THE WORKS OF
ARCHIMEDES

Translated into English, together with
Eutocius' commentaries, with commentary,
and critical edition of the diagrams

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Volume I
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is very natural for Archimedes to state not that the ratio is *given*, but that it *is* – as it were, an allowed member of the universe of discourse. The ratio is "on the table."

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To cut the given sphere so that the segments of the sphere have to each other the same ratio as the given.

Let there be the given sphere, ABΓΔ; so it is required to cut it by a plane so that the segments of the sphere have to each other the given ratio.

(a) Let it be cut by the plane AΓ. (1) Therefore the ratio of the segment of the sphere AΔΓ to the segment of the sphere ABΓ is given. (b) And let the sphere be cut through the center, and let the section be a great circle, ABΓΔ,[84] (c) and <let its> center be K, (d) and <its> diameter ΔB, (e) and let it be made: as KΔX taken together to ΔX, so PX to XB,[85] (f) and as KBX taken together to BX, so ΛX to XΔ,[86] (g) and let AΛ, ΛΓ, AP, PΓ be joined; (2) therefore the cone AΛΓ is equal to the segment of the sphere AΔΓ,[87] (3) while the <cone> APΓ <is equal> to the <segment> ABΓ;[88] (4) therefore the ratio of the cone AΛΓ to the cone APΓ is given, too. (5) And as the cone to the cone, so ΛX to XP [(6) since, indeed, they have the same base, the circle around the diameter AΓ];[89] (7) therefore the ratio of ΛX to
Eut. 310 XP is given, too. (8) And through the same <arguments> as before, through the construction, as ΛΔ to KΔ, KB to BP (9) and ΔX to XB.[90] (10) And since it is: as PB to BK, KΔ to ΛΔ,[91] (11) compoundly, as
Eut. 310 PK to KB, that is to KΔ,[92] (12) so KΛ to ΛΔ;[93] (13) and therefore

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[84] Any plane cutting through the center will produce a great circle; the force of the clause is to provide this great circle with its letters. (Note further that it is by now taken for granted that this cutting plane, producing the great circle, is at right angles to the plane AΓ.)

[85] Defining the point P. (K, Δ, B are defined by the structure of the sphere, X is taken to be defined through the make-believe of the analysis.)

[86] Analogously defining the point Λ.

[87] *SC* II.2. [88] *SC* II.2. [89] *Elements* XI.14.

[90] Translating the letters appropriately between the diagrams, the claims made here can be seen to be equivalent to *SC* II.2, Step 29 (=Step 8 here), Steps 7–8, 29 (=Step 9 here). There is the standard problem that interim conclusions are not asserted in general terms, and are therefore more difficult to carry over from one proposition to another, hence Archimedes' *explicit* reference in Step 8. Also, see Eutocius.

[91] *Elements* V.7 Cor. [92] Both KB and KΔ are radii.

[93] *Elements* V.18.

the whole PΛ is to the whole KΛ as KΛ to ΛΔ;[94] (14) therefore the
<rectangle contained> by PΛΔ is equal to the <square> on ΛK.[95]
Eut. 311 (15) Therefore as PΛ to ΛΔ, the <square> on KΛ to the <square>
on ΛΔ.[96] (16) And since it is: as ΛΔ to ΔK, so ΔX to XB, (17) it will
be, inversely and compoundly: as KΛ to ΛΔ, so BΔ to ΔX[97] [(18)
and therefore as the <square> on KΛ to the <square> on ΛΔ, so
the <square> on BΔ to the <square> on ΔX. (19) Again, since it is:
as ΛX to ΔX, KB, BX taken together to BX, (20) dividedly, as ΛΔ to
Eut. 311 ΔX, so KB to BX].[98] (h) And let BZ be set equal to KB; ((21) for it is
clear that it will fall beyond P)[99] [(22) and it will be: as ΛΔ to ΔX,
Eut. 311 so ZB to BX; (23) so that also: as ΔΛ to ΛX, BZ to ZX].[100] (24) And
since <the> ratio of ΔΛ to ΛX is given, (25) therefore <the> ratio
Eut. 312 of PΛ to ΛX is given as well.[101] (26) Now, since the ratio of PΛ to
ΛX is combined of both: the <ratio> which PΛ has to ΛΔ, and <that
Eut. 316 which> ΔΛ <has> to ΛX,[102] (27) but as PΛ to ΛΔ, the <square>
on ΔB to the <square> on ΔX,[103] (28) while as ΔΛ to ΛX, so BZ
to ZX. (29) Therefore the ratio of PΛ to ΛX is combined of both: the
<ratio> which the <square> on BΔ has to the <square> on ΔX, and
Eut. 317 <the ratio which> BZ <has> to ZX. (i) And let it be made: as PΛ to
ΛX, BZ to ZΘ.[104] (30) And <the> ratio of PΛ to ΛX is given; (31)
therefore <the> ratio of ZB to ZΘ is given as well. (32) And BZ <is>
given; (33) for it is equal to the radius; (34) therefore ZΘ is given as
well.[105] (35) Also, therefore, the ratio of BZ to ZΘ is combined of
both: the <ratio> which the <square> on BΔ has to the <square>
on ΔX, and <that which> BZ <has> to ZX. (36) But the ratio BZ to

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[94] As Eutocius explains very briefly, we have, as an implicit result of Steps 11–12, (PK:KΔ::KΛ:ΛΔ), from which can be derived, through *Elements* V.12, (PK+KΛ:KΔ+ΛΔ::KΛ:ΛΔ) – if we have a:b::c:d, we can derive (a+c):(b+d)::c:d.

[95] *Elements* VI.17.

[96] This could be derived directly from Step 13, through *Elements* VI.20 Cor.

[97] *Elements* V.7 Cor., 18. [98] *Elements* V.17.

[99] See Eutocius. The result derives from the assumption that ABΓ is the smaller segment.

[100] *Elements* V.12.

[101] A complex claim in the theory of proportions. See Eutocius, who uses *Elements* V. 7 Cor., 19 Cor., and *Data* 1, 8, 22, 25, 26.

[102] The operation of "composition of ratios" was never fully clarified by the Greeks: see Eutocius for an honest attempt. It can be connected with what we would understand as "multiplication of fractions." (The ratio a:f is composed, as it were, from two ratios b:c, d:e that satisfy (b:c)*(d:e)=a:f – whatever this multiplication and this equality actually mean. The simplest case is the one here, a:c composed of a:b and b:c.)

[103] As Eutocius shows, this can be derived from Steps 15 and 17. See textual comments.

[104] Defining the point Θ. [105] *Data* 2.

ZΘ is combined of both: the <ratio> of BZ to ZX, and of the <ratio> of ZX to ZΘ. [(37) Let the <ratio> of BZ to ZX be taken away <as>
Eut. 317 common];[106] (38) remaining, therefore, it is: as the <square> on BΔ, that is a given[107] (39) to the <square> on ΔX, so XZ to ZΘ, (40) that is to a given. (41) And the line ZΔ is given.

Eut. 317 (42) Therefore it is required to cut a given line, ΔZ, at the <point> X and to produce: as XZ to a given <line> [<namely> ZΘ], so the given <square> [<namely> the <square> on BΔ] to the <square> on ΔX.

This, said in this way – without qualification – is soluble only given certain conditions,[108] but with the added qualification of the specific characteristics of the problem at hand[109] [(that is, both that ΔB is twice BZ and that ZΘ is greater than ZB – as is seen in the analysis)], it is always soluble;[110] and the problem will be as follows:

Given two lines BΔ, BZ (and BΔ being twice BZ), and <given> a point on BZ, <namely> Θ; to cut ΔB at X, and to produce: as the <square> on BΔ to the <square> on ΔX, XZ to ZΘ.

And these <problems>[111] will be, each, both analyzed and constructed at the end.[112]

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106 We have (translating the composition of ratios into anachronistic notation): (35) BZ:ZΘ=((sq. BΔ):(sq. ΔX))*(BZ:ZX), (36) BZ:ZΘ=(BZ:ZX)*(ZX:ZΘ). From which of course we can derive, ((sq. BΔ):(sq. ΔX))*(BZ:ZX)=(BZ:ZX)*(ZX:ZΘ). Archimedes now (37) takes away the common term (BZ:ZX) and derives (38–9) the proportion (sq. BΔ):(sq. ΔX)::(ZX:ZΘ).

107 It is a square on the given diameter of the sphere.

108 "Soluble only given certain conditions:" is literally, in the Greek, "has a *diorismos*." *Diorismos* is a technical term, meaning (in this context), limits under which a problem is soluble. What Archimedes says is that, when the last statement following the analysis is stated as a general problem, where the given lines and square may vary freely – so that they may be any given lines and area whatsoever – some combinations will prove to be insoluble.

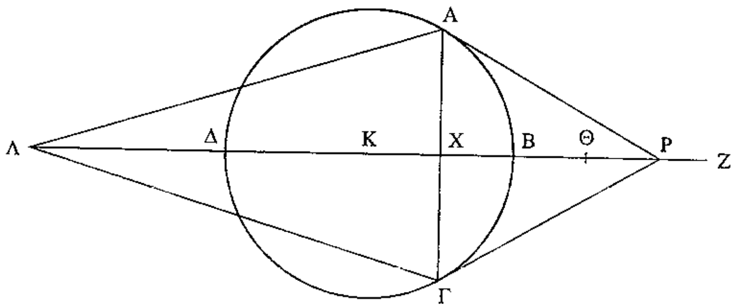
109 Literally, "with the addition of the problems at hand." The Greek for "problem" (*problema*) is wider in meaning than our modern mathematical sense, and can mean, as it does here, "*specific characteristics* of a problem."

What Archimedes means is that the specific given square and line of the problem of *SC* II.4 make the problem possible. They are not just any odd square and line. The given square is uniquely determined by one of the given lines, namely by ΔZ. It is the square on two thirds the line ΔZ. The remaining given line, ZΘ, is not uniquely determined by the given line ΔZ, but it has a boundary: it is less than a third of ΔZ. So with these specific determinations and limits, the problem can always be solved ("always" – i.e. no matter where Θ falls on the line BZ). For all of this, see Eutocius.

110 Literally, "it does not have a *diorismos*."

111 I.e. both the unqualified and the qualified problem.

112 Do not reach for the end of the treatise: this promised appendix vanished from the tradition of the *SC*. See, however, the extremely interesting note by Eutocius.



II.4
Codex C is not preserved for this diagram.

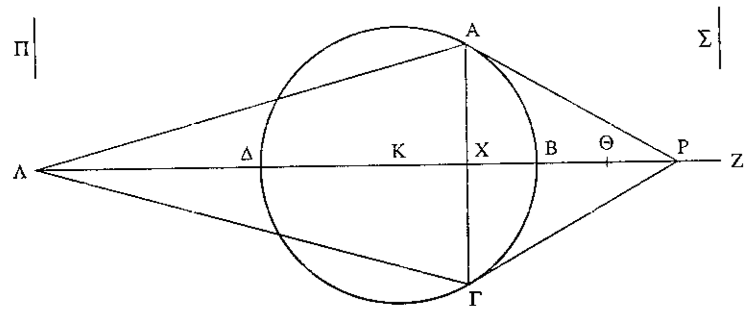
So the problem will be constructed like this:

Let there be the given ratio, the <ratio> of Π to Σ (greater to smaller), and let some sphere be given and let it be cut by a plane <passing> through the center, and let there be a section <of the sphere and the plane, namely> the circle ΑΒΓΔ, and let ΒΔ be diameter, and K center, and let BZ be set equal to KB, and let BZ be cut at Θ, so that it is: as ΘZ to ΘB, Π to Σ, and yet again let BΔ be cut at X, so that it is: as XZ to ΘZ, the <square> on BΔ to the <square> on ΔX, and, through X, let a plane be produced, right to the <line> BΔ; I say that this plane cuts the sphere so that it is: as the greater segment to the smaller, Π to Σ.

(a) For let it be made, first as KBX taken together to BX, so ΛX
Eut. 344 to ΔX, (b) second as KΔX taken together to XΔ, PX to XB, (c) and let AΛ, ΛΓ, AP, PΓ be joined; (1) so through the construction (as we proved in the analysis), the <rectangle contained> by PΛΔ will be equal to the <square> on ΛK,[113] (2) and as KΛ to ΛΔ, BΔ to ΔX;[114] (3) so that, also: as the <square> on KΛ to the <square> on ΛΔ, the <square> on BΔ to the <square> on ΔX. (4) And since the <rectangle contained> by PΛΔ is equal to the <square> on ΛK [(5) it is: as PΛ to ΛΔ, the <square> on ΛK to the <square> on ΛΔ],[115] (6) therefore it will also be: as PΛ to ΛΔ, the <square> on BΔ to the <square> on ΔX, (7) that is XZ to ZΘ. (8) And since it is: as KBX taken together to BX, so ΛX to XΔ, (9) and KB is equal to BZ, (10) therefore it will also be: as ZX to XB, so ΛX to XΔ; (11) convertedly, as XZ to ZB, so XΛ to ΛΔ;[116] (12) so that also, as ΛΔ to ΛX, so BZ to ZX.[117] (13) And since it is: as PΛ to ΛΔ, so XZ to ZΘ, (14)
Eut. 344 and as ΔΛ to ΛX, so BZ to ZX, (15) and through the equality in the perturbed proportion, as PΛ to ΛX, so BZ to ZΘ;[118] (16) therefore

[113] Step 14 in the analysis. [114] Step 17 in the analysis.
[115] Step 15 in the analysis. [116] *Elements* V. 19 Cor.
[117] *Elements* V.7 Cor.
[118] *Elements* V.23. To explain the expression: to move from A:B::C:D and B:E::D:F to conclude that A:E::C:F is to have an argument *from the equality*. Here the premises

also: as ΛX to XP, so ZΘ to ΘB.[119] (17) And as ZΘ to ΘB, so Π to Σ; (18) therefore also: as ΛX to XP, that is the cone AΓΛ to the cone APΓ,[120] (19) that is the segment of the sphere AΔΓ to the segment of the sphere ABΓ,[121] (20) so Π to Σ.



II.4
Codex C is not preserved for this diagram. Codex D has positioned the two lines Π, Σ to the two sides of the main figure, and has made Σ greater than Π.

TEXTUAL COMMENTS

Heiberg's bracketed passages (Steps 6, 18–20, 22–3, 37 and bits of 42 in the analysis, a few bits of the interlude between analysis and synthesis, and Step 5 in the synthesis) are not trivial, but are still relatively moderate given the size of the proposition. All of them, with the exception of Step 6 in the analysis (a fairly obvious, so also suspect, backward-looking justification), are bracketed because of some tensions they create when read together with Eutocius' commentary. They either state what Eutocius seems to prove separately from Archimedes, or their text disagrees with Eutocius' quotations. As usual, I doubt if such tensions are at all meaningful. Thus this fiendishly complicated proposition seems to be in relatively good textual order, which is not at all a paradox: its intricacies are such to deter the scholiast.

