this line – a harem of Eves produced from this Adam's rib. And then the postulate would be a statement about this Platonic, uniquely given "straight line." This is Heiberg's and Heath's reading, which make Postulate 1 into a general statement about the straight line as such. The temptation to adopt this reading is considerable. But I believe the temptation should be avoided. The postulates do not relate to a Platonic heaven, but are firmly situated in this world of ours where there are infinitely many straight lines. (The postulates will be employed in different propositions, with different geometrical configurations, different sets of lines.) The way to understand the point of the postulates, is, I suggest, the following:

There are many possible clusters of lines, such that: all the lines in the cluster share the same limits. Within any such cluster, certain relations of size may obtain. Postulate 2 gives a rule that holds between any two curved lines in such a given cluster (assuming the two lie in a single plane). Why do we have Postulate 1? This is because Postulate 2 cannot be generalized to cover the case of straight lines. (This is because the straight line is not contained, even partly, by "the other line and the line having the same limits as itself." See my explanation of the second postulate below.) So a special remark – hardly a postulate – is required, stating that, in any such cluster, the smallest line will be (if present in the cluster) the straight line. Thus, nothing like "a definition of the straight line" may be read into Postulate 1.

Unpacking Postulate 2

Take a limited curved line, and close it – transform it into a closed figure – by attaching a straight line between the two limits, or start and end points, of the line. This is, as it were, "sealing" the curved line with a straight line. So any curved line defines a "sealed figure" associated with it. (In the case of lines that are concave to the same direction, they even define a continuous sealed figure, i.e. one that never tapers to a point: a zigzagging line, veering in this and that direction would define a sequence of figures each attached to the next by the joint of a single point – whenever the line happened to cross the straight line between its two extremes).

Now take any such two curved lines. Assume they both have the *same* limits, and that they both lie in a single plane. Now let us have firmly before our mind's eye the sealed figure of one of those lines; and while we contemplate it, we look at the other curved line. It may fall into several parts: some that are inside the sealed figure, some that are outside the sealed figure, and some that coincide with the circumference of the sealed figure. If it has at least one part that is inside the sealed figure, and no part that is outside the sealed figure, then it has the property of the postulate. Note then that a straight line can never have this property: it will be all *on* the circumference of the sealed figure, none of it ever *inside* it (hence the need for Postulate 1).

Unpacking Postulate 3

The caveat, "when they have the limits in a plane," is slightly difficult to visualize. The point is that a couple of three-dimensional surfaces may share the same limit; yet that limit may still fail to be contained by a single plane (so this latter possibility must be ruled out explicitly). Imagine two balloons, one inside the other, somehow stitched together so that their mouths precisely coincide. Thus they have "the same limit," but the limit – the mouth – need not necessarily lie on a plane. Imagine for instance that you want to block the air from getting out of the balloons – you want a surface to block the mouth; you put the mouth next to the wall, but it just will not be blocked: the wall is a perfect plane, and the mouth does not lie on a single plane: some of it is further out than the rest. This, then, is what we do *not* want in this postulate.

The overall structure of Definitions 1–4, Postulates 1–4

This combination of definitions and postulates forms a very detailed analysis of the conditions for stating equalities between lines and surfaces. So many ideas are necessary!

1 "The same side," requiring the following considerations:
  - A generalization of "curved" to include "zigzag" lines.
  - What I call "real and ideal railroads" (Definition 1).
  - A disjunctive analysis (the real either wholly on one side of the ideal, or partly on it, but none on the other side).

2 "Concave," requiring the following considerations:
  - The idea of "lines joining any two points whatsoever."
  - The same disjunctive analysis as above.

3 "Contain," requiring the following considerations:
  - Having the same limits.
  - What I call the "sealed figure" (Postulate 2).
  - A disjunctive analysis (Whether wholly inside, or part inside and none outside).

4 Finally one must see:
  - The independence of the special case of the straight line – which requires a caveat in Postulate 1.
  - Also there is the special problem with the special case of the plane – which requires the caveat mentioned above, in Postulate 3.

There was probably no rich historical process leading to this conceptual elucidation. The only seed of the entire analysis is *Elements* I.20, that any two lines in a triangle are greater than the third. But the argument there (relying on considerations of angles in triangles) does not yield any obvious generalizations. So how did this analysis come about? A simple answer, apparently: Archimedes thought the matter through.

He is not perfectly explicit. The sense of "curved lines" must have been clear to him, but as it stands in the text it is completely misleading, and requires Eutocius' explication with his explanation of what I call "zigzag" lines. My own explications, too, with their "real and ideal" and "sealed figures," were also left by Archimedes for the reader to fill in. The use of disjunctive properties serves to make the claims even less intuitive.

Most curiously, this entire analysis of concavity will *never* be taken up in the treatise. No application of the postulates relies on a verification of its applicability, through the definitions; there is not even the slightest gesture towards such a verification.

This masterpiece had no antecedents, and no real implementation, even by Archimedes himself. A logical, conceptual *tour-de-force*, an indication of the kind of mathematical *tour-de-force* to follow. Archimedes portrayed himself as the one who sees through what others before him did not even suspect, and he gave us now a first example.

Postulate 5

This postulate, often referred to as "Archimedes' axiom," recurs, in somewhat different forms, elsewhere in the Archimedean corpus: in the introduction to the *SL* (12.7–11) [this may be a quotation of our own text], and in the introduction to the *QP* (264.9–12). As the modern appellation implies, the postulate has great significance in modern mathematics, with its foundational interests in the structure of continuity, so that one often refers to "Archimedean" or various "non-Archimedean" structures, depending on whether or not they fulfil this postulate. This is not the place to discuss the philosophical issues involved, but something ought to be said about the problem of historically situating this postulate.

Two presuppositions, I suggest, ought to be questioned, if not rejected outright:

1 "Archimedes is engaged here in axiomatics." We just saw Archimedes offering an axiomatic study (clearing up notions such as "concavity") almost for its own sake. This should not be immediately assumed to hold for this postulate as well. The postulate might also be here in order to do a specific job – as a tool for a particular geometrical purpose. In this case, it need not be seen as a contribution to axiomatic analysis as such. For instance, it is conceivable that Archimedes thought this postulate could be proved (I do not say he did; I just point out how wide the possibilities are). Nor do we need to assume Archimedes was particularly interested in this postulate; he need not necessarily have considered it "his own."

2 "Archimedes extends Euclid/Eudoxus." The significance of the postulate, assuming that it was a new discovery made by Archimedes, would depend on its precise difference from other early statements on the issue of size, ratio and excess. Indeed, the postulate relates in some ways to texts known to us through the medieval tradition of Euclid's Elements (*Elements* V Def. 4, X.1), often associated by some scholars, once again (perhaps rightly) with the name of Eudoxus. It is not known who produced those texts, and when but, even more importantly, it is absolutely unknown in what form, if any, such texts were known to Archimedes himself. (Archimedes makes clear, in both *SL* and *QP*, that the postulate – in some version – *was* known to him from earlier geometers; but we do not know *which* version). It is even less clear which texts Dositheus (or any other intended reader) was expected to know.

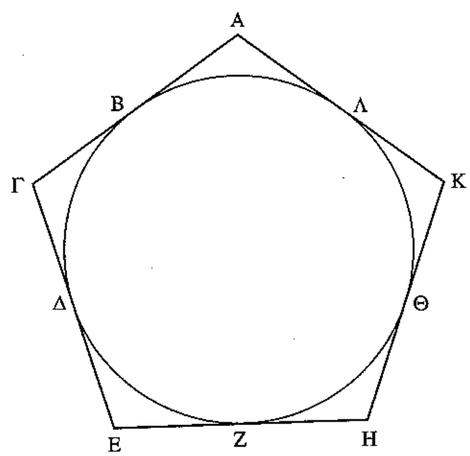
So nothing can be taken for granted. The text must be read and understood in the light of what it says, how it is used, and the related material in the Archimedean corpus. This calls for a separate study, which I shall not pursue here.

/1/

If a polygon is circumscribed around a circle, the perimeter of the circumscribed polygon is greater than the perimeter of the circle.

For let a polygon – the one set down[15] – be circumscribed around a circle. I say that the perimeter of the polygon is greater than the perimeter of the circle.

(1) For since BAΛ[16] taken together is greater than the circumference BΛ (2) through its <=BAΛ> containing the circumference <=BΛ> while having the same limits,[17] (3) similarly, ΔΓ, ΓB taken together <are greater> than ΔB, as well; (4) and ΛK, KΘ taken together <are greater> than ΛΘ; (5) and ZHΘ taken together <is greater> than ZΘ; (6) and once more, ΔE, EZ taken together <are greater> than ΔZ; (7) therefore the whole perimeter of the polygon is greater than the circumference of the circle.



I.1
In most Codices EH is parallel to base of page. Codices BG, however, both have E rather lower than H. I suspect codex A had a slight slope, ignored in most copies and exaggerated in BG.

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[15] "Set down" = "in the diagram." See general comments for this strange expression.

[16] BAΛ: an alternative way of referring to the sequence of two lines BA, AΛ. (This might be related to Archimedes' generalized notion of "line," including "zigzag" line, which was implicit in the axiomatic introduction).

[17] Post. 2; see general comments for the limited use of the axiomatic introduction.

TEXTUAL COMMENTS

The manuscripts agree that this should be numbered as the "first" proposition (i.e. the preceding passage is still "introductory"). Discrepancies between the numbering in the various manuscripts begin later (with proposition "6"). I print Heiberg's numbering, mainly for ease of reference, but it is possible that Archimedes' text had no numbers for propositions. If so, there was no break, originally, between the "introduction" and this passage. The only mark that a new type of text had begun (and the reason why all manuscripts chose to place the first number here) was the first occurrence of a diagram.

GENERAL COMMENTS

The use of the diagram

Archimedes is impatient here, and employs all sorts of shortcuts. The sentence "let the polygon – the one set down – be circumscribed around a circle" is the setting-out: the only statement translating the general enunciation in particular terms. Instead of guiding in detail the precise production of the diagram, then – as is the norm in Euclid's *Elements* – Archimedes gives a general directive. He is an architect here, not a mason. As a side-result of this, all the letters of this proposition rely, for their identification, on the diagram alone. Without looking at the diagram, there is no way you could know what the letters stand for: the text says nothing explicit about that, and instead totally assumes the diagram. Thus, the principle according to which letters are assigned to points is spatial (a counter-clockwise tour around the polygon, starting from A at the hour 12). This is instead of the standard alphabetical principle (the first mentioned letter: A, the second: B, etc. . . .). This is because there is not even the make-believe of producing the diagram through the text – as if the diagram were constructed during the reading of the text. This make-believe occurs in the standard Euclidean proposition: as the readers follow the alphabetical principle, they imagine the diagram gathering flesh gradually, as it were, as the letters are assigned to their objects.