The text refers, in the interlude between analysis and synthesis, to an appendix to the work. This appendix was lost to the main lines of transmission, it is absent from all the extant manuscripts, and was initially unknown to Eutocius. After what he implies was a long search, Eutocius was capable of finding some vestiges of this appendix, apparently in some text totally independent of the *On the Sphere and the Cylinder*. For all of this, see Eutocius.

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are not the direct sequence A–B–E and C–D–F, but rather A:B::C:D and B:E::F:C. The second sequence is not C–D–F, but F–C–D, and the conclusion is accordingly A:E::F:D. This then is a *perturbed proportion*. (None of those labels is very instructive, but they are established by tradition, and are enshrined in our text of Euclid.) Also, see Eutocius.

119 For instance: From (15) PΛ:ΛX::BZ:ZΘ, get PΛ:XP::BZ:ΘB (*Elements* V.19 Cor.), hence XP:PΛ::ΘB:BZ (*Elements* V.7 Cor.) which, with (14) again, yields the conclusion XP:ΛX::ΘB:ZΘ (*Elements* V.22). Applying *Elements* V.7 Cor. again, we get the desired conclusion: (15) ΛX:XP::ZΘ:ΘB.

120 *Elements* XI.14. 121 *SC* II.2.

GENERAL COMMENTS

A suggestion on the function of analysis in a complex solution

As noted in the textual comments, this proposition is very complicated. The complexity, however, does not stem from any deep insight gained by the proposition. The complex construction required to solve the problem is the result of a direct manipulation, through proportion theory, of the reduction of sphere segments to cones, provided in *SC* II.2. Thus the solution is in a way less than completely satisfactory: the baroque construction has no deep motivation, and stands in contrast to the extremely simple statement of the problem. Essentially, this is because Archimedes' tools here, geometrical proportions, were designed to state in clear, elegant form relations in plane geometry. Archimedes cleverly reduces the three-dimensional curvilinearity of spheres into the line segments along ΛZ, but the solid nature of the problem remains irreducible, in the form of cumbersome, non-obvious proportions. (It might perhaps be suggested that the search for ways of dealing with non-planar geometric relations, in the same elegance available for plane geometry, ultimately led to the emergence of modern mathematics.)

One way in which the solution may appear more satisfactory is, quite simply, by prefacing the synthetic solution by an analysis. The purpose of the analysis, I suggest, may be in this case a sort of *apology* for the synthesis. The analysis shows how the parameters of the problem force the author to solve the problem in this particular way and no other, and in this way make this complicated solution appear a bit more "natural." It is almost as if, to make the synthesis appear less cumbersome than it is, Archimedes prefaces it by an even more cumbersome analysis, so that, by comparison, the synthesis appears to be straightforward.

At any rate, once again: there is no reason to believe that the synthesis was *discovered* by following the analysis. It is instructive to note that the points Z, Θ appear in their natural alphabetic order in the synthesis and not the analysis, suggesting that the analysis might have been written by Archimedes only after the synthesis was already written. At any rate, the main ideas behind this solution are very clear – and have nothing to do with the method of analysis and synthesis. The solution is motivated by the desire to transform solid relations into linear relations. To do this, the relation between the segments of spheres is transformed into a relation between cones (which are then easy to translate into lines, with the tools provided in the *Elements*). Thus the main idea of the proof is simply *SC* II.2 which – crucially – was *not* offered in synthesis and analysis form. Why? Because, as a theorem, it called for no apology. Put simply: when you state the truth, its ugliness is no shame. Ugliness is a shame only (as in a problem) when you choose it among infinitely many other options.

The use of interim results

As mentioned already in n. 90 above, Steps 8–9 in the analysis show us the difficulty which arises with interim results. *SC* II.2 had reached a number of interim proportions, which were stepping-stones for further argumentation. Here the same stepping-stones are required. However, Archimedes' way of referring to them is extremely mystifying: "And through the same <arguments> as before, through the construction, as ΛΔ to ΚΔ, ΚΒ to ΒΡ, and ΔΧ to ΧΒ." (Note that the word "construction" refers not to the drawing of the diagram, but to the verbal stipulation made concerning the ratios obtaining in this proposition.) This opaque form of reference is due to a combination of two reasons. First, the stepping-stones were not enshrined at any enunciation. They were not goals in themselves, to be proved in the most general way, and hence they were never stated in general form and apart from a reference to diagrammatic letters. Second, the lettering of the two propositions, *SC* II.2 and 4, differs (although they both deal with exactly the same position). This is typical of the practice of Greek mathematics, where, at the end of each proposition, the "deck of cards is reshuffled," letters being re-assigned to the diagram according to many local factors (especially the order in which those letters are introduced into the texts of the different propositions). As a consequence, there is no specific statement Archimedes can refer to: the general statement of the interim results was never enunciated, while the particular statement was not given in a form usable in this context. All Archimedes has to refer to is the assertion: "and therefore, alternately, it is: as ΚΑ to ΑΘ, so ΑΕ to ΕΓ" – Step 29 in *SC* II.2 – which has no bearing at all on *SC* II.4 (where, for instance, there is not even an E!).

It is interesting that Archimedes did not solve this difficulty by allowing a further, interim lemma, expressed as a general enunciation. It is typical of this treatise, that the focus is throughout on the problems themselves. Once again: this is not a gradually evolving, self-sufficient treatise, like the previous book, but a series of solutions to certain striking problems, with only a very few theorems mentioned only where absolutely necessary. This is most obvious with the lemma mentioned here in the interlude between the analysis and the synthesis: perhaps the most striking result in this book, it was delegated to an appendix, set apart from the main work, and perhaps consequently lost from the main manuscript tradition.

Finally, note that, once again, we see that Archimedes does not have the tools required for making explicit references of any kind: quite simply, the propositions are not numbered, so that all he can refer to is the vague "same as before" – which could be anywhere in the treatise. Indeed, the vestigial system of numbering used in this treatise refers to problems alone: *SC* II.2, a theorem, escapes, as it were, Archimedes' coarse net.

/5/

To construct <a segment of a sphere> similar to a given segment of a sphere and, the same <segment>, equal to another given <segment>.

Let the two given segments of a sphere be ΑΒΓ, ΕΖΗ, and let the circle around the diameter ΑΒ be base of the segment ΑΒΓ, and <its> vertex the point Γ, and <let> the <circle> around the diameter ΕΖ be base of the <segment> ΕΖΗ, and <its> vertex the point Η; so it is required to find a segment of a sphere, which will be equal to the segment ABΓ, and similar to the <segement> EZH.

(a) Let it be found and let it be the <segment> ΘKΛ, and let its base be the circle around the diameter ΘK, and <its> vertex the point Λ. (b) So let there also be circles[122] in the spheres: ANBΓ, ΘΞKΛ, EOZH, (c) and their diameters, at right <angles> to the bases of the segments: ΓN, ΛΞ, HO, (d) and let Π, P, Σ be centers (e) and let it be made: as ΠN, NT taken together to NT, so XT to TΓ, (f) and as PΞ, ΞY taken together to ΞY, so ΨY to YΛ, (g) and as ΣO, OΦ taken together to OΦ, so ΩΦ to ΦH, (h) and let cones be imagined, whose bases are the circles around the diameters AB, ΘK, EZ, their vertices the points X, Ψ, Ω.

(1) So the cone ABX will be equal to the segment of the sphere ABΓ, (2) and the <cone> ΨΘK <will be equal> to the <segment> ΘKΛ, (3) and the <cone> EΩZ to the <segment> EHZ; (4) for this has been proved.[123] (5) And since the segment of the sphere ABΓ is equal to the segment ΘKΛ, (6) therefore the cone AXB is equal to the cone ΨΘK, as well [(7) and the bases of equal cones are reciprocal to the heights];[124] (8) therefore it is: as the circle around the diameter AB to the circle around the diameter ΘK, so ΨY to XT. (9) And as the circle to the circle, the <square> on AB to the <square> on ΘK;[125] (10) therefore as the <square> on AB to the <square> on
Eut. 345 ΘK, so ΨY to XT. (11) And since the segment EZH is similar to the segment ΘKΛ,[126] (12) therefore the cone EZΩ, as well, is similar to the cone ΨΘK [(13) for this shall be proved];[127] (14) therefore it is:
Eut. 346 as ΩΦ to EZ, so ΨY to ΘK. (15) And <the> ratio of ΩΦ to EZ is given;[128] (16) therefore <the> ratio of ΨY to ΘK is given as well. (h) Let the <ratio> of XT to Δ be the same; (17) and XT is given; (18)
Eut. 347 therefore Δ is given as well. (19) And since it is: as ΨY to XT, that is the <square> on AB to the <square> on ΘK, (20) so ΘK to Δ,[129] (i) let the <rectangle contained> by AB, ς be set equal to the <square> on ΘK;[130] (21) therefore it will also be: as the <square> on AB to the <square> on ΘK, so AB to ς.[131] (22) But it was also proved: as the

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[122] By "circles," Archimedes refers here to *great* circles.

[123] *SC* II.2. [124] *Elements* XII.15. [125] *Elements* XII.2.

[126] The assumption of the analysis (Step a).

[127] See Eutocius (to whom the reference probably points).

[128] This is obvious since the segment itself is given. See Eutocius for a detailed exposition.

[129] Step h, and then *Elements* V.16 (see Eutocius, who comments on this, probably not because there is any need to remind the readers of the existence of *Elements* V.16 but because of the difficult structure of Steps 19–20).

[130] I.e. the line ς is determined by this Step i to satisfy the equality rect. AB,ς =sq.ΘK.

[131] *Elements* VI.1.

the said (1) since a perpendicular (<namely>, ΔΓ) has been drawn in a right-angled triangle, AΔB; (2) and <it has been drawn> from the right <angle>, (3) it is a mean proportional between the segments of the base,[265] (4) and the triangles next to the perpendicular are similar to the whole <triangle> and to each other;[266] (5) so that it is: as BΓ to ΔΓ, BΔ to ΔA;[267] (6) therefore also the <squares> on them; (7) but as the <square> on BΓ to the <square> on ΓΔ, so the first BΓ to the third ΓA;[268] (8) therefore also: as BΓ to ΓA, the <square> on BΔ to the <square> on ΔA.[269]

Arch. 199 "And a given ratio, of AΓ to ΓB; so that the point Γ is given." (1) For since the sphere is assumed given, (2) therefore its diameter, AB, is given as well. (3) And the ratio of AΓ to ΓB is given; (4) and if a given magnitude is divided into a given ratio, each of the segments is given;[270] (5) so that AΓ is given. (6) and A is given; (7) for it is on the common section of lines given in position;[271] (8) therefore Γ is given as well.[272]

To 4

Arch. 202 "And through the same <arguments> as before, through the construction, as ΛΔ to KΔ, KB to BP and ΔX to XB." For in the <proposition> preceding this one, it was thus concluded: (1) Since it is, as KΔ, ΔX taken together to ΔX, so PX to XB,[273] (2) dividedly: as KΔ to ΔX, PB to BX;[274] (3) alternately: as KΔ, that is KB[275] (4) to BP, ΔX to XB.[276] (5) Again, since it is: as ΛX to XΔ, so KB, BX taken together to XB,[277] (6) dividedly and alternately: as ΛΔ to ΔK, ΔX to XB.[278] (7) And it was also: as ΔX to XB, KB to BP; (8) therefore as ΛΔ to ΔK, ΔX to XB and KB to BP.

Arch. 202 "And therefore the whole PΛ to the whole KΛ is as KΛ to ΛΔ." For as one to one, so all the antecedents to all the consequents.[279]

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265 *Elements* VI.8 Cor.

266 *Elements* VI.8. 267 *Elements* VI.4.

268 The reference of "first" and "third" is to three terms in a continuous proportion, first:second::second:third. That the lines in question form such a continuous proportion was asserted at Step 3 above. *Elements* VI.20 Cor. 2.

269 The manuscripts have "as BΓ to ΓA, the <line> BΔ to ΔA." (No indication of noun for ΔA.) This is most likely to be mere textual corruption, but a more interesting possibility is that a more abstract representation of the square – directly through the diagrammatic symbols, without the word "square" – is being approached.

270 *Data* 7. 271 *Data* 25. 272 *Data* 27.

273 Archimedes' construction. 274 *Elements* V.17. 275 Both radii.

276 *Elements* V.16. 277 Archimedes' construction. 278 *Elements* V.16, 17.

279 *Elements* V.12. In modern terms: given (a_1:b_1::a_2:b_2:: . . .::a_n:b_n) it can be concluded, for any k between 1 and n: (a_k:b_k::($a_1+a_2+\cdots+a_n$):($b_1+b_2+\cdots+b_n$)).

Arch. 203 "Therefore as PΛ to ΛΔ, the <square> on KΛ to the <square> on ΛΔ." (1) For since it is: as PΛ to ΛK, KΛ to ΛΔ, (2) therefore also: as the first to the second, so the <square> on the first to the <square> on the second;[280] (3) therefore it is: as PΛ to ΛΔ, so the <square> on PΛ to the <square> on ΛK. (4) But as the <square> on PΛ to the <square> on ΛK, so the <square> on ΛK to the <square> on ΛΔ; (5) for they are proportional;[281] (6) therefore as PΛ to ΛΔ, so the <square> on ΛK to the <square> on ΛΔ.

Arch. 203 "Let BZ be set equal to KB; (for it is clear that it will fall beyond P)." (1) For since it is: as XΔ to XB, so KB to BP[282] (2) and ΔX is greater than XB,[283] (3) therefore KB, as well, is greater than BP. (4) Therefore Z falls beyond P.

Arch. 203 "And since <the> ratio of ΔΛ to ΛX is given, as well as the ratio of PΛ to ΛX, therefore <the> ratio of PΛ to ΛΔ, too, is given." (1) For since it is: as KBX taken together to BX, that is ZX to XB,[284] (2) so ΛX to XΔ,[285] (3) Convertedly: as XZ to ZB, so XΛ to ΛΔ,[286] (4) also inversely: as BZ to ZX, ΛΔ to ΛX.[287] (5) And the ratio of BZ to ZX is given, (6) since ZB is equal to the radius of the given sphere, (7) while BX is given (8) as its limits B, X are given by hypothesis,[288] (9) the sphere being cut by the plane AΓ and by the <line> ΔB being at

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280 *Elements* VI.20 Cor. 2.