The use of the axiomatic introduction

The use of Postulate 2 is remarkable in its deficiency. That the various lines and circumferences are all concave in the same direction is taken for granted. Not only is this concavity property not proven – it is not even explicitly mentioned. So why have the careful exposition of the concept of "concave in the same direction" in Definition 2? There, a precise test for such concavity was formulated – *not* to be applied here. Perhaps, Archimedes' goal is not axiomatic perfection (where every axiom, and every application of an axiom, must be made explicit), but *truth*. He has discovered what he is certain is true – Postulate 2, based on Definition 2. When applying the postulate, Archimedes is much more relaxed: as long as the applicability of the postulate is sufficiently clear, there is no need to mention it explicitly.

Generality of the proof

This proposition raises the problem of mathematical generality and the complex way in which it is achieved in Greek mathematics. First, consider the choice of object for discussion. Any polygon would do, and a triangle would have been the simplest, yet Archimedes chose a pentagon. Why? Perhaps, because choosing a more complex case makes the proof appear more general. At least, this is not the simplest case (which, just because it is "simplest," is in some sense "special").

But, still, how to generalize from the pentagon to any-gon? Archimedes does not even make a gesture towards such a generalization. For instance, the selection of lines along the polygon does not follow any definite principle (e.g. clockwise or anti-clockwise). Such a definite principle could have suggested a principle of generalization ("and go on if there are more . . ."). But Archimedes suggests none, erratically jumping from line to line. Even more: Archimedes does not pause to generalize *inside* the particular proof. There are no "three dots" in this proof. He goes on and on, exhausting the polygon (instead of saying "and so on" at some stage). While there is an effort to make the particular case "as general as possible," there is no gesture towards making the generalization explicit.

/2/

Given two unequal magnitudes, it is possible to find two unequal lines so that the greater line has to the smaller a ratio smaller than the greater magnitude to the smaller.

Let there be two unequal magnitudes, AB, Δ, and let AB be greater. I say that it is possible to find two unequal lines producing the said task.

(a) Let BΓ be set out equal to Δ (1) through the second <proposition> of the first <book> of Euclid <=*Elements*>, (b) and Eut. 250 let there be set out some straight line, ZH; (2) so, ΓA being added onto itself will exceed Δ.[18] (c) So let it be multiplied,[19] and let it <=the result of multiplication> be AΘ, (d) and as many times AΘ is of AΓ, that many let ZH be of HE. (3) Therefore it is: as ΘA to AΓ, so ZH to HE;[20] (4) and inversely, it is: as EH to HZ, so AΓ to AΘ.[21] (5) And

[18] Post. 5. Note that the elided word is here often "magnitude" and not "line."

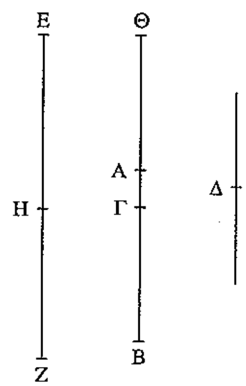
[19] "Multiplied" is taken to mean the same as "being added onto itself." It is implicit that ΓA is multiplied until it exceeds Δ.

[20] The derivation from Step 2 to Step 3 ("A is the same multiple of B as D is of C; therefore A:B::C:D") is too simple to be *proved* by Euclid. It is part of the *definition* of proportion, but only in the case of numbers (*Elements* VII. Def. 21).

[21] *Elements* V.7 Cor. Note that changing the sequence of sides, i.e. a change such as A:B::C:D → C:D::A:B is not considered as a move at all, and requires no word of the "alternately" family. The symmetry of proportion is seen as a *notational* freedom.

since AΘ is greater than Δ, (6) that is than ΓB, (7) therefore ΓA has to AΘ a ratio smaller than ΓA to ΓB.[22] (8) But as ΓA to AΘ, so EH to HZ; (9) therefore EH has to HZ a smaller ratio than ΓA to ΓB. (10) And Eut. 251 compoundly;[23] (11) [therefore] EZ has to ZH a smaller ratio than AB to BΓ [(12) through lemma]. (13) But BΓ is equal to Δ; (14) therefore EZ has to ZH a smaller ratio than AB to Δ.

(15) Therefore two unequal lines have been found, producing the said task [(16) namely the greater has to the smaller a smaller ratio than the greater magnitude to the smaller].



I.2
Codices DH: EZ=ΘB. Codex G: H permuted with E, m.2 introduced E next to both points E, Z. Codex H: E (?) instead of Γ. Heiberg permutes A/B. See general comments.

TEXTUAL COMMENTS

Step 1 is an interpolation, unbracketed by Heiberg for the bad reason that Proclus had already read a reference of Archimedes to Euclid (Proclus, *In Eucl.* 68.12) – which shows merely that the interpolation antedated Proclus. From our knowledge of Euclid, the reference should be to I.3, not to I.2, but even so, this reference is only speciously relevant. I.3 shows how to cut off, from a given line, a line equal to some other given line. There is – there can be – no generalization for magnitudes in general, even if by "magnitudes" geometrical objects alone are meant. Even if Archimedes could commit such a blunder, it remains a fact that such references are the most common scholia. Hence, most likely, this is indeed an interpolation.

This was a *sui generis* textual problem. The next three are all typical of many others we shall come across later on.

First, in Step 11, Heiberg brackets the word "therefore" because of its absence from Eutocius' quotation. This is not a valid argument, as Eutocius does not aim to copy the text faithfully. Why should he copy such words as "therefore," which have no meaning outside their context?

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22 *Elements* V.8.

23 *Elements* V.18: A:B::C:D → (A+B):B::(C+D):D (Archimedes, however, assumes an extension of the *Elements* to inequalities of ratios. This extension is supplied by Eutocius' commentary).

Second, Heiberg must be right about Step 12. Had it been Archimedes' we would certainly have a lemma, following this proposition, by Archimedes himself. This is a scholion, referring to Eutocius' own commentary.

Finally, Step 16 belongs to an important class: pieces of text which may be authentic (and then must shape accordingly our understanding of Archimedes' practices) or may be interpolated. How to tell? Only by our general understanding of Archimedes' practice – an understanding which is itself dependent upon such textual decisions! Heiberg imagined a purist, minimalist Archimedes. In this, he may have been right: my sense, too, is that Step 16 is by a later scholiast. But we should keep our minds open.

GENERAL COMMENTS

Existence and realism

The proposition is a problem: not showing the truth of an assertion (as theorems do), but performing a task. However, it is in a sense akin to a theorem. In the Euclidean norm, problems are formulated as "given X . . . to do Y." Archimedes often uses, as here, the format "given X . . . it is possible to do Y." This turns the problem into a truth-claim, more akin to a theorem.

A problem, which is rather like a truth-claim, may strike a modern reader as a *proof of existence*. This has been the subject of a modern controversy: Zeuthen (1886) had suggested that ancient problems, in general, are existence proofs, while Knorr (1983) has argued that, within geometry itself, questions of mathematical existence were often of less importance. Even when the issue of mathematical existence arose, it was handled through techniques other than those of problems. What about the present proposition, then? I would side with Knorr, and suggest that the problem does not aim to show the existence of an object, but to furnish a tool. Postulates1–4 (followed by their quick, unnumbered sequel, and by the first proposition) furnished tools for obtaining inequalities between geometrical *objects*. We now move on to develop tools (based on the fifth Postulate) for obtaining inequalities between geometrical *ratios*. Both types of *in*equalities will then be used to prove the geometrical *equalities* of this treatise. This is what the proposition does. On the philosophical side, it does not deal at all with the question of mathematical existence. The question of "existence" is basically that: do you assume that mathematical objects exist, or do you prove their existence? Archimedes reveals here what may be considered to be the usual realism of Greek mathematics, where objects are simply taken for granted.

First, let us assume that Step 1 is an interpolation. It follows then that the proposition requires an unstated postulate ("to take away a magnitude from a magnitude, so as to have left a magnitude equal to a given magnitude"). This is a strong tacit existence assumption. Further, in Step d, we need to know how many times ΓA was multiplied (in Step c) before exceeding Δ. This is because we define ZH – as so many times HE as ΓA was multiplied. But we do not *know* how many times ΓA was multiplied. On the basis of Postulate 5, we are promised that ΓA *may* exceed Δ. But there is no algorithm for finding a *specific* number of times required for exceeding Δ. Once again, we assume that we can obtain an object (the number of times X was multiplied to exceed Y), without specifying a procedure for obtaining it: its existence, once again, assures its being obtainable.

In both cases, Archimedes reveals his realism. No algorithm is required: the relevant magnitudes and ratios exist, and there is no need to spell out how exactly you get them.

Shortcuts used in exposition

Archimedes displays a certain "laziness;" it manifested itself in the preceding proposition in the setting-out of the particular case (the diagram was simply assumed), it is here manifested in the setting out of the particular *demonstrandum*: instead of saying *what* is to be possible in the particular case, Archimedes says that "it is possible to find two unequal lines *producing the said task*" (leaving it to the reader to supply just what *is* the task: hence perhaps the interpolated Step 16?).

Schematic nature and the intended generality of the diagram

I have suggested in the introduction that Greek mathematical diagrams are more "schematic" than their modern counterparts, and that they serve to display the logical structure of the geometrical configuration, rather than to provide a metrically correct picture of the geometrical objects. This, I suggest, is a strength of ancient diagrams. Here we see a remarkable example of this strength. The general issue is that, if a diagram is taken to be a metrically correct picture, then it must specify a single range of metrical values. If in the diagram one line appears greater, equal, or smaller than another, this is because, in the geometrical situation depicted, the one line is indeed, respectively, greater, equal, or smaller than the other. In a diagram that is understood to be metrically correct, there is no such thing as an indefinite relation of size. In a schematic diagram, however, the relation of size between non-overlapping lines is indefinite. Whether the one appears greater than the other, or whether they appear equal, is just irrelevant, as long as they are indeed non-overlapping. Now let us compare Archimedes' diagram with Heiberg's. Heiberg permutes the letters A/B, so that he makes a choice: $A\Theta$ is greater not only than Δ, but also than $A\Gamma$. In geometrical reality, the situation admits of a certain generality or indefiniteness: $A\Theta$ can stand in any relation to $A\Gamma$. Archimedes allows $A\Theta$ to be non-overlapping with $A\Gamma$, in this way signaling this crucial indefiniteness. For Heiberg – who took his diagrams to be metrical – indefiniteness was ruled out from the outset, hence he failed to notice the loss of generality that resulted from his transformation of the diagram.