281 Here is one of those moments when I get seriously excited and people watch on bemused as I cry aloud and foam, but please, please pay attention! Eutocius speaks about "they," meaning the "they" which are independently known to be proportional, i.e. the line segments PΛ, ΛK, ΛΔ. This use of "they" shows that these line segments are understood to be the logical subjects of this Step 4 itself. I.e., Step 4 is understood to be not on squares, but on segments of lines. This can be immediately grasped by our own expression of the type $a^2:b^2::c^2:d^2$, where clearly the expression is felt to be about (a, b, c, d), rather than about (a^2, b^2, c^2, d^2): the "^{2}" symbol is merely something we do with the main protagonists (just as we manipulate them with the ":" symbol, yet no one would think the expression is about ":"). This is at the heart of what constitutes the symbolic nature of our "^{2}," which does not yield an object so much as transforms another, separately present object. The opposite is usually the case with the fully geometrical Greek "<square> on," which is not a symbolic manipulation, but is a real geometrical expression, yielding an object completely distinct from the line segment from which we started: a square, distinct from a line. Thus the enormous significance of the expression before us: Greek "<square> on" acts and feels like a purely symbolic, modern "^{2}." This is typical of the commentary, second-order position of Eutocius: extensions of symbolism towards the fully second-order symbolism of algebra are often suggested, though never fully followed. More of this to come below.

282 See Eutocius' first comment above.

283 An assumption made explicit in Archimedes' synthesis, and at this stage – the analysis – based on the diagram alone.

284 "KBX taken together" is equal to ZX, since by Archimedes' construction ZB=KB.

285 Archimedes' construction. 286 *Elements* V.19 Cor.

287 *Elements* V.7 Cor. 288 *Data* 26.

right <angles> to the <line> AΓ,[289] (10) and through this the whole XZ, too, <is given>, (11) as well as the <ratio> of XZ to ZB;[290] (12) so that the ratio of XΛ to ΛΔ is given as well. (13) Again, since the ratio of the segments is given, (14) the ratio of the cone ΛAΓ to the cone APΓ is given, as well.[291] (15) So that the <ratio> of ΛX to XP <is given>, too; (16) for they are to each other as the heights;[292] (17) therefore <the> ratio of the whole PΛ to ΛX is given.[293] (18) Now since the ratio of each of PΛ, ΛΔ to ΛX is given, (19) therefore the ratio of PΛ to ΛΔ is given as well. (20) For the <magnitudes> which have a given ratio to the same, have also a given ratio to each other.[294]

Arch. 203 "Now since the ratio of PΛ to ΛX is combined of both: the <ratio> which PΛ has to ΛΔ, and <of> ΛΔ to ΛX." It is obvious that, once ΛΔ is taken as a mean, the synthesis of ratios is taken (as this is taken in the *Elements*, too[295]). Since, however, the discussion of the subject has been somewhat confused, and not such as to make the concept satisfactory, (as can be found reading Pappus[296] and Theon[297] and Arcadius[298] who, in many treatises, present the operation not by arguments, but by examples), there will be no incongruity if we linger briefly on this subject so as to present the operation more clearly.[299]

So: I say that if some middle term is taken between two numbers (or magnitudes), the ratio of the initially taken numbers[300] is composed of

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289 *Data* 25. 290 *Data* 1. 291 *SC* II.2. 292 *Elements* XI.14.
293 *Data* 22. 294 *Data* 8. 295 Reference to *Elements* VI.23.

296 The reference must be to a work, or works, no longer extant.

297 Theon of Alexandria, late fourth century AD, Hypatia's father, known especially through his (extant) commentary on the Almagest. The reference here is to this commentary, pp. 61 ff. Basil. and possibly, to other, lost, works (Eutocius' plural "many treatises" is very emphatic).

298 Known only through this reference. One wonders if the sequence Pappus – Theon is not meant to be chronological, in which case Arcadius is probably a very late author, not much earlier than Eutocius himself – which could help to explain how Eutocius knows him but we don't. See Knorr (1989) 166, however, for a suggestion linking Arcadius with a known (and unattributed) *Introduction to the Almagest*, containing a passage on the composition of ratios.

299 What Eutocius says is that as far as the mathematical consensus is concerned, Archimedes' argument is clear and even obvious. However, since the mathematical consensus itself seems to be at fault here, a commentary is required. First we had a spirit of philological enterprise, in the catalogue of two mean proportionals, and now a mathematical independence. Eutocius has grown considerably since the commentary to the first book. The composition of ratios is indeed a sore point in Greek mathematics: let's see how much sense he will make out of it (Eutocius himself clearly was happy with his own discussion, and he has recycled it in his later commentary to Apollonius' *Conics*, II. pp. 218 ff.).

300 So the immediately preceding "or magnitudes" is an afterthought.

the ratio, which the first has to the mean and of the <ratio> which the mean has to the third.

So first it ought to be recalled how a ratio is said to be composed of ratios. For as in the *Elements*: "when the quantities of the ratios, multiplied, produce a certain <quantity>,"[301] where "quantity" clearly stands for "the number" whose cognate is the given ratio[302] (as say several authors as well as Nicomachus[303] in the first book of *On Music* and Heronas[304] in the commentary to the *Arithmetical Introduction*[305]), which is the same as saying: "the number which, multiplied on[306] the consequent term of the ratio, produces the antecedent as well." And the quantity would be taken in a more legitimate way in the case of multiples,[307] while in the case of superparticulars, superpartients,[308] it is no longer possible for the quantity to be taken with the unit remaining undivided;[309] so that in these cases the unit must be divided – which, even if this does not belong to what is proper in arithmetic, yet it does belong to what is proper in calculation. And the unit is divided by the part or by the parts by which the ratio is called,[310] so that (to say this in a clearer way), the quantity of the half-as-large-again is, added to the unit, half the unit; and <the quantity of the> four-thirds is, added to the unit, one third the unit, so that, as has been said above as well, the quantity of the ratio, multiplied on the consequent term, produces the

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301 *Elements* V. Def. 5, bracketed in Heiberg's edition of the *Elements* but apparently in Eutocius' own text.

302 The idea is that a typical ratio is, for instance, the multiplicative "twice," whose cognate is the cardinal "two." So the term for ratio "twice" is the cognate of the term for number "two."

303 A first–second centuries AD Pythagorean philosopher-mathematician. His *On Music* does not survive.

304 Known only from this reference.

305 An extant treatise written by Nicomachus.

306 Standard English usage has X multiplied by Y, not on Y. I prefer to stick to the literal translation of the Greek particle ἐπί since, a few pages below, in a geometrical context, Archimedes and Eutocius are about to employ the same expression so as to suggest a similar calculation, applied to geometry.

307 I.e. integer multiplicatives such as "twice," "three times," etc.

308 Kinds of what we call non-integer, positive rational numbers. Here they are mentioned as kinds of ratio, not as kinds of numbers. See Eutocius' comment to *SC* I.2 for the terms.

309 The quantity of non-integer ratios is not a cardinal. This seemingly trivial point must be stressed by Eutocius, since, in trying to make sense of "ratios as quantities" he starts from the relationship between ratios and their cognate number, which exists only in the case of integers.

310 For instance: 2 is a sixth of 12, therefore it is a "part" of 12 (namely, a sixth part). 9 to 12 cannot be expressed by such a single term. It is three quarters of 12, three parts. Therefore it is "parts" of 12 (*Elements* VII. Deff. 3–4).

antecedent. For the quantity of nine to six, being the unit and the half, multiplied on 6, produces 9, and it is possible to observe the same in the other cases as well.

Having clarified these first, let us return to the enunciated proposition. For let the two given numbers[311] be A, B, and let a certain mean be taken between them, Γ. So it is to be proved that the ratio of A to B is combined of the <ratio> which A has to Γ, and Γ to B.

(a) For let the quantity of the ratio A, Γ[312] be taken, <namely> Δ, (b) and <let the quantity> of the <ratio> Γ, B <be taken, namely> E; (1) therefore Γ, multiplying Δ, produces A, (2) while B, multiplying E, <produces> Γ. (c) So let Δ, multiplying E, produce Z. I say that Z is a quantity of the ratio of A to B, that is, that Z, multiplying B, produces A. (d) For let B, multiplying Z, produce H. (3) Now since B, multiplying Z, has produced H, (4) and, multiplying E, <it has produced> Γ, (5) it is therefore: as Z to E, H to Γ.[313] (6) Again, since Δ, multiplying E, has produced Z (7) while, multiplying Γ, it has produced A, (8) it is therefore: as E to Γ, Z to A. (9) Alternately: as E to Z, Γ to A,[314] (10) inversely also: as Z to E, so A to Γ.[315] (11) But as Z to E, H was proved <to be> to Γ; (12) therefore also: as H to Γ, A to Γ; (13) therefore A is equal to H.[316] (14) But B, multiplying Z, has produced H; (15) therefore B, multiplying Z, produces A as well; (16) therefore Z is a quantity of the ratio of A to B.[317] (17) And Z is: Δ, multiplied on E, (18) that is: the quantity of the ratio A, Γ, <multiplied> on the quantity of the ratio Γ, B; (19) therefore the ratio of A to B is composed of both: the <ratio>, which A has to Γ, and Γ to B; which it was required to prove.[318]

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311 "Numbers:" any pretence at generality is by now dropped. In what follows, Eutocius consistently uses the masculine article, referring to "number." My translation "A," abbreviated to avoid excessive tedium, thus stands for the original "the <number> A."

312 "The ratio A, Γ:" a revolutionary expression.

313 This can be related to *Elements* VII.17 – which is apparently what Eutocius conceives of as the basis for his own argument. Eutocius seems to discuss the subject matter neither of *Elements* V (geometrical magnitudes), nor that of *Elements* VII (integers), but the subject matter of what he has called "calculation," which we call positive rational numbers. Since, however, his tool box is so heavily based on Euclid, he probably finds it natural to deal with (what we call) positive rational numbers as if they were integers.

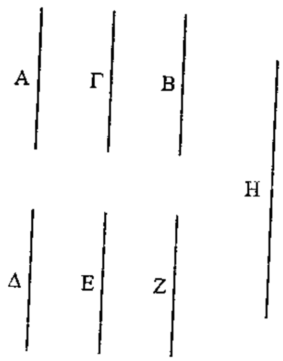
314 *Elements* V.16 (magnitudes) or VII.13 (numbers)?

315 *Elements* V.7 Cor. (No separate *Elements* proof for numbers).

316 *Elements* V.9.

317 Eutocius does not stop here – the interim definition of goal – but goes on to obtain the original goal, referring explicitly to the composition of ratios.

318 The proof is valid for the rational numbers Eutocius has in mind. But its limited generalizability is fundamental. While it is easy to provide a clear sense of the-ratio-of-two-rational-numbers as a rational number, there is no such simple way of defining, say, the-ratio-of-two-lines as, say, a line, let alone as any numerical magnitude of the



In II.4
Codex D has A greater than Γ, in turn greater than B; and Δ greater than E, in turn greater than Z. Codex E has A greater than B, in turn equal to Γ; and Δ greater than E, in turn equal to Z. Codex G has the three lines Δ, E, Z (equal to each other) greater than the three lines A, B, Γ (equal to each other).

And so that the thing said shall be clarified with an example, too,[319] let some mean number, <namely> 4, be inserted between 12 and 2. I say that the ratio of 12 to 2, that is the six-times, is composed of the thrice (12 to 4) and of the twice (4 to 2).

For if we multiply the quantities of the ratios on each other, that is 3 on 2, 6 results, being the quantity of the ratio of 12 to 6, and it is six-times, which is also what was put forth to prove.

And even if the inserted mean happens not to be (first) smaller than the greater and (second) greater than the smaller, but is instead the opposite of that,[320] or it is greater than both, or smaller than both; even so the composition mentioned above follows. Let some mean be inserted between 9 and 6, greater than both, <namely> 12. I say that from both: the converse-of-a-third-as-large-again ratio (9 to 12) and from the twice (12 to 6), the half-as-large-again is composed (9 to 6).

(1) For the quantity of the ratio 9 to 12 is three fourths, (2) that is half and a fourth, (3) and the quantity of 12 to 6 is 2. (4) Now if we multiply 2 on half and fourth, a one unit and a half results, (5) which is a quantity of the half-as-large-again ratio, (6) which 9 has to 6, as well. And similarly, if 4, as well, is inserted <as> a mean between 9 and 6: from the 9 to 4 (twice-cum-converse-of-four-times) and <from> the 4 to 6 (converse-of-half-as-large-again), the half-as-large-again is composed. For again, when we multiply the quantity of the

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kinds the Greeks knew. The ratio of lines is just that, a ratio. What we need however for Eutocius' purposes (who after all deals here with the geometrical magnitudes of Archimedes' treatise) is to see any ratio whatsoever as some sort of single object, not just as a relation between two objects. In other words, we need modern mathematics which, for better or worse, is not what Eutocius is offering us.

319 Unlike his predecessors, Eutocius explicates the concept with a proof, rather than an example. Having done so, he now goes on to add the examples.

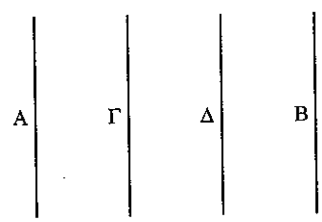
320 That is, smaller than the smaller and greater than the greater. This, as Heiberg notes in his textual apparatus, is not possible. Heiberg suggests Eutocius may have nodded off here, but I would be even happier to believe this is some sort of an attempt at humor.

twice-cum-converse-of-four-times, <namely> $2\frac{1}{4}$,[321] on the quantity of the converse-of-half-as-large-again, that is the two thirds, we shall get the one <and> a half, the quantity of the half-as-large-again ratio, as has been said. And the same argument will apply similarly in all other cases.

From the things said it is also clear that if not one mean term is inserted between two given numbers or magnitudes, but many, the ratio of the extremes is composed of all the ratios which the terms have, arranged in sequence, starting from the first and terminating in the last one in the order of the <terms> standing in ratios.[322]

For, there being two terms, A, B, let more than one <terms, namely> Γ, Δ, be inserted. I say that the ratio of A to B is composed of the <ratio> which A has to Γ, and Γ to Δ, and Δ to B.

(1) For since the <ratio> of A to B is composed of the <ratio>, which A has to Δ, and Δ to B, (2) as has been said above, (3) and the ratio of A to Δ is composed of the <ratio> which A has to Γ, and Γ to Δ, (4) therefore the ratio of A to B is combined of the <ratio> which A has to Γ, and Γ to Δ, and Δ to B. And similarly it will be proved in the remaining <cases? means?>.



In II.4 Second diagram

Arch. 203

Further in the text he says:[323] "but as PΛ to ΛΔ, <so> the <square> on ΔB was proved to be to the <square> on ΔX." (1) For since it has been proved: as PΛ to ΛΔ, the <square> on ΛK to the <square> on ΔΛ,[324] (2) and as the <square> on KΛ to the <square> on ΛΔ, so the <square> on BΔ to the <square> on ΔX ((3) for it was proved: as KΛ to ΛΔ, BΔ to ΔX, through the "compoundly"[325]); (4) therefore as PΛ to ΛΔ, the <square> on BΔ to the <square> on ΔX.

321 Greek strictly speaking does not have the symbol "¼." Instead, the symbol "4'" is used to mean "fourth part."

322 Eutocius is at pains to clarify that the order need not be that of quantity (from great to small) but can be any order whatsoever.

323 The word "text" is a free translation of what means roughly "what is said." Eutocius' point is that the lemma immediately follows the preceding lemma: the commentary follows one Archimedean sentence, and then the next.

324 Step 15 in Archimedes' analysis.

325 Step 17 of Archimedes' analysis. This time "compoundly" refers not to the composition-of-ratios operation, but to the "compoundly" proportion argument, *Elements* V.18.

Arch. 203 "And let it be made, as PΛ to ΛX, BZ to ZΘ." Wherever the point Θ be positioned, as far as the logical consequence of the proof is concerned, no obstacle to the argument may arise. But it shall be clear that it always falls (just as it is positioned in the diagram) between B, P, as follows: (1) for since it is, as ΛK to ΔK, that is to KB,[326] (2) so KP to PB,[327] (3) and therefore also as one to one, so all to all:[328] (4) as ΛP to PK, KP to PB. (5) But ΛP has to PX a greater ratio than ΛP to PK;[329] (6) therefore ΛP has to PX a greater ratio than KP to BP, too, (7) that is ZB to BP. (8) Conversely, PΛ has to ΛX a smaller ratio than BZ to ZP.[330] (9) Therefore if we make: as PΛ to ΛX, so BZ to some other <line>, it shall be to a <line> greater than ZP.

And it is at once apparent that ZΘ is greater than ΘB.[331] (1) For since it has been proved: as ΛΔ to ΔK, ΔX to XB (2) and KB to BP,[332] (3) and ΔX is greater than XB, (4) therefore ΛΔ, too, is greater than ΔK, (5) and KB <is greater> than BP; (6) so that ΛΔ <is greater> than BP, as well.[333] (7) Therefore the whole ΛX is greater than XP, as well; (8) so that ΘZ <is greater> than ΘB, as well.

Arch. 204 "Remaining, therefore, it is: as the <square> on BΔ, that is a given, to the <square> on ΔX, so ZX to ZΘ." (1) For since the <ratio> composed of the <square> on BΔ to the <square> on ΔX and of BZ to ZX was proved to be the same as the ratio of BZ to ΘZ,[334] (2) and the same ratio (of BZ to ZΘ) is the same also as the <ratio> composed of the <ratio> of BZ to ZX and of XZ to ZΘ, (3) therefore, also, the ratio composed of the <ratio> of the <square> on BΔ to the <square> on ΔX and of the <ratio> of BZ to ZX is the same as the <ratio> composed of the <ratio> of BZ to ZX and of the <ratio> of XZ to ZΘ. (4) Now if we take away the <ratio> common to both ratios, <namely> the <ratio> of BZ to XZ, the remaining ratio of the <square> on BΔ to the <square> on ΔX is the same as the <ratio> of XZ to ZΘ.

Arch. 204 And "So it is required to cut a given line, ΔZ, at the <point> X, and to produce: as XZ to a given <line>" (that is ZΘ) "so the given <square>" (that is the <square> on BΔ) "to the <square> on ΔX. And this, said in this way – without qualification – is soluble only given certain conditions, but with the added qualification of the specific characteristics of the problem at hand" (that is, both that ΔB is twice BZ

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[326] Both radii.
[327] Step 10 of Arcihmedes' analysis, plus an implicit use of *Elements* V.18.
[328] *Elements* V.12. [329] *Elements* V.8.
[330] See Eutocius' commentary to *SC* I.2.
[331] So we get an even firmer grasp of where the point Θ is.
[332] Steps 10–11 in Archimedes' analysis.
[333] Remember ΔK=KB (radii). [334] Step 34 in Archimedes' analysis.

and that ZΘ is greater than BZ – as is seen in the analysis) "it is always soluble; and the problem will be as follows: given two lines ΔB, BZ (and ΔB being twice BZ), and a point on BZ, <namely> Θ; to cut ΔB at X, and to produce: as the <square> on ΔB to the <square> on ΔX, XZ to ZΘ; and these <problems> will be, each, both analyzed and constructed at the end." While he promised to prove the aforementioned claim at the end, it is impossible to find the promised thing in any of the manuscripts. Which is why, as we found, Dionysodorus, too, failing to get to the same proofs – being unable to lay hands on the lost lemma – went on another route to the entire problem, which we shall write down in the following. And Diocles too, in the book he composed *On Burning Mirrors*, also in the belief that Archimedes promised, but had not delivered the promise, attempted to fill the gap himself; and we shall write down the attempt in the following. (Indeed, this again has nothing resembling the lost argument, but, similarly to Dionysodorus, he constructs the problem through a different proof.)

But – in a certain old book (for we did not cease from the search for many books), we have read theorems written very unclearly (because of the errors), and in many ways mistaken about the diagrams. But they had to do with the subject matter we were looking for, and they preserved in part the Doric language Archimedes liked using, written with the ancient names of things: the parabola called "section of a right-angled cone," the hyperbola "section of an obtuse-angled cone." From which things we began to suspect, whether these may not in fact be the things promised to be written at the end. So we read more carefully the content itself (since we have found – as had been said – that it has been an uneasy piece of writing, because of the great number of mistakes), taking apart the ideas one by one.[335] We write it down, as far as possible, word-for-word (but in a language that is more widely used, and clearer).

The first theorem is proved for the general case, so that his claim, concerning the limits on the solution, becomes clearer. Then it is also applied to the results of the analysis in the problem.[336]

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335 Note how narrative form is kept throughout, beginning from the romantic quest for books, following the commentator in his study – the moment of sudden conversion, and then the long work of taking the treatise apart.

336 The textual and mathematical commentary on the following passage – on Archimedes' problem in the lost appendix and on its later solutions – could not be contained within the boundaries of this volume. I publish them separately in Netz (forthcoming b). Within this volume, I limit myself to immediate comments on the details of the text.

Given a line, AB, and another, AΓ, and an area, Δ: let it first be put forth:[337] to take a point on AB, such as E, so that it is: as AE to AΓ, so the area Δ to the <square> on EB.

(a) Let it come to be, (b) and let AΓ be set at right <angles> to AB, (c) and, having joined ΓE, (d) let it be drawn through to Z, (e) and let ΓH be drawn through Γ, parallel to AB, (f) and let ZBH be drawn through B, parallel to AΓ, meeting each of the <lines> ΓE, ΓH, (g) and let the parallelogram HΘ be filled in, (h) and let KEΛ be drawn through E parallel to either ΓΘ or HZ, (i) and let the <rectangle contained> by ΓHM be equal to the <area> Δ.

(1) Now since it is: as EA to AΓ, so the <area> Δ to the <square> on EB,[338] (2) but as EA to AΓ, so ΓH to HZ,[339] (3) and as ΓH to HZ, so the <square> on ΓH to the <rectangle contained> by ΓHZ,[340] (4) therefore as the <square> on ΓH to the <rectangle contained> by ΓHZ, so the <area> Δ to the <square> on EB, (5) that is to the <square> on KZ;[341] (6) alternately also: as the <square> on ΓH to the <area> Δ, that is to the <rectangle contained> by ΓHM, (7) so the <rectangle contained> by ΓHZ to the <square> on ZK.[342] (8) But as the <square> on ΓH to the <rectangle contained> by ΓHM, so ΓH to HM;[343] (9) therefore also: as ΓH to HM, so the <rectangle contained> by ΓHZ to the <square> on ZK. (10) But as ΓH to HM, so (HZ taken as a common height) the <rectangle contained> by ΓHZ to the <rectangle contained> by MHZ;[344] (11) therefore as the <rectangle contained> by ΓHZ to the <rectangle contained> by MHZ, so the <rectangle contained> by ΓHZ to the <square> on ZK; (12) therefore the <rectangle contained> by MHZ is equal to the <square> on ZK.[345] (13) Therefore if a parabola is drawn through H around the axis ZH, so that the lines drawn down <to the axis> are in square the <rectangle applied> along HM,[346] it shall pass through K,[347] (14) and it <=the parabola> shall be given in position, (15) through HM's being given in magnitude (16) as it contains, together with the given HΓ, the given <area> Δ;[348] (17) therefore K touches a parabola given in position.

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[337] I.e. "let the geometrical task be." [338] The assumption of the analysis.
[339] *Elements* VI.2, 4, and I.34. [340] *Elements* VI.1. [341] *Elements* I.34.
[342] *Elements* V.16. [343] *Elements* VI.1.
[344] *Elements* VI.1. [345] *Elements* V.7.
[346] For any point Z on the axis, the square on the line drawn from the parabola to the point Z, i.e. the square on KZ, is equal to the rectangle contained by ZH (i.e. the line to the vertex of the parabola) and by the constant line HM (the *latus rectum*) – i.e. to the rectangle ZHM.
[347] Converse of *Conics* I.11.
[348] For Step 15 to derive from Step 16, *Data* 57 is required. Step 14 derives from Step 15 in the sense that there is a unique parabola given an axis, a vertex and a *latus*

(j) Now, let it <=the parabola> be drawn, as has been said, and let it be as HK. (18) Again, since the area ΘΛ is equal to the <area> ΓB,[349] (19) that is the <rectangle contained> by ΘKΛ is equal to the <rectangle contained> by ABH, (20) if a hyperbola is drawn through B, around the asymptotes ΘΓ, ΓH, it shall pass through K, (21) through the converse of the 8th theorem of the second book of the *Conic Elements* of Apollonius,[350] (22) and it shall be given in position (23) through <the fact> that each of ΘΓ, ΓH is <given in position>, as well, (24) further yet – <through the fact> that B is given in position, too. (k) Let it be drawn, as has been said, and let it be as KB; (25) therefore K touches a hyperbola given in position; (26) and it also touched a parabola given in position. (27) Therefore K is given. (28) And KE is a perpendicular drawn from it to a <line> given in position, <namely> to AB; (29) therefore E is given.[351] (30) Now since it is: as EA to the given <line> AΓ, so the given <area> Δ to the <square> on EB: (31) two solids, whose bases are the <square> on EB and the <area> Δ, and whose heights are EA, AΓ, have the bases reciprocal to the heights; (32) so the solids are equal;[352] (33) therefore <the solid produced by> the <square> on EB, on EA <as the solid's height> is equal to <the solid produced by> the given <area> Δ on the given <line> ΓA <as the solid's height>.[353] (34) But <the solid produced by> the <square> on BE on EA <as the solid's height> is the greatest of all similarly taken <solids> on BA, when BE is twice EA, as shall be proved;[354] (35) therefore <the solid produced by> the given <area> on the given <line as the solid's height> must be not greater than <the solid produced by> the <square> on BE on EA <as the solid's height>.[355]

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rectum, once again through an obvious converse of *Conics* I.11 (any other conic section must yield two unequal lines drawn to the axis, both producing an equal rectangle when applied to the same *latus rectum*).

349 *Elements* I.43.

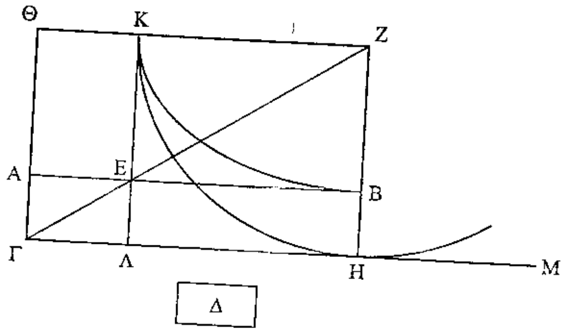
350 What we have as *Conics* II.12. Notice that this type of reference is most probably due to Eutocius.

351 *Data* 30. 352 *Elements* XI.34.

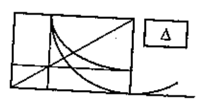
353 The expression "plane on line" has here a geometrical significance, yet it can be also interpreted as the multiplicative "on" used in the examples of calculation earlier, where we had "number on number." For this ambiguity of meaning, see Netz (forthcoming b).

354 For the modern reader: the maximum of $x^2(a - x)$ for $a>x>0$ is at $x = 2/3a$. Archimedes indeed proves this below, obviously, as we shall see, following a geometrical route.

355 That is, assuming BE is twice EA. The idea is the following. You take the original line BA, divide it at the point where BE is twice EA, derive the solid BE^2*EA, and now you've got a maximum for the solid $\Delta*\Gamma A$. Since both Δ and ΓA are independently given, they could theoretically be given in such a way that $BE^2*EA<\Delta*\Gamma A$. This is the limit on the conditions of solubility.



In II.4 Third diagram Codices DH have the rectangle Δ to the right of the main figure, as in the thumbnail. In these two codices, it is also a near square. Codex D has a redundant line parallel to ΘΓ, KΛ, between them; codex 4 had a redundant line AZ, erased (perhaps by the same scribe, immediately correcting a trivial error). Codex E has the line KΛ slightly slanted, so that K is to the right of Λ.



And it will be constructed like this: let the given line be AB, and some other given <line> ΑΓ, and the given area Δ, and let it be required to cut AB, so that it is: as one segment to the given AB, so the given <area> Δ to the <square> on the remaining segment.

(a) Let AE be taken, a third part of AB; (1) therefore the <area> Δ, on the <line> ΑΓ is either greater than the <square> on BE, on EA, or equal, or smaller.

(2) Now then, if it is greater, the problem may not be constructed, as has been proved in the analysis; (3) and if it is equal, the point E produces the problem. (4) For, the solids being equal, (5) the bases are reciprocal to the heights,[356] (6) and it is: as the <line> EA to the <line> ΑΓ, so the <area> Δ to the <square> on BE.