/3/

Given two unequal magnitudes and a circle, it is possible to inscribe a polygon inside the circle and to circumscribe another, so that the side of the circumscribed polygon has to the side of the inscribed polygon a smaller ratio than the greater magnitude to the smaller.

Let the given two magnitudes be A, B, and the given circle the one set down.[24] Now, I say that it is possible to produce the task.

(a) For let there be found two lines, Θ, ΚΛ, (b) of which let the greater be Θ, so that Θ has to ΚΛ a smaller ratio than the greater magnitude to the smaller, (c) and let ΛΜ be drawn from Λ at right
Eut. 252 <angles> to ΛΚ, (d) and let ΚΜ be drawn down from Κ, equal to Θ
[(1) for this is possible],[25] (e) and let two diameters of the circle be drawn, at right <angles> to each other, ΓΒ, ΔΖ.[26] (f) Now, bisecting the angle <contained> by ΔΗΓ, and bisecting its half, and doing the same ever again, (2) we will have left some angle smaller than twice the <angle contained> by ΛΚΜ.[27] (g) Let it be left, and let it be ΝΗΓ,
Eut. 253 (h) and let ΝΓ be joined. (3) Therefore ΝΓ is a side of an equilateral
polygon [(4) Since in fact the angle <contained by> ΝΗΓ measures the <angle contained> by ΔΗΓ,[28] (5) which is right, (6) and therefore the circumference ΝΓ measures the <circumference> ΓΔ (7) which is a quarter of a circle; (8) so that it <=ΝΓ> measures the circle, too, (9) therefore it is a side of an equilateral polygon.[29] (10) For this is obvious]. (i) And let the angle <contained by> ΓΗΝ be bisected by the line ΗΞ, (j) and, from Ξ, let ΟΞΠ touch the circle, (k) and let ΗΝΠ, ΗΓΟ be
Eut. 253 produced; (11) so that ΠΟ, too, is a side of the polygon circumscribed
around the circle, <which is> also equilateral[30] [(12) it is obvious that it is also similar to the inscribed, whose side is ΝΓ].[31] (13) And

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24 I.e. in the diagram.

25 See Eutocius. Also see Steps 2–3 in the following proposition and the footnote there.

26 Confusingly, the letter B is reduplicated in this proposition, serving once as a given magnitude and once as a point on the circle. See textual comments.

27 An extension of *Elements* X.1.

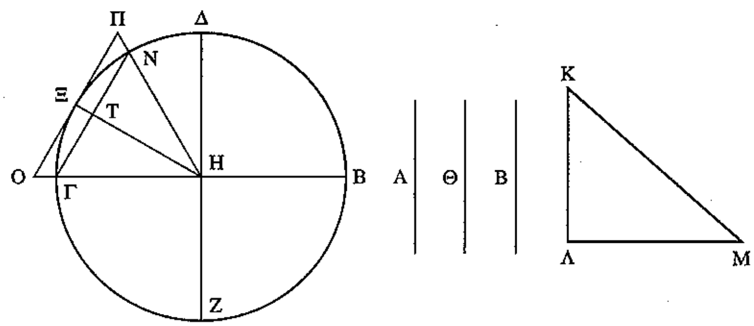
28 "To measure" is to have the ratio of a unit to an integer. Step e: ΝΗΓ has been produced by bisecting ΔΗΓ, recursively; hence their ratio is that of a unit to an integer (we will say it is $1:2^n$).

29 If this circumference measures the circle, the circle can be divided into a whole number of such circumferences. Dividing it in this way, and drawing the chords for each circumference, we will get a polygon inscribed inside the circle. All its sides are chords subtending equal circumferences, hence through *Elements* III.29 they are all equal: an equilateral polygon.

30 See Eutocius.

31 We want to show that ΝΓ is parallel to ΠΟ. (1) The angle at Ξ is right (*Elements* III.18). (2) The angle NHT is equal to the angle ΓHT (Step i). (3) NH and ΓH are equal (both radii, *Elements* I Def. 15). (4) And TH is common to the triangles NHT, ΓHT; (5) which are therefore congruent (2–4 in this argument, *Elements* I.4), (6) so the angle at T is right (5 in this argument, *Elements* I.13), (7) and so ΝΓ is parallel to ΠΟ (1, 6 in this argument, *Elements* I.28).

since the <angle contained> by NHΓ is smaller than twice the <angle contained> by ΛKM, (14) but it is twice the <angle contained> by THΓ, (15) therefore the <angle contained> by THΓ is smaller than the <angle contained> by ΛKM. (16) And the <angles> at Λ, T are
Eut. 254 right;[32] (17) therefore MK has to ΛK a greater ratio than ΓH to HT.[33] (18) And ΓH is equal to HΞ; (19) so that HΞ has to HT a smaller ratio (that is ΠO to NΓ)[34] (20) than MK to KΛ; (21) further, MK has to KΛ a smaller ratio than A to B.[35] (22) And ΠO is a side of the circumscribed polygon, (23) while ΓN <is a side> of the inscribed; (24) which it was put forward to find.



TEXTUAL COMMENTS

In Step e, the text refers to lines ΓB, ΔZ, a labeling that agrees with the diagram and which I follow. Heiberg has changed the letter B to E, in both text and diagram, first, because the letter B, in the manuscripts' reading, is reduplicated (used once for a given magnitude and another time for the end of a line), and second, because the letter E, in the manuscripts' reading, is not used at all, creating a gap in the alphabetical sequence. (Otherwise, the only gap is the missing letters P, Σ prior to the final T). The argument for Heiberg's correction is almost compelling, yet it does require making two separate transformations, in text and diagram. Generally speaking, there are enough scribal errors in the letters of both diagram and text to suggest that neither was systematically corrected to agree with the other, so that it is not very probable that a mistake in one could influence the other (though, of course, this remains a possibility). Finally, our overall judgment that letters in Greek diagrams are not reduplicated is based precisely on such textual decisions (see also Proposition I.44 below). With little certainty either way, I keep the manuscripts' reading.

I.3
All codices except B have B twice, on a line and on the circle. Thus certainly A. Codex B, and Heiberg following him, has changed the B on the circle to E. See textual comments.
Codices DG: Θ somewhat smaller than A, B. Codex H: Θ somewhat greater than A, B. Codex B exchanges the positions of B, Θ, and makes Θ considerably greater than A, B. Codices E, 4, have A=Θ=B. My conjecture is that, in codex A, the three lines were drawn rather freely, with small size differences (which, in truth, we now cannot reconstruct).
Codices BD have the side KΛ a little longer than the side ΛM, but this clearly represents bad judgment of the margins, as in all other manuscripts the triangle is as in the figure printed.
A strange mistake in Heiberg: he claims mistakenly that he has added a Π which is missing from the codices' diagrams (there is some confusion with I.4).

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[32] Λ right: Step c. T right: see note to Step 12. [33] See Eutocius.

[34] We would expect the word order "so that HΞ has to HT (that is ΠO to NΓ) a smaller ratio . . ." (see general comments). Then the content of Step 19 would have been clearer: it asserts that HΞ:HT::ΠO:NΓ (*Elements* VI.2).

[35] Steps b ($\Theta:K\Lambda < A:B$), d ($KM = \Theta$).

Steps 4–10 are probably rightly bracketed. Had they been in Eutocius' text, he would not have given his own commentary (besides, the subjective judgment that this piece of mathematics is of low quality, seems particularly strong here). Step 1 seems strange, but could be Archimedean (see also my footnotes on Eutocius' commentary on this step). Step 12 is not directly useful, but it is the *kind* of thing required by many assumptions of the proof, and does not have the look of a scholion (a scholion would prove, or gesture at a proof of such a claim). Finally, the strange word order of Step 19 may represent a textual problem. An interesting option (no more than an option) is that Archimedes completely left out the words "that is ΠΟ to ΝΓ" (accentuating the "hide-and-seek" aspect of this stage of the proof),[36] and then an honest interpolator inserted them at a *strange* location – signaling, perhaps, the interpolation *as such* by inserting it in the "wrong" position? – But this is *sheer* guesswork.

GENERAL COMMENTS

The scholiast's regress

The scholiast of 4–10 offers a good illustration of the scholiast's paradoxical position. This is the paradox of Carol (1895): you can never prove anything. You are arguing from P to Q; but you really need an extra premise, that P entails Q; and then you discover the extra premise, that P and "P entails Q" entail Q; and so on. So where to stop arguing? Mathematicians stop when they are satisfied (or when they think their audience will be) that the result is convincing enough. But scholiasts – for instance, a translator who offers also a brief commentary – face a tougher task. They should explain everything. Exasperating – and we sympathize with the author of Step 10. Having given the explanation, the scholiast wrings his hands in despair, realizing that this is not yet *quite* a *final*, decisive *proof*, and exclaims: "for this is obvious!"

/4/

Again, there being two unequal magnitudes and a sector, it is possible to circumscribe a polygon around the sector and to inscribe another, so that the side of the circumscribed has to the side of the inscribed a smaller ratio than the greater magnitude to the smaller.

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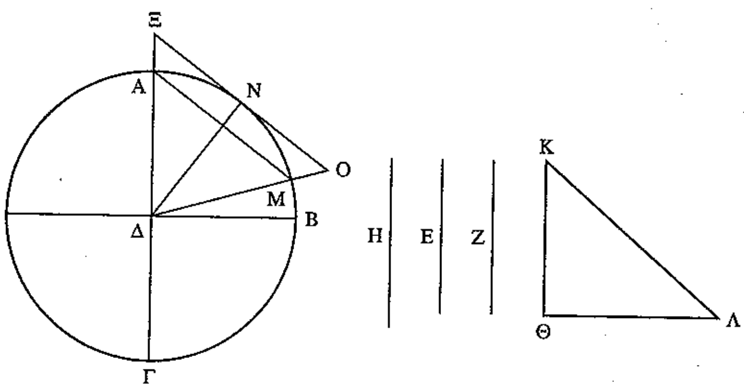
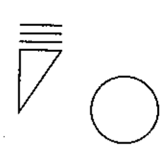
36 I refer to these features of the ending: Step 21 takes for granted Steps b and d, made much earlier (so that their tacit assumption is somewhat tricky); Step 24 asserts that the task has been produced, but to see this we actually need to piece together all of the Steps 19–23 (of which, Step 19 is doubly buried, in this "that is" clause which in turn is awkwardly placed).

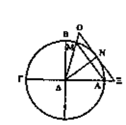
For let there be again two unequal magnitudes, E, Z, of which let the greater be E, and some circle ABΓ having Δ <as> a center, and let a sector be set up at Δ, <namely> AΔB; so it is required to circumscribe and inscribe a polygon, around the sector ABΔ, having the sides equal except BΔA, so that the task will be produced.