(7) And if the <area> Δ, on ΑΓ, is smaller than the <square> on BE, on EA, it shall be constructed like this:

(a) Let ΑΓ be set at right <angles> to AB, (b) and let ΓZ be drawn through Γ parallel to AB, (c) and let BZ be drawn through B parallel to the <line> ΑΓ, (d) and let it meet ΓE (<itself> being produced) at H, (e) and let the parallelogram ZΘ be filled in, (f) and let KEΛ be drawn through E parallel to ZH. (8) Now, since the <area> Δ, on ΑΓ, is smaller than the <square> on BE, on EA, (9) it is: as EA to ΑΓ, so the <area> Δ to some <area> smaller than the <square> on BE,[357] (10) that is, <smaller> than the <square> on HK.[358] (g) So let it be: as EA to ΑΓ, so the <area> Δ to the <square> on HM, (h) and let the <rectangle contained> by ΓZN be equal to the <area> Δ.[359] (11) Now since it is: as EA to ΑΓ, so the <area> Δ, that is the <rectangle contained> by ΓZN (12) to the <square> on HM,

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[356] *Elements* XI.34.

[357] The closest foundation in Euclid is *Elements* VI.16, proving that if $a*b = c*d$, then $a{:}d{::}c{:}b$ (for a, b, c and d being lines).

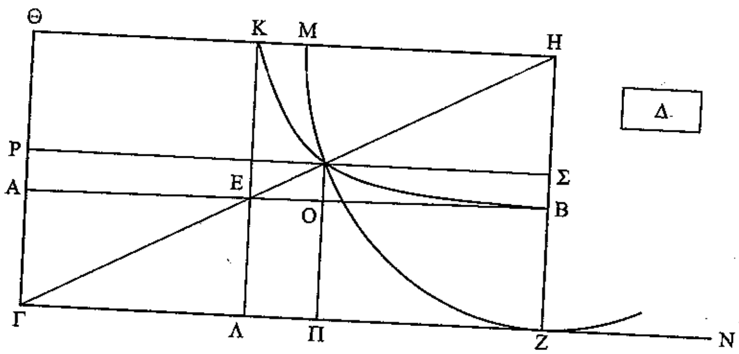
[358] Steps b, e, f, *Elements* I.34.

[359] Steps g and h define the points M, N respectively, by defining areas that depend upon those points.

(13) but as EA to ΑΓ, so ΓΖ to ΖΗ,[360] (14) and as ΓΖ to ΖΗ, so the <square> on ΓΖ to the <rectangle contained> by ΓΖΗ,[361] (15) therefore also: as the <square> on ΓΖ to the <rectangle contained> by ΓΖΗ, so the <rectangle contained> by ΓΖΝ to the <square> on ΗΜ; (16) alternately also: as the <square> on ΓΖ to the <rectangle contained> by ΓΖΝ, so the <rectangle contained> by ΓΖΗ to the <square> on ΗΜ.[362] (17) But as the <square> on ΓΖ to the <rectangle contained> by ΓΖΝ, ΓΖ to ΖΝ,[363] (18) and as ΓΖ to ΖΝ, (taking ΖΗ as a common height) so is the <rectangle contained> by ΓΖΗ to the <rectangle contained> by ΝΖΗ;[364] (19) therefore also: as the <rectangle contained> by ΓΖΗ to the <rectangle contained> by ΝΖΗ, so the <rectangle contained> by ΓΖΗ to the <square> on ΗΜ; (20) therefore the <square> on ΗΜ is equal to the <rectangle contained> by ΗΖΝ.[365] (21) Therefore if we draw, through Z, a parabola around the axis ΖΗ, so that the lines drawn down <to the axis> are, in square, the <rectangle applied> along ΖΝ – it shall pass through M.[366] (i) Let it be drawn, and let it be as the <parabola> ΜΞΖ. (22) And since the <area> ΘΛ is equal to the <area> ΑΖ,[367] (23) that is the <rectangle contained> by ΘΚΛ to the <rectangle contained> by ΑΒΖ,[368] (24) if we draw, through B, a hyperbola around the asymptotes ΘΓ, ΓΖ, it shall pass through K[369] (through the converse of the 8th theorem of <the second book of> Apollonius' *Conic Elements*). (j) Let it be drawn, and let it be as the <hyperbola> ΒΚ, cutting the parabola at Ξ, (k) and let a perpendicular be drawn from Ξ on ΑΒ, <namely> ΞΟΠ, (l) and let the <line> ΡΞΣ be drawn through Ξ parallel to ΑΒ. (25) Now, since ΒΞΚ is a hyperbola (26) and ΘΓ, ΓΖ are asymptotes,[370] (27) and ΡΞΠ[371] are drawn parallel to ΑΒΖ, (28) the <rectangle contained> by ΡΞΠ is equal to the <rectangle contained> by ΑΒΖ;[372] (29) so that the <area> ΡΟ, too, <is equal> to the <area> ΟΖ. (30) Therefore if a line is joined from Γ to Σ, it shall pass through Ο.[373] (m) Let it pass, and let it be as ΓΟΣ. (31) Now, since it is: as ΟΑ to ΑΓ, so ΟΒ to ΒΣ,[374] (32) that is ΓΖ to ΖΣ,[375] (33) and as ΓΖ to ΖΣ (taking ΖΝ as a

[360] Steps b, e, f, *Elements* I.29, 32, VI.4. [361] *Elements* VI.1.
[362] *Elements* V.16. [363] *Elements* VI.1. [364] *Elements* VI.1.
[365] *Elements* V.7. [366] The converse of *Conics* I.11.
[367] Based on *Elements* I.43.
[368] As a result of Step a (the angle at A right), all the parallelograms are in fact rectangles.
[369] Converse of what we call *Conics* II.12. [370] Steps 25–6: based on Step j.
[371] An interesting way of saying "the <lines> ΡΞ, ΞΠ." [372] *Conics* II.12.
[373] Step 30 is better put as: "The diagonal of the parallelogram ΡΣΖΓ passes through Ο," which can then be proved as a converse of *Elements* I.43.
[374] *Elements* I.29, 32, VI.4. [375] *Elements* VI.2.

common height) the <rectangle contained> by ΓZN to the <rectangle contained> by ΣZN,[376] (34) therefore as OA to AΓ, too, so the <rectangle contained> by ΓZN to the <rectangle contained> by ΣZN. (35) And the <rectangle contained> by ΓZN is equal to the area Δ,[377] (36) while the <rectangle contained> by ΣZN is equal to the <square> on ΣΞ, (37) that is to the <square> on BO,[378] (38) through the parabola.[379] (39) Therefore as OA to AΓ, so the area Δ to the <square> on BO. (40) Therefore the point O has been taken, producing the problem.



In II.4 Fourth diagram Codices BDG have the line AB parallel to ΓZ. Codex D has Δ as nearly a square. It also fails to have the points H, Σ, B, Z aligned on a single line, and does not draw a line HZ. Codex E has the lines ΛK, ΠΞ divergent so that K is to the left of Λ, Ξ is to the right of Π. Codex G has a straight line instead of the arc segment MΞ; H has an arc segment instead of the straight line ΞO. The letter N is omitted in codex A. Heiberg restores it on the line HZ, between the points Σ, B (he also removes the line segment that continues from ΓZ beyond Z).

And it will be proved like this that, BE being twice EA, the <square> on BE, on EA, is <the> greatest of all <magnitudes> similarly taken on BA.[380]

For let there be, as in the analysis, again: (a) a given line, at right <angles> to AB, <namely> AΓ, (b) and, having joined ΓE, (c) let it be produced and let it meet at Z the <line> drawn through B parallel to AΓ, (d) and, through the <points> Γ, Z, let ΘZ, ΓH be drawn parallel to AB, (e) and let ΓA be produced to Θ, (f) and, parallel to it, let KEΛ be drawn through E, (g) and let it come to be: as EA to AΓ, so the <rectangle contained> by ΓHM to the <square> on EB; (1) therefore the <square> on BE, on EA, is equal to the <rectangle contained> by ΓHM, on AΓ, (2) through the <fact> that the bases of the two solids are reciprocal to the heights.[381] Now I say that the <rectangle

376 *Elements* VI.1.

377 Step h. The original Greek is literally: "To the <rectangle contained> by ΓZN is equal the area Δ" (with the same syntactic structure, inverted by my translation, in the next step).

378 Steps a, e, k, l, *Elements* I.34.

379 A reference to *Conics* I.11 – the "symptom" of the parabola.

380 Here we reach the proof for the limits of solubility, promised at the end of the analysis.

381 *Elements* XI.34.

contained> by ΓHM, on AΓ, is <the> greatest of all <magnitudes> similarly taken on BA.[382]

(h) For let a parabola be drawn through H, around the axis ZH, so that the <lines> drawn down <to the axis> are in square the <rectangle applied> along HM;[383] (3) so it will pass through K, as has been proved in the analysis, (4) and, produced, it will meet ΘΓ (5) since it is parallel to the diameter of the section, ((6) through the twenty-seventh theorem of the first book of Apollonius' *Conic Elements*[384]). (i) Let it <=the parabola> be produced and let it meet <the line ΓΘ produced> at N, (j) and let a hyperbola be drawn through B, around the asymptotes NΓH; (7) therefore it will pass through K, as was said in the analysis. (k) So let it pass, as the <hyperbola> BK, (l) and, ZH being produced, (m) let HΞ be set equal to it <=to ZH>, (n) and let ΞK be joined, (o) and let it be produced to O; (8) therefore it is obvious, that it <=ΞO> will touch the parabola, (9) through the converse of the thirty-fourth theorem of the first book of Apollonius' *Conic Elements*.[385] (10) Now since BE is double EA ((11) for so it is assumed[386]) (12) that is ZK <is twice> KΘ,[387] (13) and the triangle OΘK is similar to the triangle ΞZK,[388] (14) ΞK, too, is twice KO.[389] (15) And ΞK is double KΠ, as well, (16) through the <facts> that ΞZ, too, is double KH,[390] (17) and that ΠH is parallel to KZ;[391] (18) therefore OK is equal to KΠ. (19) Therefore OKΠ, being in contact with the hyperbola, and lying between the asymptotes, is bisected <at the point of contact with the hyperbola>; (20) therefore it touches the hyperbola[392] (21) through the converse of the third theorem of the second book of Apollonius' *Conic Elements*. (22) And it touched the parabola, too, at the same <point> K. (23) Therefore the parabola touches the hyperbola at K.[393] (p) So let the

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382 The point E is taken implicitly to satisfy the relation mentioned in the introduction to the proof: EB is equal to twice EA.

383 For every point taken on the parabola (say, in this diagram, K): (sq.(KZ) = rect.(ZH, HM)). (The point Z is obtained by KZ being, in this case, at right angles to the axis of the parabola and, in general, by its being parallel to the tangent of the parabola at the vertex of the diameter considered for the property.)

384 What we call *Conics* I.26. 385 What we know as *Conics* I.33.

386 This is the implicit assumption of the entire discussion.

387 Step d, *Elements* I.30, 34. 388 Step c, *Elements* I.29, 32.

389 *Elements* VI.4. 390 Step m.

391 Step d, *Elements* I.30. Finally, Step 15 derives from 16, 17 through *Elements* VI.2.

392 In the sense of "being a tangent."

393 As far as the extant corpus goes, this is a completely intuitive statement. Not only in the sense that we do not get a proof of the implicit assumption ("if two conic sections have the same tangent at a point, they touch at that point"), but also in a much more fundamental way, namely, we never have the concept of two conic sections being tangents even *defined*.

hyperbola, produced, as towards P, be imagined as well,[394] (q) and let a chance point be taken on AB, <namely> Σ, (r) and let TΣY be drawn through Σ parallel to KΛ, (s) and let it meet the hyperbola at T, (t) and let ΦTX be drawn through T parallel to ΓH. (24) Now since (through the hyperbola and the asymptotes)[395] (25) the <area> ΦY is equal to the <area> ΓB; (26) taking the <area> ΓΣ away <as> common, (27) the <area> ΦΣ is then equal to the <area> ΣH, (28) and through this, the line joined from Γ to X will pass through Σ.[396] (u) Let it pass, and let it be as ΓΣX. (29) And since the <square> on ΨX is equal to the <rectangle contained> by XHM[397] (30) through the parabola,[398] (31) the <square> on TX is smaller than the <rectangle contained> by XHM.[399] (v) So let the <rectangle contained> by XHΩ come to be equal to the <square> on TX.[400] (32) Now since it is: as ΣA to AΓ, so ΓH to HX,[401] (33) but as ΓH to HX (taking HΩ as a common height), so the <rectangle contained> by ΓHΩ to the <rectangle contained> by XHΩ,[402] (34) and <the rectangle contained by ΓHΩ> to the <square> on XT (which is equal to it <=to the rectangle contained by XHΩ>)[403] (35) that is to the <square> on BΣ,[404] (36) therefore the <square> on BΣ, on ΣA, is equal to the <rectangle contained> by ΓHΩ, on ΓA.[405] (37) But the <rectangle contained> by ΓHΩ, on ΓA, is smaller

[394] In Step k it has been drawn only as far as K.

[395] Refers to *Conics* II.12, already invoked in setting-up the hyperbola. For the theorem to apply in the way required here, it is important that the asymptotes are at right angles to each other (as indeed provided by the setting-out of the theorem).

[396] Converse of *Elements* I.43.

[397] The point Ψ is the intersection of the parabola with the line ΦX. Since this line had not yet come to existence when the parabola was drawn, this point could not be made explicit then, and it is left implicit now, to be understood on the basis of the diagram – this, the most complex of diagrams!

[398] *Conics* I.11.

[399] Archimedes effectively assumes that, inside the "box" KZHΛ, the hyperbola is always "inside" the parabola. This is nowhere proved by Apollonius. Greeks could prove this, e.g. on the basis of *Conics* IV.26.

[400] This step does not construct a rectangle (this remains a completely virtual object). Rather, it determines the point Ω.

[401] Steps c, d, *Elements* I.29, 30, 32, VI.4. [402] *Elements* VI.1.

[403] Step v. [404] Steps r, t, *Elements* I.34.

[405] *Elements* XI.34. The structure of Steps 32–6 being somewhat involved, I summarize their mathematical gist: (32) ΣA:AΓ::ΓH:HX, but (33) ΓH:HX::rect.(ΓHΩ):rect.(XHΩ), (34) rect.(XHΩ) = sq.(XT) hence (from 33–4) the result (not stated separately): (34′) ΓH:HX::rect.(ΓHΩ):sq.(XT); then (35) sq.(XT)=sq.(BΣ) hence the result (not stated separately): (35′) ΓH:HX::rect.(ΓHΩ):sq.(BΣ) and, with 32 back in the argument, the result (not stated separately): (35″) ΣA:AΓ::rect.(ΓHΩ):sq.(BΣ) whence finally: (36) sq.(BΣ) on ΣA = rect.(ΓHΩ) on ΓA.

than the <rectangle contained> by ΓHM, on ΓA;[406] (38) therefore the <square> on BΣ, on ΣA, is smaller than the <square> on BE, on EA.

(39) So it will be proved similarly also in all the points taken between the <points> E, B.

But then let a point be taken between the <points> E,A, <namely> ς. I say that like this, too, the <square> on BE, on EA, is greater than the <square> on Bς, on ςA.[407]

(w) For, the same being constructed, (x) let ϘςP be drawn through ς parallel to KA, (y) and let it meet the hyperbola at P; (40) for it meets it, (41) through its being parallel to the asymptote;[408] (z) and, having drawn A′PB′ through P, parallel to AB, let it meet HZ (being produced), at B′. (42) And since, again, through the hyperbola, (43) the <area> Γ′Ϙ is equal to <the area> AH,[409] (44) the line joined from Γ to B′ will pass through ς.[410] (a′) Let it pass, and let it be ΓςB′. (45) And since, again, through the parabola, (46) the <square> on A′B′ is equal to the <rectangle contained> by B′HM.[411] (47) Therefore the <square> on PB′ is smaller than the <rectangle contained> by B′HM.[412] (b′) Let the <square> on PB′ come to be equal to the <rectangle contained> by B′HΩ.[413] (48) Now since it is: as ςA to AΓ, so ΓH to HB′,[414] (49) but as ΓH to HB′ (taking HΩ as a common height), so the <rectangle contained> by ΓHΩ to the <rectangle contained> by B′HΩ,[415] (50) that is to the <square> on PB′, (51) that is to the <square> on Bς,[416] (52) therefore the <square> on Bς, on ςA, is equal to the <rectangle contained> by ΓHΩ, on ΓA.[417] (53) But the <rectangle contained> by ΓHM is greater than the <rectangle contained> by ΓHΩ;[418] (54) therefore the <square> on BE on EA is greater than the <square> on Bς, on ςA, as well.

(55) So it shall be proved similarly in all the points taken between the <points> E, A, as well. (56) And it was also proved for all the <points>

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406 Step v, *Elements* XI.32.

407 In Netz (1999) I suggest that this part of the argument may be due to Eutocius, rather than Archimedes.

408 *Conics* II.13. 409 *Conics* II.12.

410 Converse to *Elements* I.43. 411 *Conics* I.11.

412 Steps 40–7 retrace the ground covered earlier at 24–31. Step 47 is unargued, like its counterpart 31.

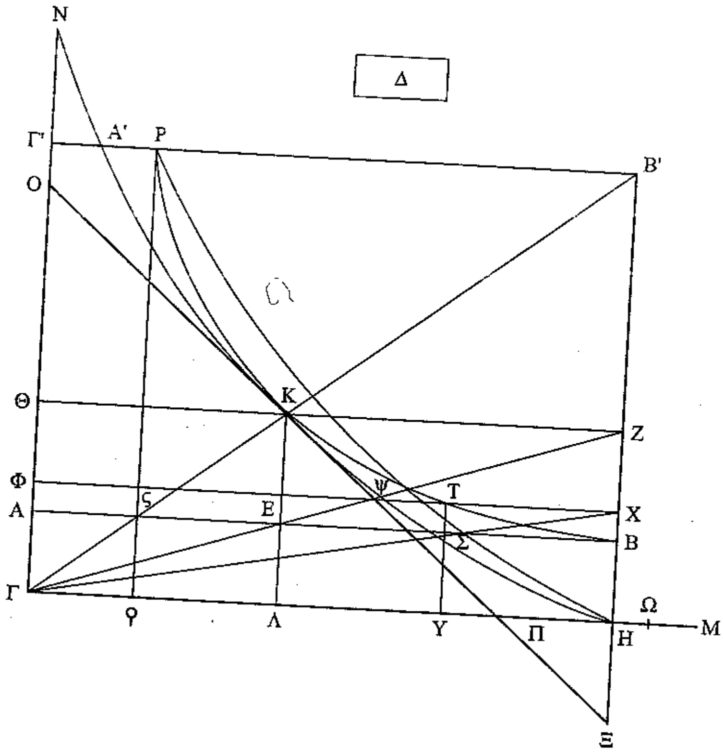
413 This is a very strange moment: an already determined point (Ω, determined at Step v above) is now being re-determined.

414 *Elements* I.29, 32, VI.4. 415 *Elements* VI.1.

416 Steps w, x, z, *Elements* I.30, 34. 417 *Elements* XI.34.

418 Step b′, *Elements* VI.1. The implicit result of: (52) sq.(Bς) on ςA = rect.(ΓHΩ) on ΓA and (53) rect.(ΓHM) > rect.(ΓHΩ) is (53′) sq.(Bς) on ςA < rect.(ΓHM) on ΓA. This implicit Step 53′ (together with Step 1!) is the basis of the next, final step.

between the <points> E, B; (57) therefore, of all the <magnitudes> taken similarly on AB, the greatest is the <square> on BE on EA, when BE is twice EA.



Now one must understand also the consequences of the diagram above.[419] For since it has been proved that the <square> on BΣ, on ΣA, and the <square> on Bς, on ςA, are smaller than the <square> on BE, on EA: <therefore> it is possible to produce the task assigned by the original problem, by cutting the <line> AB at two points (when the given area on the given <line> is smaller than the <square> on BE on EA).

(a) And this comes to be, if we imagine a parabola drawn around the diameter XH, so that the lines drawn down <to the diameter> are in square the <rectangle applied> along HΩ; (1) for such a parabola certainly passes through the <point> T.[420] (2) And since it <=the parabola> must meet ΓN ((3) being parallel to the diameter[421]), (4) it is clear that it cuts the hyperbola at some point above K, ((b) as, here,[422] at P), (5) and <it is clear that> a perpendicular drawn from P on AB

In II.4 Fifth diagram Codex A had the two curves – parabola and hyperbola – meet at a point higher, and to the left of P, as in the thumbnail. This is preserved in all copies. Codices E4 have the lower segment of the hyperbola (below K) drawn twice to the point B, once passing correctly at the point T in its correct position (the intersection TY/XΦ), once passing above it. Codex H has a similar arrangement, only it has the higher branch of the hyperbola terminate not at the point B, but above it, between B and X; codex D has only the higher branch, drawn to the point B. In all these codices DEH4, the letter T is consequently repositioned to be at the intersection of the higher branch and the line TY. It appears likely that codex A had the arrangement of E4. In all copies, the parabola passes rather near the point Σ (and not, as in the printed diagram, somewhat below the point). Perhaps it is intended to pass through that point. In codices DG, the rectangle Δ has the sides somewhat longer

419 Here we almost certainly have Eutocius, rather than Archimedes speaking.
420 From Step v of the proof, and then the converse to *Conics* I.11.
421 Step 2 derives from Step 3, through *Conics* I.26.
422 I.e. "in this diagram."

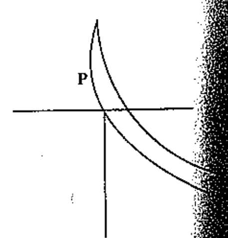
((c) as, here, Pς), cuts AB at ς, so that the point ς produces the task assigned by the problem, (6) and so that the <square> on BΣ on ΣA is then equal to the <square> on Bς on ςA (7) as is self-evident from the preceding proofs.

So that – it being possible to take two points on BA, producing the required task – one may take whichever one wishes, either the <point> between the <points> E, B, or the <point> between the <points> E, A. For if <one takes> the <point> between the <points> E, B, then, as has been said, one draws a parabola through the points H, T, which cuts the hyperbola at two points. <Of these two points,> the <point> closer to H, that is to the axis of the parabola, will procure[423] the <point> between the <points> E, B (as here T has procured Σ), while the point more distant <from the diameter will procure> the <point> between the <points> E, A (as here P procures ς).

Now, <taken> generally, the problem is analyzed and constructed like this. But, in order that it may also be applied to Archimedes' text,[424] let the diameter of the sphere ΔB be imagined (just as in the diagram of the text), and the radius BZ,[425] and the given <line> ZΘ.[426] Therefore he says the problem comes down to:

"To cut ΔZ at X, so that it is: as XZ to the given <line> so the given <square> to the <square> on ΔX. This, said in this way – without qualification – is soluble only given certain conditions."

In II.4 (*cont.*)
than the base. Codex G has the rectangle Δ to the left of the main figure. Many lines do not appear parallel. Horizontals: codex D has AB, ΦX climb to the right, ΘZ fall a little to the right; codex E has ΦX fall a little to the right; codex H has ΦX climb a little to the right; codex 4 has AB fall a little to the right. Verticals: codex E has P rather to the left of ς, K slightly to the left of Λ. Codex A had Φ instead of Ψ (so in all copies). Codex E omits Λ and Γ′, Codex G omits P. I think I might see a ϡ where Heiberg (whom I follow) prints a Γ′. The character is so rare either way that no real decision is possible.



[423] The verb *heurisko*, better known to mathematicians for its first perfect singular used by an animate subject (*hêurêka*, translated "I have found," "I've got it"), commonly used in the infinitive with an animate logical subject understood (in the definition of goal inside problems: "*dei heurein* . . ." translated "it is required to find . . ." i.e. by the mathematician). Here, a third-person present/future with an *inanimate* subject, the translation must be different, and mine is only one of many possible guesses.

[424] I.e., the text of *SC* II.4 (all this, after all, is a commentary to that proposition!). Archimedes has promised (*SC* II.4, passage following Step 40) to analyze and construct "both problems," meaning 1. The general problem, given any two lines and an area, 2. The problem required in this proposition (the given lines and area are limited within certain parameters). The lost text found by Eutocius contained only the first, general problem. Perhaps we have lost the particular case. (It is clear that Eutocius' source was not another text of the *SC* – he would have told us that – but rather, some compilation of mathematical results. In such a context, the particular problem would have been of no interest.) Perhaps Archimedes never did give a particular solution; perhaps he meant it to be implicit in the general solution. It is so, in a sense, and Eutocius' business here is to make this implicit particular solution explicit.

[425] Another case where identity and equality are not distinguished. Eutocius' intention is not that BZ is the radius, but that it is equal to the radius.

[426] Note that Eutocius' new diagram does not come directly from the original diagram of *SC* II.4. Eutocius produced a mirror-inversion of the original diagram, putting the greater segment to the right. This is done in order to make this new diagram fit the diagrams for the solution of the problem.

For if the given <area>, on the given <line> turns out to be greater than the <square> on ΔB, on BZ, the problem would be impossible, as has been proved.[427] And if it is equal, the point B would produce the task assigned by the problem, and in this way, too, the solution would have no relevance to what Archimedes originally put forward; for the sphere would not be cut according to the given ratio.[428] Therefore, said in this way, without qualification, it was only soluble given certain added conditions. "But with the added qualification of the specific characteristics of the problem at hand" (that is, both that ΔB is twice BZ and that BZ is greater than ZΘ), "it is always soluble." For the given <square> on ΔB, on ZΘ, is smaller than the <square> on ΔB, on BZ (through BZ's being greater than ZΘ), and, when this is the case, we have shown that the problem is possible, and how it then unfolds.

It should also be noticed that Archimedes' words fit with our analysis. For previously (following his analysis) he stated, in general terms, that which the problem came down to, saying: "it is required to cut a given <line>, ΔZ, at the <point> X, and to produce: as XZ to a given <line>, so the given <square> to the <square> on ΔX." Then he says that, in general, said in this way, this is soluble only given certain conditions, but with the addition of specific characteristics of the problem that he has obtained (that ΔB is twice BZ, and that BZ is greater than ZΘ), it is always soluble. And so he takes this problem in particular, and says this: "And the problem will be as follows: given two lines ΔB, BZ (and ΔB being twice BZ), and <given> a point on BZ, <namely> Θ; to cut ΔB at X . . ." – and no longer saying, as previously, that it is required to cut ΔZ, but <to cut> ΔB, instead – because he knew (as we ourselves have proved above) that there are two points which, taken on ΔZ, produce the task assigned by the problem, one between the <points> Δ, B, and another between the <points> B, Z. Of these, the <point> between the <points> Δ, B would be of use for what Archimedes put forward originally.[429]

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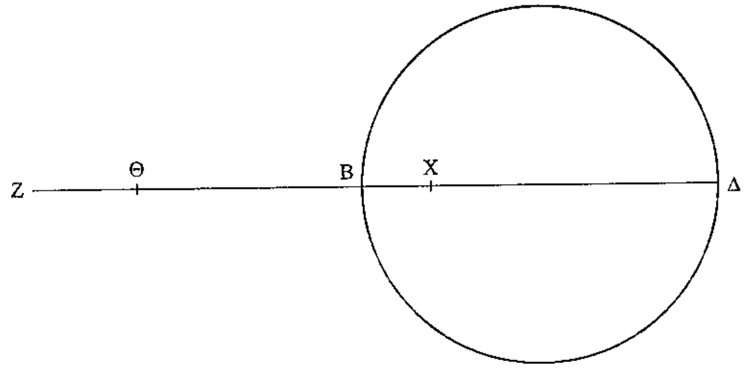
427 Since BZ is equal to the radius, and BΔ is the diameter, obviously BΔ is twice BZ. Hence the point B is the maximum for the solid, as shown in the lemma to the analysis. If the solid required by the terms of the problem is greater than this maximum, it simply cannot be constructed.

428 What we are looking for is a point at which to cut the sphere, so that its two segments then have a given ratio. The point B, on the surface of the sphere, can be said to produce no cutting into two segments at all. (Or, if it is said to cut the sphere, the two "segments" – one a sphere, one a point – do not have a ratio.)

429 A phrasing reminiscent of the point made above, why B would not do as a solution (it does not produce a cut in the sphere). The same consideration applies here: we require only that solution which picks a point inside the sphere.

In II.4 Sixth diagram



So we have copied this down, in conformity with Archimedes' words, as clearly as possible.[430]

Dionysodorus, too, as has been said above, did not get to read what Archimedes promised to have written at the end, and was incapable of discovering again, as it were, the unpublished proofs. Taking another route to the whole problem,[431] the method of solution he uses in his treatise is not without grace. We therefore thought it incumbent upon us to add him to the above, correcting the text as best we could. For with him, too, in all the manuscripts we had come across, as a result of men's massive carelessness, much of the proofs was difficult to understand, with the sheer number of mistakes.

As Dionysodorus

To cut the given sphere by a plane, so that its segments will have to each other the given ratio.

Let there be the given sphere, whose diameter is AB, and <let> the given ratio be that which ΓΔ has to ΔE. So it is required to cut the sphere by a plane, right to AB, so that the segment whose vertex is A has to the segment whose vertex is B the ratio which ΓΔ has to ΔE.