(a) For let there be found two unequal lines H, ΘK, the greater H, so that H has to ΘK a smaller ratio than the greater magnitude to the smaller [(1) for this is possible],[37] (b) and similarly, after a line is drawn from Θ at right <angles> to KΘ, (c) let KΛ be produced equal to H [(2) for <this is> possible, (3) since H is greater than ΘK].[38] (d) Now, the angle <contained> by AΔB being bisected, and the half bisected, and the same being made forever, (4) there will be left a certain angle, which is smaller than twice the <angle contained> by ΛKΘ.[39] (e) So let it be left <as> AΔM; (5) so AM is then a side of a polygon inscribed inside the circle.[40] (f) And if we bisect the angle <contained> by AΔM by ΔN (g) and, from N, draw ΞNO, tangent to the circle, (6) that <tangent> will be a side of the polygon circumscribed around the same circle, similar to the one mentioned;[41] (7) and similarly to what was said above (8) ΞO has to AM a smaller ratio than the magnitude E to Z.



I.4
Here is the first diagram where we begin to see codex B having a radically different lay-out – see unlabelled thumbnail. This has no consequence for reconstructing codex A. Moerbeke has changed the basic page layout, so that his diagrams were in the margins (instead of inside the columns of writing) forcing very different economies of space. I shall mostly ignore codex B's lay-out in the following. Codex G has a different arrangement for the circle, for which see labelled thumbnail; codex D rotates the circle slightly counterclockwise (probably for space reasons). Codices DGH have H extend a little lower than E, Z, which I follow, but codices BE4 have E=Z=H. In codex D, Z>E as well. Codex D has ΘK>ΘΛ. Codex G has Θ instead of E. Codex H has omitted Γ, has both K and H (!) instead of M. Heiberg has introduced, strangely, the letter Π at the intersection of NΔ/MA.





37 *SC* I.2.

38 *Elements* I.32: the sum of angles in a triangle is two right angles (so a right angle must be the greatest angle). *Elements* I.19: the greater angle is subtended by the greater side (so the right angle must subtend the greater side). Since the angle at Θ is right, KΛ must be greater than ΘK, which is guaranteed, indeed, by the relations KΛ=H (Step c), H>ΘK (Step a).

39 *Elements* X.1 Cor.

40 See Step 3 in *SC* I.3 (and the following Steps 4–10 there).

41 See Step 11 in *SC* I.3, and Eutocius on that step.

TEXTUAL COMMENTS

Step 1 is reminiscent of Step 1 in I.3 above. Both assert the possibility of an action. In Proposition 3, the possibility was guaranteed by facts external to this work. Here, the possibility is guaranteed by Archimedes' own Proposition 2. If Step 1 in this proposition were to be considered genuine, this would throw an interesting light on Archimedes' references to his own proximate results – but we can not say that this is genuine.

Steps 2–3 are even more problematic. Here, again, the steps assert the possibility of an action – the very same action whose possibility is asserted in Step 1 of the preceding proposition. There the text was no more than "for this is possible." Here, there is some elaboration, explaining *why* this is possible. The elaboration is the right sort of elaboration – better in fact than Eutocius' commentary on Step 1 of the preceding proposition. Everything makes sense – except for the fact that this elaboration comes only *the second time* that this action is needed. Why not give it earlier? There is something arbitrary about giving it here. But who says Archimedes was not arbitrary? In fact, he is perhaps more likely to be arbitrary than a commentator; but once again, we simply cannot decide.

GENERAL COMMENTS

Repetition of text, and virtual mathematical actions

Here starts the important theme of *repetition*. Many propositions in this book contain partial repetitions of earlier propositions. This proposition partially repeats Proposition 3.

Repetitions arise because the same argument is applied to more than a single object. In this case, the argument for circles is repeated for sectors. Modern mathematicians will often "generalize" – look for the genus to which the argument applies ("circles or sectors," for instance), and argue in general for this genus. This is not what is commonly done in Greek mathematics, whose system of classification to genera and species is taken to be "natural" – objects are what they are, a circle is a circle and a sector is a sector, and if a proof is needed for both, one tends to have a separate argument for each.

There are many possible ways of dealing with repetitions. One extreme is to pretend it is not there: to have precisely the same argument, without the slightest hint that it was given earlier in another context. This is then *repetition simpliciter*. Less extreme is a full repetition of the same argument, which is at least honest about it, i.e. giving signals such as "similarly," "again," etc. This is *explicit repetition*. Or repetitions may involve an abbreviation of the argument (on the assumption that the reader can now fill in the gaps): this is *abbreviated repetition*. And finally the entire argument may be abbreviated away, by e.g. "similarly, we can show . . . ," the readers are left to see for themselves that the same argument can be applied in this new case. This may be called the *minimal repetition*.

Usually what we have is some combination of all these approaches – which is strange. Once the possibility of a minimal repetition is granted, anything else is redundant. And yet the Greek mathematician labors through many boring repetitions, goes again and again through the same motions, and then airily remarks "and then the same can be shown similarly" – so why did you go through all those motions? Consider this proposition. First, there are many signals of repetition: "again" at the very start of both enunciation and setting-out; "similarly" at Steps b, 7. Also much is simply repeated: the basic construction phase (i.e. the construction up to and excluding the construction of the polygons themselves). (This may even be *more* elaborate here than in the preceding proposition – see textual comments.) The main deductive action, on the other hand, is completely abbreviated away: Steps 13–24 of the preceding proposition are abbreviated here into the "similarly" of Step 7.

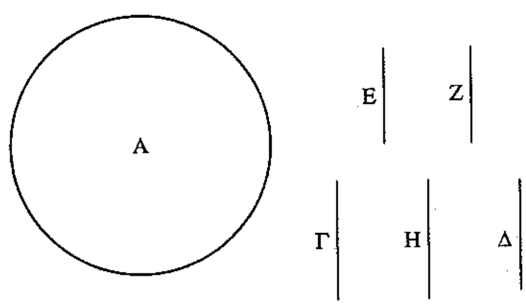
The most interesting part is sandwiched between the full repetition and the full abbreviation: the construction of the polygons. In Step d the angle is bisected. The equivalent in Proposition 3 is Step f, where *we bisect it*. The difference is that of passive and active voice, and it is meaningful. The active voice of Proposition 3 signifies the real action of bisecting. The passive voice of Proposition 4 signifies the virtual action of contemplating the *possibility* of an action. Going further in the same direction is the following: "(f) And if we bisect the angle <contained> by AΔM by ΔN (g) and, from N, draw ΞNO, tangent to the circle, (6) that <tangent> will be a side of the polygon . . ." The equivalent in the preceding proposition (Steps i–k, 11) has nothing conditional about it. Instead, it is the usual sequence of an action being done and its results asserted. The conditional of Proposition 4, Steps f–g, 6, is very different. Instead of doing the mathematical action, we argue through its *possibility* – through its virtual equivalent. So these two examples together (passive voice instead of active voice, conditional instead of assertion) point to yet another way in which the mathematical action can be "abbreviated:" not by chopping off bits of the text, but by standing one step removed from it, contemplating it from a greater distance – by substituting the virtual for the actual. This substitution, I would suggest, is essential to mathematics: the quintessentially mathematical way of abbreviating the infinite repetition of particular cases through a general argument, virtually extendible *ad infinitum*.

/5/

Given a circle and two unequal magnitudes, to circumscribe a polygon around the circle and to inscribe another, so that the circumscribed has to the inscribed a smaller ratio than the greater magnitude to the smaller.

Let a circle be set out, A, and two unequal magnitudes, E, Z, and <let> E <be> greater; so it is required to inscribe a polygon inside the polygon and to circumscribe another, so that the task will be produced.

(a) For I take two unequal lines, Γ, Δ, of which let the greater be Γ, so that Γ has to Δ a smaller ratio than E to Z;[42] (b) and, taking H as a mean proportional of Γ, Δ,[43] (1) therefore Γ is greater than H, as well. (c) So let a polygon be circumscribed around the circle, (d) and another inscribed, (e) so that the side of the circumscribed polygon has to that of the inscribed a smaller ratio than Γ to H [(2) as we learned];[44] (3) so, through this, (4) the duplicate ratio, too, is smaller than the duplicate.[45] (5) And the <ratio> of the polygon to the polygon is duplicate that of the side to the side [(6) for <the polygons are> similar],[46] (7) and <the ratio> of Γ to Δ is <duplicate that> of Γ to H;[47] (8) therefore the circumscribed polygon, too, has to the inscribed a smaller ratio than Γ to Δ; (9) much more, therefore, the circumscribed has to the inscribed a smaller ratio than E to Z.



I.5
The diagram follows the consensus of codices EH4. Codex D has all five lines in a single row (E, Z to the left of G, H, D), while codex G has the three line Γ, H, Δ more to the left (so that H is to the left of E, Δ is between E, Z), while codex B, of course, has a different layout altogether. In codex D, H=Γ=Δ; in codex G, Γ>Δ>H; in Codex B, Γ>H>Δ (so Heiberg).

TEXTUAL COMMENTS

Step 2 is an obvious interpolation (the verb "learn" must come from a scholiast, not from Archimedes). There is no compelling reason, however, to suspect Step 6. Heiberg tended to doubt any backwards-looking argument (everything starting with a "for"), as if they were all notes by scholia, whereas Archimedes himself only used forward-looking, "therefore" arguments. Heiberg *could* have been right: once again, our view of Archimedes' practice on this matter will have to depend on our reconstruction of Archimedes' text.

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42 *SC* I.2.

43 "Mean proportional:" X is mean proportional between A and B when A:X::X:B; here, Γ:H::H:Δ. *Elements* VI.13.

44 *SC* I.3.

45 "Duplicate ratio:" in our terms, if the original ratio is a:b, then the duplicate ratio is $a^2:b^2$.

46 *Elements* VI.20. 47 Step b.

GENERAL COMMENTS

Impatience revealed in exposition

The proof proper starts with a very remarkable first person. Are we to imagine extreme impatience: "now listen, I just do this, and then that, and that's all; clear now?" – impatience is a constant feature of the style throughout this introductory sequence, and this proposition, in particular, is a variation on the preceding ones, no more. The proof, once again, is very abbreviated. Possibly, Step 6 is by Archimedes, and if so, it would be an interesting case of abbreviation, as the text is literally "for similar" – no more than an indication of the kind of argument used; almost a footnote to *Elements* VI. 20.

What makes this preliminary sequence of problems important is not their inherent challenge, but their being required, later, in the treatise. These are mere stepping-stones. Briefly: later in the treatise, Archimedes will rely upon "compressing" circular objects between polygons, and these interim results are required to secure that the "compression" can be as close as we wish. Effectively, this sequence unpacks Postulate 5 to derive the specific results about different kinds of compressions. It is natural that a work of this kind shall start with such "stepping-stones," but the natural impatience with this stage of the argument favors a certain kind of informality that will remain typical of the work as a whole.