(a) Let BA be produced to Z, (b) and let AZ be set <as> half of AB, (c) and let ZA have to AH <that> ratio which ΓE has to EΔ, (d) and let AH be at right <angles> to AB, (e) and let AΘ

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[430] At face value, this seems to suggest that so far we had only Archimedes' words. But of course this is not the meaning. For the sake of the transition, from Archimedes to Dionysodorus, Eutocius lumps together all the preceding text as "Archimedes." It is always salutary to realize how careless are ancient commentators in signposting their text and dividing lemmas from commentary.

[431] Meaning now the main problem of *SC* II.4.

be taken <as> a mean proportional between ZA, AH; (1) therefore AΘ is greater than AH.[432] (f) And let a parabola be drawn through the <point> Z around the axis ZB, so that the <lines> drawn down <on the axis> are in square <the rectangles applied> along AH;[433] (2) therefore it shall pass through Θ, (3) since the <rectangle contained> by ZAH is equal to the <square> on AΘ.[434] (g) So let it be drawn, and let it be as the <line> ZΘK, (h) and let BK be drawn down through B, parallel to AΘ, (i) and let it cut the parabola at K, (j) and let a hyperbola be drawn through H, around ZBK <as> asymptotes; (4) so it cuts the parabola between the <points> Θ, K.[435] (k) Let it cut <the parabola> at Λ, (l) and let ΛM be drawn <as> a perpendicular from Λ on AB, (m) and let HN, ΛΞ be drawn through H, Λ parallel to AB. (5) Now since HΛ is a hyperbola, (6) and ABK are asymptotes, (7) and MΛΞ are parallel to AHN, (8) the <rectangle contained> by AHN is equal to the <rectangle contained> by MΛΞ, (9) through the 8th theorem of the second book of Apollonius' *Conic Elements*.[436] (10) But HN is equal to AB,[437] (11) while ΛΞ <is equal> to MB; (12) therefore the <rectangle contained> by ΛMB is equal to the <rectangle contained> by HAB, (13) and through the <fact> that the <rectangle contained> by the extremes is equal to the <rectangle contained> by the means, (14) the four lines are proportional;[438] (15) therefore it is: as ΛM to HA, so AB to BM; (16) therefore also: as the <square> on ΛM to the <square> on AH, so the <square> on AB to the <square> on BM. (17) And since (through the parabola), the

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[432] AΘ is greater than AH, because it is the mean proportional in the series ZA – AΘ – AH (Step e). ZA is greater than AH, because ZA, AH have the same ratio as ΓE, EΔ (Step c), and ΓE is greater than EΔ – which, finally, we know from the diagram.

[433] Notice that the *latus rectum* is here not at the vertex of the parabola.

[434] Step e, *Elements* VI.17. Step 2 derives from Step 3 on the basis of the converse of *Conics* I.11.

[435] The key insight of Archimedes' solution was that the parabola contained the hyperbola in the relevant "box." The key insight of Dionysodorus' solution is that the hyperbola cuts the parabola at the relevant "box." Both insights are stated without proof, typical for such topological insights in Greek mathematics. Dionysodorus' understanding of the situation may have been like this. Concentrate on the wing of the hyperbola to the right of AΘ. It must get closer and closer to the line BK, without ever touching that line (BK is an asymptote to the hyperbola: the relevant proposition is *Conics* II.14). So the hyperbola cannot pass wholly below or above the point K; at some point, well before reaching the line BK, it must pass higher than the point K. Since at the stretch ΘK, the parabola's highest point is K (this can be shown directly from the construction of the parabola, *Conics* I.11), what we have shown is that the hyperbola, starting below the parabola (H below Θ), will become higher than the parabola, well before either reaches the line BK. Thus they must cut each other.

[436] What we call *Conics* II.12. [437] *Elements* I.34. [438] *Elements* VI.16.

<square> on ΛM is equal to the <rectangle contained> by ZM, AH,[439] (18) therefore it is: as ZM to MΛ, so MΛ to AH;[440] (19) therefore also: as the first to the third, so the <square> on the first to the <square> on the second and the <square> on the second to the <square> on the third;[441] (20) therefore as ZM to AH, so the <square> on ΛM to the <square> on HA. (21) But as the <square> on ΛM to the <square> on AH, so the <square> on AB was proved <to be> to the <square> on BM; (22) therefore also: as the <square> on AB to the <square> on BM, so ZM to AH. (23) But as the <square> on AB to the <square> on BM, so the circle whose radius is equal to AB to the circle whose radius is equal to BM;[442] (24) therefore also: as the circle whose radius is equal to AB to the circle whose radius is equal to BM, so ZM to AH; (25) therefore the cone having the circle whose radius is equal to AB <as> base, and AH <as> height, is equal to the cone having the circle whose radius is equal to BM <as> base, and ZM <as> height;[443] (26) for such cones, whose bases are in reciprocal proportion to the heights, are equal.[444] (27) But the cone having the circle whose radius is equal to AB <as> base, and ZA <as> height, is to the cone having the same base, but <having> AH <as> height, as ZA to AH,[445] (28) that is ΓE to EΔ ((29) for, being on the same base, they are to each other as the heights[446]); (30) therefore the cone, too, having the circle whose radius is equal to AB <as> base, and ZA <as> height, is to the cone having the circle whose radius is equal to BM <as> base, and ZM <as> height, as ΓE to EΔ. (31) But the cone having the circle whose radius is equal to AB <as> base, and ZA <as> height, is equal to the sphere,[447] (32) while the cone having the circle whose radius is equal to BM <as>

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439 *Conics* I.11. 440 *Elements* VI.17. 441 *Elements* VI.20 Cor. 2.

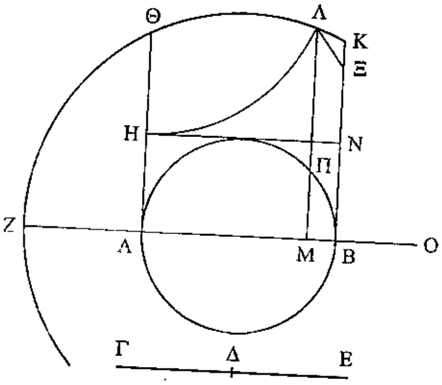
442 *Elements* XII.2. Those are curious circles. We are not quite given them, since we do not know their exact radii. (We know what their radii are *equal to*, but this is not yet knowing what they *are*.) On the other hand, these are fully fledged individuals: they are "the" circles of their kind, not just "a" circle whose radius is equal to a given line. Over and above the semi-reality of the diagram, we are asked to invent another toy reality, where certain unnamed circles subsist. More of this to follow.

443 The toy circles, introduced in Step 23, spring out of their boxes, now as cones. These cones have a particularly funny spatial location: their heights are not merely equal to certain lines, but *are* in fact those certain lines. Hence they are half in the toy universe of the circles, half in the more tangible universe of the diagram. Or more precisely: the sense of location has been eroded, and we are faced with purely hypothetical geometrical objects.

444 *Elements* XII.15. 445 *Elements* XII.14.

446 This belated explicit reference to *Elements* XII.14 is meant to support Step 27, not Step 28. It is probably Eutocius' contribution and, if so, so are probably the other references to the *Elements* and the *Conics*.

447 *SC* I.34.



In II.4 Seventh diagram Codex E has the line ΛΞ drawn to the point K, so that the points Ξ, K coalesce; meanwhile, it repositions the letter Ξ so that it appears to belong to the point Λ (!), right beneath it. The letter Ξ is similarly, at least ambiguously positioned in codices GH. All codices omit the letter N.

base, and ZM <as> height, is equal to the segment of the sphere, whose vertex is B, and <whose> height is BM, (33) as shall be proved further on; (34) therefore the sphere, too, has to the said segment the ratio which ΓE has to EΔ; (35) dividedly, also: the segment, whose vertex is A, and <whose> height is AM, has to the segment, whose vertex is B, and <whose> height is BM, this ratio, which ΓΔ has to ΔE.[448] (36) Therefore the plane produced through ΛM, right to AB, cuts the sphere according to the given ratio; which it was required to do.

And it shall be proved like this, that the cone having the circle whose radius is equal to BM <as> base, and ZM <as> height, is equal to the segment of the sphere whose vertex is B, and whose height is BM:

(n) For let it come to be: as ZM to MA, so OM to MB;[449] (37) therefore the cone having the same base as the segment, and OM <as> height, is equal to the segment.[450] (38) And since it is: as ZM to MA, so OM to MB, (39) alternately also: as ZM to MO, so AM to MB,[451] (40) but as AM to MB, so the <square> on ΠM to the <square> on MB,[452] (41) and as the <square> on ΠM to the <square> on MB, so the circle whose radius is equal to ΠM, to the circle whose radius is equal to MB,[453] (42) therefore as the circle whose radius is equal to ΠM, to the circle whose radius is equal to MB, so MZ to MO. (43) Therefore the cone having the circle whose radius is equal to MB <as> base, and ZM <as> height, is equal to the cone having the circle whose radius is equal to ΠM <as> base, and MO <as> height; (44) for their bases are in reciprocal proportion to the heights;[454] (45) so

[448] *Elements* V.17. [449] This is the definition of the point O.
[450] *SC* II.2. [451] *Elements* V.16.
[452] The point Π is defined by the diagram alone. It is the intersection of the line ΛM with the circle. The claim is based on *Elements* III.31, VI.8 Cor., VI.20 Cor. 2.
[453] *Elements* XII.2. [454] *Elements* XII.15.

that it <=the cone having the circle whose radius is equal to MB as base, and ZM as height> is equal to the segment, too.

As Diocles in *On Burning Mirrors*[455]

And Diocles, too, gives a proof, following this introduction:

Archimedes proved in *On Sphere and Cylinder* that every segment of a sphere is equal to a cone having the same base as the segment, and, <as> height, a line having a certain ratio to the perpendicular <drawn> from the vertex of the segment on the base: <namely, the ratio> that: the radius of the sphere, and the perpendicular of the alternate segment, taken together, have to the perpendicular of the alternate segment.[456] For instance, if there is a sphere ABΓ, and if it is cut by a certain plane, <namely> the circle around the diameter ΓΔ,[457] and if (AB being diameter, and E center) we make: as EA, ZA taken together to ZA, so HZ to ZB, and yet again, as EB, BZ taken together to ZB, so ΘZ to ZA, it is proven: that the segment of the sphere ΓBΔ is equal to the cone whose base is the circle around the diameter ΓΔ, while its height is ZH, and that the segment ΓAΔ is equal to the cone whose base is the same, while its height is ΘZ. So he set himself the task of cutting the given sphere by a plane, so that the segments of the sphere have to each other the given ratio, and, making the construction above, he says: "(1) Therefore the ratio of the cone whose base is the circle around the diameter ΓΔ, and whose height is ZΘ, to the cone whose base is the same, while its height is ZH, is given, too;"[458] (2) and indeed, this too was proved;[459] (3) and cones which are on equal bases are to each other as the heights;[460] (4) therefore the ratio of ΘZ to ZH is given. (5) And since it is: as ΘZ to ZA, so EBZ taken together to ZB, (6) dividedly: as ΘA to AZ, so EB to ZB.[461] (7) And so through the same <arguments> also: as HB to ZB, so the same line <=EB> to ZA.

So a problem arises like this: with a line, <namely> AB, given in position, and given two points A, B, and given EB, to cut AB at Z and

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455 The following text corresponds to Propositions 7–8 of the Arabic translation of Diocles' treatise (Toomer [1976] 76–86, who also offers in 178–92 a translation of the passage in Eutocius with a very valuable discussion, 209–12).

456 *SC* II.2.

457 The circle meant is that perpendicular to the "plane of the page," or to the line AB.

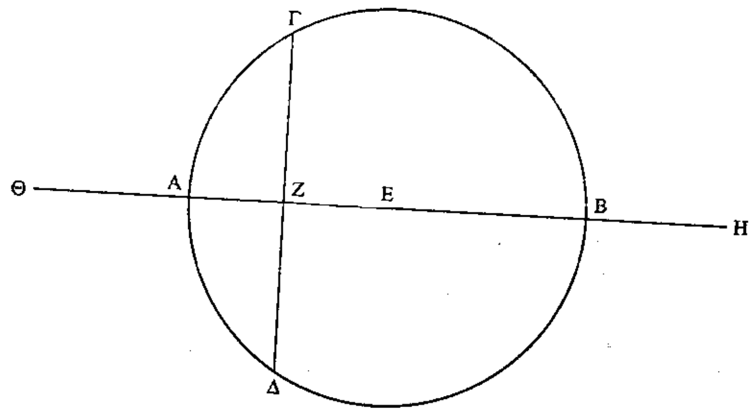
458 This text is part Diocles' own analysis, part a re-creation of Archimedes' analysis, now in the terms of Diocles' diagram. Step 1 here corresponds to *SC* II.4 Step 4.

459 Step 2 probably means: "by proving *SC* II.2, we thereby prove the claim of step 1."

460 *Elements* XII.14.

461 *Elements* V.17.

to add ΘA, BH so that the ratio of ΘZ to ZH will be <the> given, and also, so that it will be: as ΘA to AZ, so the given line to ZB, while as HB to BZ, so the same given line to ZA.



In II.4 Eighth diagram

And this is proved in what follows. For Archimedes, having proved the same thing, rather long-windedly, even so he then reduced it to another problem, which he does not prove in the *Sphere and Cylinder*![462]

Given in position a line AB, and given two points A, B, and the ratio, which Γ has to Δ, to cut AB at E and to add ZA, HB, so that it is: as Γ to Δ, so ZE to EH; and also that it is: as ZA to AE, so a certain given line to BE, and as HB to BE, so the same given line to EA.[463]

(a) Let it come to be, (b) and let ΘAK, ΛBM be drawn at right <angles> to AB, (c) and let each of AK, BM be set equal to the given line. (d) Joining the <lines> KE, ME, let them be produced to Λ, Θ, (e) and let KM be joined, as well, (f) and let ΛN be drawn through Λ, parallel to AB, (g) and let ΞEOΠ <be drawn> through E, <parallel> to NK. (1) Now since it is: as ZA to AE, so MB to BE; (2) for this is assumed; (3) and as MB to BE, so ΘA to AE (4) through the similarity of the triangles,[464] (5) therefore as ZA to AE, so ΘA to AE. (6) Therefore

[462] Archimedes transformed the problem of *SC* II.4 into another, more general problem, at a certain remove from the sphere to be cut. To solve that problem, conic sections were required, but since the problem was much more general, and since its solution was removed to an appendix, *Sphere and Cylinder* remained cordoned off from conic sections, preserving a certain elementary aspect. Diocles, on the other hand, applies conic sections to the terms of the problem arising directly from *Sphere and Cylinder*. This makes his approach at the same time more direct, but less elegant.

[463] The "certain given line" remains unlabeled.