/6/

So similarly we shall prove that given two unequal magnitudes and a sector it is possible to circumscribe a polygon around the sector and to inscribe another similar to it, so that the circumscribed has to the inscribed a smaller ratio than the greater magnitude to the smaller. And this is obvious, too: that if a circle or a sector are given, and some area, it is possible, by inscribing equilateral polygons inside the circle or the sector, and ever again inside the remaining segments, to have as remainders some segments of the circle or the sector, which are smaller than the area set out. For these are given in the *Elements*.[48]

But it is to be proved also that, given a circle or a sector, and an area, it is possible to circumscribe a polygon around the circle or the sector, so that the remaining segments of the circumscription[49] are smaller

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[48] *Elements* X.1; or this may be a wider reference to the "method of exhaustion," where polygons are inscribed in this way, first used in the *Elements* in XII.2. (This assumes – which need not necessarily be true – that this reference is by Archimedes, and that the reference is to something largely akin to the *Elements* as we have them.)

[49] "Circumscription" stands for περιγραφή, a deviation from the standard περιγραφέν, "the circumscribed <polygon>."

"The remaining segments of the circumscription" are the polygon minus the circle.

than the given area. (For, after proving for the circle, it will be possible to transfer a similar argument to the sector, as well.)

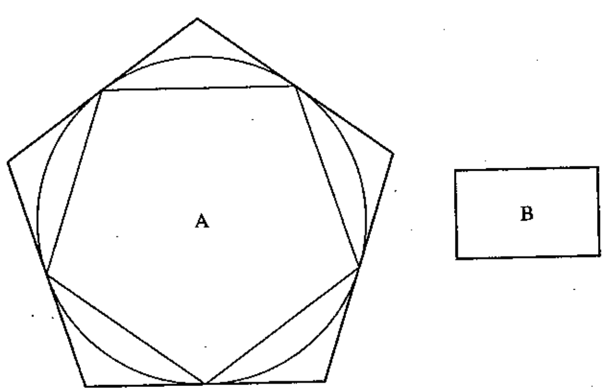
Let there be given a circle, A, and some area, B. So it is possible to circumscribe a polygon around the circle, so that the segments left between the circle and the polygon are smaller than the area B; (1) for <this>, too: <that>, there being two unequal magnitudes – the greater being the area and the circle taken together, the smaller being the circle – (a) let a polygon be circumscribed around the circle and another inscribed, so that the circumscribed has to the inscribed a smaller ratio than the said greater magnitude to the smaller.[50] (2) Now, this is the circumscribed polygon whose remaining <segments> will be smaller than the area set forth, B.

(3) For if the circumscribed has to the inscribed a smaller ratio than: both the circle and the area B taken together, to the circle itself, (4) the circle being greater than the inscribed, (5) much more will the circumscribed have to the circle a smaller ratio than: both the circle and the area B taken together, to the circle itself; (6) and therefore, dividedly, the remaining <segments> of the circumscribed polygon have to the circle a smaller ratio than the area B to the circle;[51] (7) therefore the remaining <segments> of the circumscribed polygon are smaller than the area B.[52]

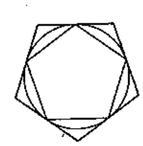
(8) Or like this: since the circumscribed has to the circle a smaller ratio than: both the circle and the area B taken together, to the circle,

Eut. 254 (9) through this, then, the circumscribed will be smaller than <them> taken together;[53] (10) and so the whole of the remaining <segments> will be smaller than the area B.

(11) And similarly for the sector, too.



I.6
Codex G has the pentagons upside-down, as in the thumbnail. The codices, except for codices DG, introduce a letter E at the top vertex of the circumscribing pentagon. Codex D has the height of the rectangle greater than its base.



[50] *SC* I.5.

[51] *Elements* V.17 shows A:B::C:D → (A–B):B::(C–D):D which is called "division." This is extended here to cover inequalities (as can be supplied also from Eutocius' commentary to Proposition 2).

[52] *Elements* V.10. [53] *Elements* V.10.

TEXTUAL COMMENTS

The number "6" for the proposition probably appeared at the top of this proposition, in the lost codex A. It appears there in all the manuscripts dependent upon A – with the exception of Moerbecke's translation. There is a gap here in the Palimpsest. At any rate, the manuscripts begin to diverge in their numbering; I will not report the further divergences, which are considerable. More important, we begin to see why they diverge: the text really does not come in clear units. The text makes digressions, repetitions, alternations: it is not the simple sequence of claim and proof, as in the Euclidean norm.

We have here another rare reference to the *Elements* (cf. Prop. 2, Step 1), at the end of the second paragraph. It is not to be dismissed straight away, since it is *functional*. The reference to the *Elements* is this time meant to explain why one thing is obvious (it's in the *Elements*!) while the other requires proof (it isn't in the *Elements*!). It thus may perhaps be authorial.

GENERAL COMMENTS

Lack of pedagogic concerns

The second paragraph has the strange word "circumscription," which, as explained in the footnote, is the same as "circumscribed." This shift in vocabulary is insignificant except for betraying a certain looseness – a looseness which can be seen with more serious logical points. Most important, the text is vague: what are the "remaining segments" mentioned again and again? Remaining from *what*? Archimedes just takes it for granted that his meaning is understood (namely that they are what is left after we take away the circle). Possibly, such cases may show that Archimedes did not seriously try to put himself in the place of the prospective reader. Archimedes betrays a certain haughtiness, even, towards such a reader: the "impatience" towards the argument easily becomes an impatience towards the reader. Rigor and clarity are sacrificed for the sake of brevity.

Meta-mathematical interests displayed in the text

The second paragraph ends remarkably, with a meta-mathematical statement "after proving for the circle, it will be possible to transfer a similar argument to the sector as well." A corollary following the proof is normal (as in fact we get in Step 11), but to anticipate the corollary in such a way is a remarkable intrusion of the second-order discourse inside the main, first-order discourse: even before getting down to the proof itself, Archimedes notes its possible extendability. The same is true, of course, for the whole of the beginning of this proposition.

Further, consider the alternative proof in Steps 8–10. It may of course be a later scholion, but it may also be authentic – and was already known to Eutocius. Assume then that this is by Archimedes: why should he have it? Now, the ratio-manipulation with the "dividedly" move in Step 6 is a rigorous way to derive the inequality there – based on the tools of proportion with which the Greek mathematician is always at home. But it is also an artificial move and one strongly feels that many other manipulations could do as well. The alternative offered in Steps 8–10 follows, I would suggest, a more direct visual intuition. The decision to keep both proofs is interesting, and may reveal Archimedes wavering between two ideals of proof.

The impression is this. The first few propositions are less interesting in their own right; they are obviously anticipatory. Archimedes gradually resorts to abbreviation, to expressions of impatience. In this proposition, it is as if he finally moves away from the sequence of propositions, looking at them from a certain distance, discoursing not so much *through* them as *about* them. As we shall have occasion to see further below, such modulations of the authorial voice are used by Archimedes to guide us through the text; in this case, this final modulation of voice, from first order to second order, serves to signal the conclusion of this preliminary sequence.

/7/

If a pyramid having an equilateral base is inscribed in an equilateral cone, its <=the pyramid's> surface without the base is equal to a triangle having a base equal to the perimeter of the base <of the pyramid> and, <as> height, the perpendicular drawn from the vertex on one of the sides of the base <of the pyramid>.

Let there be an isosceles cone, whose base is the circle ΑΒΓ, and let an equilateral pyramid be inscribed inside it, having ΑΒΓ <as> base; I say that its <=the pyramid's> surface without the base is equal to the said triangle.

(1) For since the cone is isosceles, (2) and the base of the pyramid is equilateral, (3) the heights of the triangles containing the pyramid[54] are equal to each other.[55] (4) And the triangles have <as> base the lines ΑΒ, ΒΓ, ΓΑ, (5) and, <as> height, the said; (6) so that the triangles are equal to a triangle having a base equal to ΑΒ, ΒΓ, ΓΑ, and, <as> height, the said line[56] [(7) that is the surface of the pyramid without the triangle ΑΒΓ].[57]

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54 By "triangles containing the pyramid" are meant the faces of the pyramid (excluding the base). By "their heights" are meant those drawn from the vertex of the cone to the sides of the base.

55 The move from Steps 1–2 to 3 can be obtained in several ways; see general comments.

56 *Elements* VI.1.

57 Step 7 refers by the words "that is" to "the triangles" mentioned at the beginning of Step 6 (and not to the "triangle having a base equal to ΑΒ, ΒΓ, ΓΑ" mentioned later in Step 6). This is as confusing in the original Greek as it is in the translation.

Note that the expression does not follow naturally from the definitions. If anything, the definitions provide a special meaning for the adjective "solid" (when applied to "rhombus," it refers to a special kind of object). This adjective, however, is precisely what gets dropped in the actual use of the expression. Further, the definitions introduce a general object – any co-based and co-axial pair of cones – whereas the development of the treatise demands a special kind of object (the isosceles – where, incidentally, co-axiality follows from co-basedness). Thus the definitions introduce an expression that is not used later on, and refer to an object that is not used later on. In both sides of the semiotic equation – both sign and signified – the definitions do not fit easily with the treatise as a whole. This is indeed similar to the way in which the axiomatic material (especially concerning the notion of "concave in the same direction") never gets directly applied. In general, then, the introductory material does not really govern the main text, which is instead governed by internal, "natural" processes, such as the evolution of formulaic expressions.

/21/

If an even-sided and equilateral polygon is inscribed inside a circle, and lines are drawn through, joining the sides of the polygon[212] (so that they are parallel to one – whichever – of the lines subtended by two sides of the polygon),[213] all the joined <lines> have to the diameter of the circle that ratio, which the <line> (subtending the <sides, whose number is> smaller by one, than half <the sides>) <has> to the side of the polygon.

Let there be a circle, ABΓΔ, and let a polygon be inscribed in it, AEZBHΘΓMNΔΛK, and let EK, ZΛ, BΔ, HN, ΘM be joined; (1) so it is clear that they are parallel to the <line> subtended by two sides of the polygon;[214] Now I say that all the said <lines>[215] have to the diameter of the circle, AΓ, the same ratio as ΓE to EA.

(a) For let ZK, ΛB, HΔ, ΘN be joined; (2) therefore ZK is parallel to EA (3) while BΛ <is parallel> to ZK (4) and yet again ΔH to BΛ, (5) and ΘN to ΔH (6) while ΓM <is parallel> to ΘN [(7) and since

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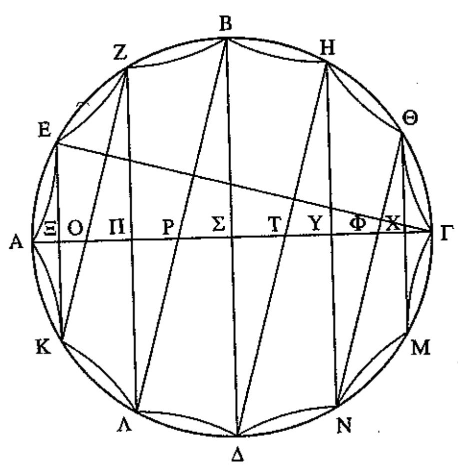
212 In this formula, "sides" mean "vertices" (or, as Heiberg suggests, "angles").

213 This means, in diagram terms, that the choice of EK is arbitrary: any other line "subtended by two sides of the polygon," e.g. AZ, could do as a starting-point for the parallel lines.

214 Why is this clear? Perhaps through the equality of angles on equal chords (*Elements* III.27, 28), also the equality of alternate angles in parallels (*Elements* I.27). However obtained, an equality such as "(angle EKZ) = (angle KZΛ)" establishes "EK parallel to ZΛ."

215 The reference is to all the *parallel* lines.

EA, KZ are two parallels, (8) and EK, AO are two lines drawn through]; (9) therefore it is: as EΞ to ΞA, KΞ to ΞO.[216] (10) But as KΞ to ΞO, ZΠ to ΠO, (11) and as ZΠ to ΠO, ΛΠ to ΠP, (12) and as ΛΠ to ΠP, so BΣ to ΣP, (13) and yet again, as BΣ to ΣP, ΔΣ to ΣT, (14) while as ΔΣ to ΣT, HY to YT, (15) and yet again, as HY to YT, NY to YΦ, (16) while as NY to YΦ, ΘX to XΦ, (17) and yet again, as ΘX to XΦ, MX to XΓ[217] [(18) and therefore all are to all, as one of the ratios to one];[218] (19) and therefore as EΞ to ΞA, so EK, ZΛ, BΔ, HN, ΘM to the diameter AΓ. (20) But as EΞ to ΞA, so ΓE to EA;[219] (21) therefore it will be also: as ΓE to EA, so all the <lines> EK, ZΛ, BΔ, HN, ΘM to the diameter AΓ.



I.21
Codices BG, followed by Heiberg, have straight lines instead of arcs in the polygon. Codices EH4 have Σ instead of E. Codex E has K instead of X.

TEXTUAL COMMENTS

Heiberg brackets Steps 7 and 8. If this is indeed an interpolation then it is noteworthy: an interpolated argument *preceding* its result – which is also a very *cryptic* interpolation. It seems to state something like "if two lines are drawn through two parallel lines, so that the two lines drawn through cut each other between the parallels, the two resulting triangles are similar" – which may be proved through *Elements* I.29, 32, VI.4. At any rate, the result implicit

216 *Elements* I.29, 32, VI.4.

217 In Steps 9–17, the odd steps are based on *Elements* I.29, 32, VI.4 (besides, of course, a conceptualization of the configuration similar to that suggested at Steps 7–8). The even steps are based on Step 1, as well.

218 *Elements* V.12. The formulation is literally meaningless: instead of "one of the ratios," it should have been "one of the terms" or "one of the lines."

219 *Elements* VI.4 (but more than this is required to see that the triangles are similar, e.g. we may notice that, since AB is one-quarter the circle, so following *Elements* III.31 the angle at Σ must be right, and since the lines are parallel it follows that the angle at Ξ, too, is right, and then we apply *Elements* VI.8).

here is *not* in the *Elements*. Are we to imagine a late interpolator assuming extra-Euclidean knowledge? Perhaps, then, this could be Archimedean (and similar to Proposition 16, Step 4). Once again, then, this would be a result, not from some lost, extra-Euclidean *Elements*, but simply one that Archimedes had no patience to go through (even though it was nowhere proved as such), opting instead for a vague indication of its possible grounds.

Step 18 is strange as well. "All" appears in the neuter – signifying what? (Antecedents to consequents? Anyway, not "ratios," which are masculine.) Further, "a ratio to a ratio" is meaningless: we should have had "as one of the lines to one" or "as one of the terms to one." Such solecism may be the mark of the clumsy interpolator – unless, of course, it is a mark of the careless author.

GENERAL COMMENTS

The strange nature of the proposition

Now to the main feature of this proposition, already mentioned in the textual comments: it is strange.

Take the diagram. In this proposition a new, striking diagrammatic practice appears for the first time: the sides of the polygon are represented by curved, concave lines. This is probably done to aid the resolution of the arcs and the chords (a considerable problem with dodecagons), but at any rate, this novel practice marks a radical departure from simple, "pictorial" representation. It is, I think, somewhat improbable that a scribe would invent such a practice, in defiance of his sources. If so, we may have in this practice a hint of Archimedes' own diagrammatic practices. At any rate, the strange diagram marks a strange proposition.

The diagram leads immediately to another strange feature of this piece of text, this time in the reference to the diagram in the name of the dodecagon with its twelve letters. The longest we had so far was five letters, in Proposition 11, Steps 19–20; in general, names are usually within the range of one to three letters, with occasionally four letters: longer names are very rare. (Thus, for instance, the polygon of the first proposition is nowhere named, partly perhaps to avoid an extremely long name.) I believe this long name is intentionally playful – a long, strange name, in a strange proposition.

More important is the logical structure of the proposition. It enumerates facts, in long lists (similar to the strange, long name of the dodecagon); beyond listing the facts, the proposition practically does not *argue*. Its arguments are implicit (and difficult to reconstruct): that the lines are parallel (see n. 214 above), that this entails a certain set of proportions (see textual comments), and that those proportions are reducible to the ratio ΓE:EA (see n. 219 above). So a strange logical flow, as well.

Most importantly, the subject matter itself is strange: it has nothing to do with anything we know, from this book or from, say, the *Elements*. Instead, this proposition is a complete break. Specifically, it has nothing to do with sphere and cylinder. We had thought we had got closer, with conic solids treated by "chapter 4" (Propositions 17–20), but we now move suddenly to an unexpected, unconnected interlude, once again signaling a major transition in the work.

/33/

The surface of every sphere is four times the greatest circle of the <circles> in it.

For let there be some sphere, and let the <circle> A be four times the great circle; I say that A is equal to the surface of the sphere.

(1) For if not, it is either greater or smaller. (a) First let the surface of the sphere be greater than the circle <=A>. (2) So there are two unequal magnitudes: the surface of the sphere, and the circle A; (3) therefore it is possible to take two unequal lines, so that the greater <line> has to the smaller a ratio smaller than that which the surface of the sphere has to the circle.[290] (b) Let the lines B, Γ be taken, (c) and let Δ be a mean proportional between B, Γ, (d) and also, let the sphere be imagined cut by a plane <passing> through the center, at the circle EZHΘ, (e) and also, let a polygon be imagined inscribed inside the circle, and circumscribed, so that the circumscribed is similar to the inscribed polygon, and the side of the circumscribed has <to the inscribed> a smaller ratio than that which B has to Δ[291] [(4) therefore the duplicate ratio, too, is smaller than the duplicate ratio.[292] (5) And the duplicate <ratio> of B to Δ is the <ratio> of B to Γ,[293] (6) while, duplicate <the ratio> of the side of the circumscribed polygon to the side of the inscribed, <is> the <ratio> of the surface of the circumscribed solid to the surface of the inscribed];[294] (7) therefore the surface of the figure circumscribed around the sphere has to the surface of the inscribed figure a smaller ratio than the surface of the sphere to the circle A;[295] (8) which is absurd; (9) for the surface of the circumscribed is greater than the surface of the sphere,[296] (10) while the surface of the inscribed <figure> is smaller than the circle A [(11) for the surface of the inscribed has been proved to be smaller than four times the greatest circle of the <circles> in the sphere,[297] (12) and the

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290 *SC* I.2. 291 *SC* I.3.

292 In modern terms, the step argues from (circumscribed side:inscribed side<B:Δ) to ((circumscribed side:inscribed side)^2<(B:Δ)^2). (This intuitive derivation is not proved in the *Elements*.)

293 Step c, *Elements* VI. 20 Cor. 2. 294 *SC* I.32.

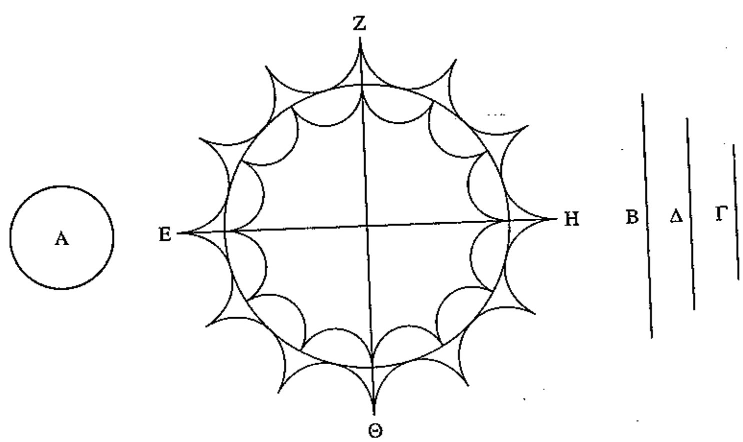
295 By applying the two substitutions of Steps 5, 6 on the claim of Step 4, we get the (implicit) claim that ((surface of circumscribed solid:surface of inscribed solid)<(B:Γ)), but by Step b (interpreted through Step 3) ((B:Γ)<(surface of sphere:circle A)), and the claim of Step 7 is seen to hold. As we were used to do in Propositions 13–14, the absurdity argued in the following few steps is better seen if we apply an implied "alternately" operation (*Elements* V.16) on Step 7, to yield (7*) ((surface of circumscribed solid:surface of sphere)<(surface of inscribed solid:circle A)).

296 *SC* I.28. 297 *SC* I.25.

circle A is four times the great circle]. (13) Therefore the surface of the sphere is not greater than the circle A.

So I say that neither is it smaller. (f) For if possible, let it be <smaller>. (g) And similarly let the lines B, Γ be found, so that B has to Γ a smaller ratio than that which the circle A has to the surface of the sphere, (h) and <let> Δ <be> a mean proportional between B, Γ, (i) and again let it be inscribed and circumscribed,[298] so that the <side> of the circumscribed has <to the inscribed> a smaller ratio than the <ratio> of B to Δ [(14) therefore the duplicates, too];[299] (15) therefore the surface of the circumscribed has to the surface of the inscribed[300] a smaller ratio than [B to Γ.[301] (16) But B has to Γ a smaller ratio than] the circle A to the surface of the sphere; (17) which is absurd; (18) for the surface of the circumscribed is greater than the circle A,[302] (19) while the <surface> of the inscribed is smaller than the surface of the sphere.[303]

(20) Therefore neither is the surface of the sphere smaller than the circle A. (21) And it was proved, that neither is it greater; (22) therefore



I.33
Codex B, followed by Heiberg, has straight lines instead of arcs in the polygon. Codex D inserts an extra, circumscribing circle. Codex G has the three lines (similarly arranged internally) between the two circles (and not to their right). Codex D has the circle A lower relative to the main figure, while Codex B has a different layout altogether. Codices AC have omitted Θ.

298 No subject for the verb is specified in the original. The text as it stands is very confusing, failing to distinguish between sides, areas, or volumes, or between circumscribed/inscribed polygons and circumscribed/inscribed solids. The reference would be to the sides of the polygons, if we follow the line of argument from Step e above but, alternatively, the text may assume that all the inequalities can be achieved simultaneously, so that no need is felt to distinguish between the various circumscribed and inscribed pairs.

299 The same as Step 4 above.

300 This time the reference is clearly to circumscribed and inscribed *solids* (since only solids have *surfaces*).

301 This is the same as the *implicit* result of Steps 4–6 above, as explained in note to Step 7.

302 *SC* I.30. 303 *SC* I.23.

the surface of the sphere is equal to the circle A, (23) that is to four times the great circle.

TEXTUAL COMMENTS

There are two problems here. The first is Steps 11–12, which are a very elementary unpacking of known results and constructions. They could be genuine, but are likely to be scholiastic.

The second, very complex problem arises with the main argument in both parts of the proof. Heiberg's minimal reading gets rid of Steps 4–6 and 14, and reduces Steps 15–16 to a single step. So in both cases there is, according to Heiberg, no argument at all: the construction gives rise, directly, to the absurd result (in Steps 7, 15+16).

The strongest evidence against Heiberg's hypothesis is that, even with all the bracketed steps considered genuine, the argument is still very sketchy. The passage from 4–6 to 7 leaves out an important implicit claim (see note to Step 7); Step 14 is extremely condensed (to the point of being misleading and ambiguous Greek: "the duplicates," neuter, does not refer to anything clear); Steps 15+16, as I shall explain, are elliptic as well. I therefore tend to believe all those steps are genuine.

Steps 15–16 form an especially intriguing structure. The Palimpsest has: "(15) therefore the surface of the circumscribed has to the surface of the inscribed a smaller ratio than B to Γ. (16) But B has to Γ a smaller ratio than the circle A to the surface of the sphere," while A had: "(15) therefore the surface of the circumscribed has to the surface of the inscribed a smaller ratio than the circle A to the surface of the sphere."

The claim made by manuscript A is left implicit in the Palimpsest. A, on the other hand, omitted the argument for this claim (namely, Step 16 of the Palimpsest). The Palimpsest leaves a *result* unsaid; A left an *argument* unsaid. Mathematically speaking, A seems the more likely reading: Steps 18–19 refer to the claim asserted at A. Purely textually, however, the case of the Palimpsest seems stronger – and textual considerations must take precedence.

The textual argument is this. There are no parallels of a scholiastic expansion of this kind in the text of C, while, on the other hand, the omission of A can be easily understood as an homoeoteleuton ("a smaller ratio than:" in the Greek, a sequence of twenty-one letters, repeated exactly!).

It then seems that Archimedes left a result unsaid at the second part of the proof, while, at the first part of the proof, he (or the interpolator?) has left an argument unsaid (at Step 7). This might be intentional: one thing we have seen throughout this work is a tendency to have parallel passages display as much variation between them as possible.

GENERAL COMMENTS

A sense of accomplishment

Note that the enunciation is much briefer than many preceding enunciations. Mathematical significance tends to be in inverse proportion to length: the key results are important just because they state some direct, simple relation, whereas interim results, of no inherent significance, may often involve cumbersome, ungainly complications. And indeed we have now finally reached a result whose interest is self-evident.

The sense of accomplishment – of the book having reached a goal – is conveyed in several ways. First, the texture of the language is particularly crisp, the brevity of the enunciation being carried over to the argument itself: I return to this in a note below.

Secondly, the narrative placement of the proposition is remarkable. To begin with, we return to use Propositions 2–3 (not directly used since Proposition 5!). The return, here, to such an early proposition, is a majestic display of design. Further, Propositions 2–3 gave rise to a specific procedure, of proof through a double absurdity, manipulating proportions (last used in Proposition 14). We thought this too was a secondary tool, necessary to produce no more than some interim results. Now we discover how crucial the tool is for the main result itself. The entire mechanism is recalled: assuming both "greater" and "smaller;" obtaining absurdities through proportion inequalities; relying on the assumption that the ratio of the greater to the smaller is greater than the ratio of the smaller to the greater. Nothing superfluous: all the threads of the argument are gathered together – a pulling together that contrasts with the seemingly haphazard, centrifugal structure of the first half of the book. For, besides the use of Propositions 2–3, and the recalling of the procedure of Propositions 13–14, one notes the use of many recent results on surfaces and solids, such as *SC* I.23, 25, 28, 30, 32, which in turn of course depended on previous results for polygons and cones. This is the culmination of "chapter 5," the main chapter of the book, and the sphere has been measured: the surface here and, in the following proposition, the even more important volume.

Elegance of expression

Consider the following two details, both typical of the way in which this proposition, while involving so many threads, is still presented as a single, unified argument. Mathematical elegance is obtained through some specific verbal tools.

First, the setting-out: "let the <circle> A be four times the great circle." Notice that the only purpose of this construction is to allow Archimedes to avoid repeating again and again the cumbersome expression "four times the great circle" (or, worse, "four times the greatest circle of the <circles> in the sphere"). It is thus more a verbal than a geometrical construction.

Second, consider Step e: ". . . the side of the circumscribed has <to the inscribed> a smaller ratio than that which B has to Δ" (similarly, Step i). That the words "to the inscribed" can be omitted is a mark of how formulaic this expression has become by now. (Besides revealing, once again, that a ratio is felt to be a property belonging to the antecedent. Incidentally, the repetition of the omission in both Steps e and i proves that this is not a textual corruption: see also Step 8 in the following proposition.) Once again, a certain brevity is allowed. Because we can sometimes predict what the ratio will be *to*, such predictable ratios – the main ratios negotiated in this proposition – may be directly assigned to individual objects, simplifying the usual complication arising from the fact that ratio and proportion are many term relations.

In many previous propositions, Archimedes, it seemed, practically reveled in cumbersome objects standing to each other in complex relations. (Recall the line equal to all lines parallel to one line – whichever – of the lines subtended by two sides of the polygon! Remember the proportions in which it participated! Consider, e.g. the enunciation of Proposition 21.) The elegance of this proposition is designed to contrast with such cumbersome propositions – just as its pulling together of earlier results is designed to contrast with an earlier, seemingly haphazard structure.

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Every sphere is four times a cone having a base equal to the greatest circle of the <circles> in the sphere, and, <as> height, the radius of the sphere.

For let there be some sphere and in it a great circle ΑΒΓΔ. (1) Now, if the sphere is not four times the said cone, (a) let it be, if possible, greater than four times; (b) and let there be the cone Ξ, having a base four times the circle ΑΒΓΔ, and a height equal to the radius of the sphere. (2) Now, the sphere is greater than the cone Ξ.[304] (3) So there will be two unequal magnitudes: the sphere and the cone. (4) Now, it is possible to take two unequal lines, so that the greater has to the smaller a smaller ratio than that which the sphere has to the cone Ξ.[305] (c) Now,
Eut. 265 let them <=the two unequal lines> be K, H, (d) and I, Θ taken so that they exceed each other, K <exceeding> I, and I <exceeding> Θ, and Θ <exceeding> H, by an equal <difference>,[306] (e) and let also a polygon be imagined inscribed inside the circle ΑΒΓΔ (let the number of its sides be measured by four), (f) and another, circumscribed, similar to the inscribed, as in the earlier <constructions>, (g) and let the side of the circumscribed polygon have to the <side> of the inscribed a smaller ratio than that which K has to I,[307] (h) and let ΑΓ, ΒΔ be diameters in right <angles> to each other. (i) Now, if the plane, in which are the polygons, is carried in a circular motion (the diameter ΑΓ remaining fixed), (5) there will be figures, the one inscribed in the sphere, the other circumscribed, (6) and the circumscribed will have to the inscribed a ratio triplicate of the side of the circumscribed

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[304] Steps a, b; Interlude, recalling *Elements* XII.11. [305] *SC* I.2.

[306] The resulting sequence of lines is an *arithmetical* progression, with equal *differences*. See Eutocius' commentary for further discussion.

[307] *SC* I.3.

<polygon> to the <side> of the <polygon> inscribed inside the circle ABΓΔ.[308] (7) But the side has to the side a smaller ratio than K to I; (8) so that the circumscribed figure[309] has a smaller ratio than triplicate <of the ratio> of K to I. (9) But, also, K has to H a greater ratio than triplicate that which K has to I[310] [(10) for this is obvious through the lemmas];[311] (11) much more, therefore, that which was circumscribed has to the inscribed a smaller ratio than that which K has to H. (12) But K has to H a smaller ratio than the sphere to the cone Ξ;[312] (13) and alternately;[313] (14) which is impossible; (15) for the circumscribed figure is greater than the sphere,[314] (16) while the inscribed is smaller than the cone Ξ [(17) through the fact that the cone Ξ is four times the cone having a base equal to the circle ABΓΔ, and a height equal to the radius of the sphere,[315] (18) while the inscribed figure is smaller than four times the said cone].[316] (19) Therefore the sphere is not greater than four times the said <cone>.

(j) Let it be, if possible, smaller than four times; (20) so that the sphere is smaller than the cone Ξ.[317] (k) So let the lines K, H be taken, so that K is greater than H and has to it a smaller ratio than that which the cone Ξ has to the sphere,[318] (l) and let Θ, I be set out, as before, (m) and let a polygon be imagined inscribed inside the circle ABΓΔ, and another circumscribed, so that the side of the circumscribed has to the side of the inscribed a smaller ratio than K to I, (n) and the rest constructed in the same way as before; (21) therefore, the circumscribed solid figure will also have to the inscribed a ratio triplicate the side of the <polygon> circumscribed around the circle ABΓΔ to the <side> of the inscribed.[319] (22) But the side has to the side a smaller ratio than K to I; (23) so the circumscribed figure will have to the inscribed a smaller ratio than triplicate that which K has to I. (24) And K has to H a greater ratio than triplicate that which K has to I;[320] (25) so that the circumscribed figure has to the inscribed a smaller ratio than

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308 *SC* I.32.

309 Again, the words "to the inscribed" are omitted. (Compare Steps e, i in the preceding proposition.)

310 See Eutocius' commentary on Step d above.

311 Most probably, a reference to Eutocius.

312 Step c. The implicit result of Steps 11–12 taken together is (circumscribed: inscribed<sphere:cone). It is to this implicit result that Step 13 refers.

313 I.e: (circumscribed:sphere<inscribed:cone) (*Elements* V.16).

314 *SC* I.28.

315 Step b; Interlude recalling *Elements* XII.11.

316 *SC* I.27.

317 Steps b, j; Interlude, recalling *Elements* XII.11.

318 *SC* I.2.

319 *SC* I.32. As for the "also," it refers to the earlier use of the same principle in the first part of this proposition.

320 See Eutocius' commentary.

K to H. (26) But K has to H a smaller ratio than the cone Ξ to the sphere[321]; (27) which is impossible; (28) for the inscribed is smaller than the sphere,[322] (29) while the circumscribed is greater than the cone Ξ.[323] (30) Therefore neither is the sphere smaller than four times the cone having the base equal to the circle ABΓΔ and, <as> height, the <line> equal to the radius of the sphere. (31) And it was proved that neither is it greater. (32) Therefore <it is> four times.

/Corollary/

And, these being proved, it is obvious that every cylinder having, <as> base, the greatest circle of the <circles> in the sphere, and a height equal to the diameter of the sphere, is half as large again as the sphere, and its surface with the bases is half as large again as the surface of the sphere.

(1) For the cylinder mentioned above is six times the cone having the same base, and a height equal to the radius <=of the sphere>,[324] (2) and the sphere has been proved to be four times the same cone;[325] (3) so it is clear that the cylinder is half as large again as the sphere. (4) Again, since the surface of the cylinder (without the bases) has been proved equal to a circle whose radius is a mean proportional between: the side of the cylinder, and the diameter of the base,[326] (5) and the side of the said cylinder (which is around the sphere) is equal to the diameter of the base [(6) it is clear that their mean proportional will then be equal to the diameter of the base],[327] (7) and the circle having the radius equal to the diameter of the base, is four times the base,[328] (8) that is <four times> the greatest circle of the <circles> in the sphere, (9) therefore the surface of the cylinder without the bases, too, will be four times the great circle; (10) therefore the whole surface of

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[321] Step k. The implicit result of Steps 25–6 is: ((the circumscribed figure:the inscribed)<(cone Ξ:sphere)). It is to this implicit result that Step 27 refers.

[322] *SC* I.28. [323] *SC* I.31.

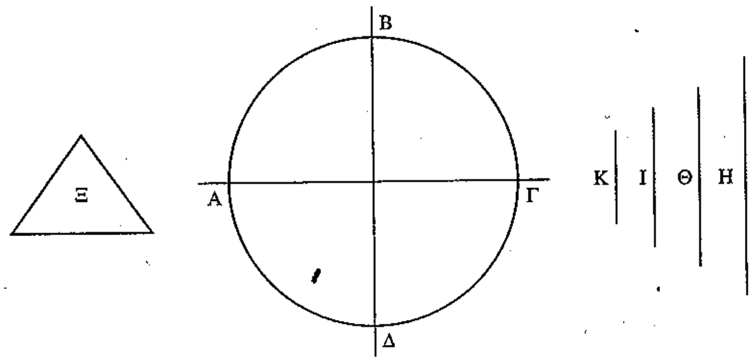
[324] *Elements* XII.10: the cone with the same base and height as the cylinder is one third the cylinder. Interlude, recalling *Elements* XII.14: cones with the same base are to each other as their height (the cylinder mentioned here has the *diameter* as its height, the cone has the *radius* as its height).

[325] *SC* I.34. [326] *SC* I.13.

[327] The assumption is that if A:B::B:C, and A=C, then A=B=C. (See general comments.) The implicit result of Steps 4–6 is that "the surface of the cylinder (without the bases) is equal to a circle, whose radius is the diameter of the base." It is this result which, together with Step 8, yields Step 9.

[328] *Elements* XII.2.

the cylinder, with the bases, will be six times the great circle. (11) And also, the surface of the sphere is four times the great circle.[329] (12) Therefore the whole surface of the cylinder is half as large again as the surface of the sphere.



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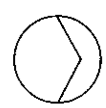
Codex C is not preserved for this diagram. Codex B has switched, in a sense correctly, the order of sizes among the four lines. Heiberg naturally does the same. A mistake may have been made on codex A; or perhaps all the diagram sets out to do is to represent "four lines in a sequence of sizes," the actual metrical relations being ignored. Codex B represents the cone Ξ, ingeniously, as in the thumbnail. Heiberg introduces circumscribing and inscribed polygons.



TEXTUAL COMMENTS

As usual, the title "Corollary" is a modern editorial intervention, and the relation between the main proposition and its obvious conclusion is an issue we need to investigate based on the text itself.

Step 10 is clearly an interpolation (probably later than Eutocius). Steps 17–18 are "trivial" and worrying in another way too: they are a backwards-looking justification to a backwards-looking justification. Such a double break of the generally onwards-pressing argument is remarkable, and would be much easier to understand if they were scholiastic. Still, they *could* be genuine.

Heiberg argues that Step 6 in the corollary is unclear and leaves the result implicit, and so non-Archimedean: but would omitting it make things any more clear or explicit? If an image of Archimedes emerges from this treatise, it is of a mathematician who enjoyed the enigmatic: the unexpected result emerging from the shadows. (The Roman soldier was asked to extinguish his lamp, not to interfere with the darkness – and then lost his temper.)

GENERAL COMMENTS

Why state the result in a corollary?

The corollary of this proposition is the climax of the book. The words at the start of the corollary, "these being proved, it is obvious . . . ," should therefore be perceived in their full sense of drama: as referring back to *everything proved so far* – to the entire book. Having gone this way, the result is not merely convincing: it is obvious. The amazing has become obvious, and we have

[329] *SC* I.33.

achieved the essential goal of Greek science. Thus the decision to place the main result in a corollary is in a sense natural. The book is a tool, designed to make us see the inevitability of the main result: the entire demonstrative apparatus is a tool for overcoming the need for demonstration.

Strict proposition structure versus ease of geometrical manipulation

This proposition is on the whole "complete" – with a clear enunciation and setting-out, and a real proof. One element is missing, though: the definition of goal. In other words, Archimedes does not stop before the proof to say "it is to be proved, that. . ." This can be explained as follows. The enunciation requires a sphere, a great circle, and a cone (with base=*great circle*, height=radius of the sphere). The brief setting-out specifies only the sphere and the great circle. The cone of the enunciation, however, is not constructed in this proof. What is constructed is another cone (base=*4*great circle*, height=radius of the sphere). Archimedes needs this other cone, because (1) the proposition deals with a relation 1:4, (2) his basic tool, Proposition 2, deals with inequalities of the form a>b, not a>4*b. He therefore prefers to utilize in the demonstration proper a cone different from that of the general enunciation (there, the cone (base=great circle) was naturally preferred, as being meaningful in terms of the geometrical configuration). The bottom line is that Archimedes needs one cone for the purposes of the geometrical configuration, and another cone for the purposes of the tools of proportion-theory. He is economic – in both enunciation and demonstration he uses only the cones he really requires. And he prefers this economy to a strict adherence to the ideal structure of the proposition.

Various ways of "taking as obvious"

As in the preceding proposition, there are several cases where important interim results remain implicit. Take for instance the structure of Steps 11–13: (11) circumscribed:inscribed<K:H. (12) K:H<sphere:cone Ξ. (13) "Alternately." We are asked to supply two acts of demonstrative imagination: first, to get from Steps 11–12 to the unasserted (circumscribed:inscribed<sphere: cone); second, to unpack the "alternately" at Step 13 to mean (circumscribed: sphere<inscribed:cone).

Each act of imagination is in itself manageable, a trivial omission. This is like reading the chess column: the diagram of the position is given, and the first one or two moves are clearly "read" inside the diagram. But at some stage most readers begin to lose touch. It is one thing to imagine an operation on a *present* position, it is quite another to imagine an operation on an *imagined* position. Most chess readers would therefore at some point reach for their boards. Did Archimedes want the ancient readers to reach for their geometrical board – did he want his readers to *work* – so that, by working through the argument, they will all the more appreciate it?

This brings us back to one of our main themes: what does the reader take for granted? What does Archimedes take for granted? I offer footnotes with

"Archimedes' tool-box," that is, results *implied* by the arguments. Some of these are clearly assumed explicitly – for instance, *Elements* VI.16 – that if four magnitudes are in proportion, the rectangle contained by the extremes is equal to that contained by the means. But some other results may be just all too obvious for Archimedes: for instance, the extension of proportion-theory to inequalities of ratios (the assumption, for instance, that if a:b>c:d, then a:c>b:d).

The text signals varying degrees of obviousness. Some arguments are spelled out; some are skipped completely. For example, take the corollary, and the multiplication table. Step 1 implies 3*2=6; Step 3 implies (an equivalent of) 6/4=1.5, as does Step 12. So the multiplication table is indeed taken for granted. However, it is not completely implicit – the corollary works hard to state, explicitly, the numbers six and four. One is tempted to conclude, therefore, that the multiplication table was not as directly accessible to the Greek reader as it is to the modern reader.

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The surface of the figure inscribed inside the segment of the sphere[330] is equal to a circle, whose radius is equal, in square, to the <rectangle> contained by: one side of the polygon inscribed in the segment of the great circle, and <by> the <line> equal to: all the <lines> parallel to the base of the segment, with the half of the segment's base.

Let there be a sphere and in it a segment whose base <is> the circle around the <diameter> AH [. Let a figure be inscribed inside it[331] – as has been said – contained by conical surfaces], and <let there be> a great circle AHΘ, and an even-sided polygon AΓEΘZΔH, (without the side AH),[332] and let a circle be taken – Λ – whose radius is equal, in square, to the <rectangle> contained by the side AΓ and by: all the <lines> EZ, ΓΔ, and also the half of the base, that is AK; it is to be proved that the circle is equal to the surface of the figure.

(a) For let the circle M be taken, whose radius is, in square, the <rectangle> contained by the side EΘ and <by> the half of EZ; (1) so the circle M will then be equal to the surface of the cone whose base <is> the circle around EZ, and <whose> vertex <is> the point

330 This expression, "the figure inscribed inside the segment of the sphere" is an extension of "the figure inscribed inside the sphere," with the rotation now not of a polygon, but of a segment of a polygon.

331 "Inside it"="inside the segment" (*not* inside the *sphere*; this is seen from Greek genders).

332 "Without:" i.e. for the purposes of counting the number of sides and getting an even number AH is ignored.