[464] The triangles referred to are ΘAE, BEM. That they are similar can be seen through Step b, *Elements* I.27, I.29, I.15 (or I.29, I.32). Step 3 derives from Step 4 through *Elements* VI.4.

ZA is equal to ΘA.[465] (7) So, through the same <arguments>, BH, too, <is equal> to BΛ.[466] (8) And since it is: as ΘAE taken together to MBE taken together, so KAE taken together to ΛBE taken together; (9) for each of the ratios is the same as the <ratio> of AE to EB;[467] (10) therefore the <rectangle contained> by ΘAE taken together and by ΛBE taken together, is equal to the <rectangle contained> by KAE taken together and by MBE taken together;[468] (h) Let each of AP, BΣ be set equal to KA.[469] (11) Now since ΘAE taken together is equal to ZE, (12) while ΛBE taken together is equal to EH, (13) and KAE taken together is equal to PE, (14) and MBE taken together is equal to ΣE, (15) and the <rectangle contained> by ΘAE taken together and by ΛBE taken together was proved to be equal to the <rectangle contained> by KAE taken together and by MBE taken together, (16) therefore the <rectangle contained> by ZEH is equal to the <rectangle contained> by PEΣ. (17) So through this, whenever P falls between the <points> A, Z, then Σ falls outside H, and vice versa.[470] (18) Now since it is: as Γ to Δ, so ZE to EH, (19) and as ZE to EH, so the <rectangle contained> by ZEH to the <square> on EH,[471] (20) therefore: as Γ to Δ, so the <rectangle contained> by ZEH to the <square> on EH. (21) And the <rectangle contained> by ZEH was proved equal to the <rectangle contained> by PEΣ; (22) therefore it is: as Γ to Δ, so the <rectangle

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465 *Elements* V.9.

466 The setting-out and Step a, again, provide the proportion HB:BE::KA:AE and, through the similarity of the triangles KAE, ΛEB the argument is obvious.

467 By "each of the ratios" Diocles refers to the ratios of the separate lines making up the "taken together" objects. So we have four ratios: ΘA:MB, AE:BE, KA:ΛB, AE:BE (AE:BE occurs twice). All, indeed, are the same as AE:BE, through the similarities of triangles we have already seen. Step 8 follows from Step 9 through successive applications of *Elements* V.18.

468 *Elements* VI.16. Notice a possible source of confusion. The rectangles are each contained by two lines, and each of these lines is a sum of two lines, denoted by three characters. This is confusing, because often we have a rectangle contained by two lines, and these containing two lines are directly denoted by three characters. Here the summation happens not between the sides of the rectangles, but inside each of the sides.

469 Thus all lines AP, BΣ, KA, BM are now equal to the unlabeled, given line – this anonymous line is cloned, as it were, all through the diagram.

470 The "vice versa" means that, conversely to what has been mentioned, also when H falls between B, Σ, then P falls outside Z. ("Outside" here means "away from the center of the diagram" – imagine the diagram as an underground network, and imagine that the lines have two directions, "Inbound" and "Outbound).") This is a remarkable moment. Diocles (or Eutocius?) is aware both of topological considerations, and of a functional relation between variables. But the basic thought is very simple: it is impossible to have two equal rectangles, if the sides of one of the rectangles are both greater than the sides of the other. If one side is greater, the other must be smaller. This is not stated in the *Elements*, but it is implicit in *Elements* VI.16. (That P, Σ, must both be "outside" AB, is implicit in the construction of the points and is learned from the diagram.)

471 *Elements* VI.1.

contained> by PEΣ to the <square> on EH. (i) Let EO be set equal to BE, (j) and, joining BO, let it be produced to either side, (k) and, drawing ΣT, PY from Σ, P at right <angles to the line AB>, (l) let them meet it <=the line BO, produced> at T, Y. (23) Now since the <line> TY has been drawn through a given <point> B, (24) producing, to a <line> given in position, <namely> to AB, an angle (<namely>, the <angle contained> by EBO), half of a right <angle>,[472] (25) TY is given in position.[473] (26) And the <lines> ΣT, PY, <given> in position, are drawn from given <points,> Σ, P, cutting it <=the line TY, given in position,> at T, Y; (27) therefore T, Y are given;[474] (28) therefore TY is given in position and in magnitude. (29) And since, through the similarity of the triangles EOB, ΣTB,[475] (30) it is: as TB to BO, so ΣB to BE,[476] (31) it is compoundly, also: as TO to OB, so ΣE to EB.[477] (32) But as BO to OY, so BE to EP.[478] (33) Therefore also, through the equality: as TO to OY, so ΣE to EP.[479] (34) But as TO to OY, so the <rectangle contained> by TOY to the <square> on OY, (35) and as ΣE to EP, so the <rectangle contained> by ΣEP to the <square> on EP;[480] (36) therefore also: as the <rectangle contained> by TOY to the <square> on OY, so the <rectangle contained> by ΣEP to the <square> on EP; (37) alternately also: as the <rectangle contained> by TOY to the <rectangle contained> by ΣEP, so the <square> on OY to the <square> on EP. (38) And the <square> on OY is twice the <square> on EP, (39) since the <square> on OB is twice the <square> on BE, too;[481] (40) therefore the <rectangle contained> by TOY, too, is twice the <rectangle contained> by ΣEP. (41) And the <rectangle contained> by ΣEP was proved to have, to the <square> on EH, the ratio which Γ has to Δ; (42) and therefore the <rectangle contained> by TOY has to the <square> on EH the ratio, which twice Γ has to Δ. (43) And the <square> on EH is equal to the <square> on ΞO; (44) for each of the <lines> EH, ΞO is equal to ΛBE taken together;[482] (45) Therefore the <rectangle contained> by TOY has to the <square> on ΞO <the> ratio, which twice Γ has to Δ. (46) And the ratio of twice Γ to Δ is given; (47) therefore the ratio of the <rectangle contained> by TOY to the <square> on ΞO is given as well.

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[472] From Step i, OE = EB. From Steps b, g, OEB is a right angle. Then the claim of Step 24 is seen through *Elements* I.32.

[473] *Data* 30.

[474] *Data* 25.

[475] Steps b, g, k, *Elements* I.27, 29, 15 (or 32).

[476] *Elements* VI.4.

[477] *Elements* V.18.

[478] Steps b, g, k, *Elements* I.27, VI.2.

[479] *Elements* V.22.

[480] Steps 34–5: both from *Elements* VI.1.

[481] This is through the special case of Pythagoras' theorem (*Elements* I.47) for an isosceles right-angled triangle.

[482] ΞE=ΛB (through Steps b, f, g, *Elements* I.27, 30, 34). EO = EB through Step i. So this settles ΞO = ΛBE. EH = ΛBE can be seen through Step 7.

(48) Therefore if we make: as Δ to twice Γ, so TY to some other <line>, as Φ, and if we draw an ellipse around TY, so that the <lines> drawn down <on the diameter>, inside the angle ΞOB (that is <inside> half a right <angle>), are in square the <rectangles applied> along Φ, falling short by a <figure> similar to the <rectangle contained> by TY, Φ,[483] <the ellipse> shall pass through the point Ξ, (49) through the converse of the twentieth theorem of the first book of Apollonius' *Conic Elements*.[484] (m) Let it be drawn and let it be as YΞT; (50) therefore the point Ξ touches an ellipse given in position. (51) And since ΛK is a diagonal of the parallelogram NM,[485] (52) the <rectangle contained> by NΞΠ is equal to the <rectangle contained> by ABM.[486] (53) Therefore if we draw a hyperbola through the <point> B, around ΘKM <as> asymptotes, it shall pass through Ξ,[487] (54) and it shall be given in position ((55) through the <facts> that the point B, too, is given in position, (56) as well as each of the <lines> AB, BM, (57) and also, through this, the asymptotes ΘKM). (n) Let it be drawn and let it be as ΞB; (58) therefore the point Ξ touches a hyperbola given in position. (59) And it also touched an ellipse given in position; (60) therefore Ξ is given.[488] (61) And ΞE is a perpendicular <drawn> from it; (62) therefore E is given.[489] (63) And since it is: as MB to BE, so ZA to AE, (64) and AE is given, (65) therefore AZ is given, as well.[490] (66) So, through the same <arguments>, HB is given as well.[491]

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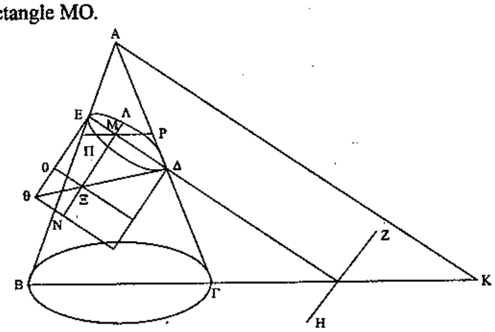
483 This is the Apollonian way of stating that Φ is the parameter of the ellipse. Imagine that Φ is set at the point T, at right angles to the line YT. You get a configuration similar to that of *Conics* I.13 (see figure. Φ here is transformed into EΘ), for which Apollonius proves that for any point Λ taken on the ellipse EΛΔ, the square on ΛM is equal to the associated rectangle MO.



484 What we call *Conics* I.21.

485 Steps b, c, f, *Elements* I.27, 33.

486 Based on *Elements* I.43.

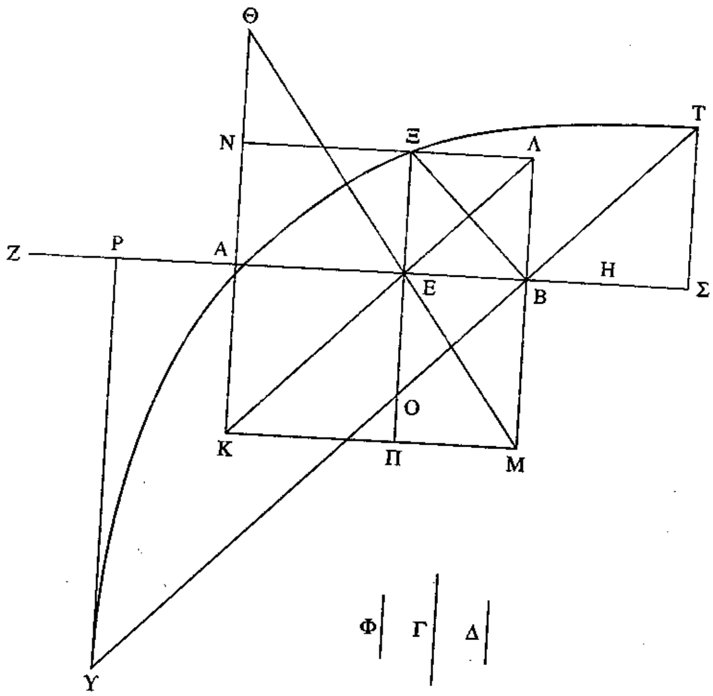
487 Converse of *Conics* II.12.

488 *Data* 25.

489 *Data* 30.

490 With E given, BE is given as well. BM is given from setting-out, Step c, hence BM:BE is given. Step 65 then derives from *Data* 2.

491 The only difference will be that instead of using the proportion MB:BE::ZA:AE, we use the proportion HB:BE::BM:EA (both from setting-out, Step c).



In II.4 Ninth diagram Codex D omits the line ΞB. (Since the line represents a hyperbola, Heiberg draws it as a curved line). Codex D further has the lines KΛ, TY, not parallel, meet at the point K. It also positions the three lines Φ, Γ, Δ at the "slot" beneath BΣ, and has Φ and Δ greater (rather than smaller) than Γ. Codex A had X instead of K, and has permuted the letters H, N. All are corrected in the codices BG. Codex D has both X, K where K is required. Codex H has E instead of Σ.

And it shall be constructed like this:

For (as in the same diagram), let the given line, which it is required to cut, be AB, and let the other given <line> be AK, and <let> the given ratio <be> the <ratio> of Γ to Δ.

(a) Let BM (being equal to AK) be drawn at right <angles> to AB, (b) and let KM be joined, (c) and let AP and BΣ be set equal to KA, (d) and let PY, ΣT be drawn from P, Σ at right <angles to AB>, (e) and let half a right <angle> be constructed at the point B, <namely> the angle <contained> by ABO, (f) and, producing BO to either side, let it cut ΣT, PY at T, Y, (g) and let it come to be: as Δ to twice Γ, so TY to Φ, (h) and let an ellipse be drawn around TY, so that the <lines> drawn down inside half a right angle are in square the <rectangles> applied along Φ, falling short by a figure similar to the <rectangle contained> by TY, Φ, (i) and let a hyperbola be drawn through B, around AK, KM <as> asymptotes, <namely> the <hyperbola> BΞ, (j) cutting the ellipse at Ξ, (k) and let a perpendicular, <namely> ΞE, be drawn from Ξ on AB, (l) and let it be produced to Π, (m) and let ΛΞN be drawn through Ξ parallel to AB, (n) and let KA, MB be produced to Λ, Θ, (o) and, joining ME, let it be produced and let it meet KN at Θ. (1) Now since BΞ is a hyperbola, and ΘK, KM are asymptotes, (2) the <rectangle contained> by NΞΠ is equal to the <rectangle contained> by ABM, (3) through the 8th theorem of the second book of Apollonius' *Conic Elements*,[492] (4) and, through this, KEΛ is a straight <line>.[493] (p) So let AZ be set equal to ΘA, (q) and <let> BH <be set> equal to ΛB. (5) Now since it is: as twice Γ to Δ, so Φ to TY, (6) and as Φ to TY, so the <rectangle contained> by TOY to the <square> on ΞO, (7) through the 20th theorem of the first book of Apollonius' *Conic Elements*,[494] (8) therefore as twice Γ to Δ, so the <rectangle contained> by TOY to the <square> on ΞO. (9) And since it is: as TB to BO, so ΣB to BE,[495] (10) compoundly also: as TO to OB, so ΣE to EB.[496] (11) But as BO to OY, so BE to EP;[497] (12) therefore through the equality, also: as TO to OY, so ΣE to EP.[498] (13) Therefore also: as the <rectangle contained> by TOY to the <square> on OY, so the <rectangle contained> by ΣEP to the <square> on EP;[499] (14) Alternately: as the <rectangle contained> by TOY to the <rectangle contained> by ΣEP, so the <square> on OY to the <square> on EP.[500] (15) But the <square> on OY is twice the <square> on EP (16) through <the fact> that the <square> on BO, too, is <twice> the <square> on BE;[501] ((17) for BE is equal to EO,[502] (18) each of the <angles> at B, O being half right);[503] (19) therefore the <rectangle contained> by TOY, too, is twice the <rectangle contained> by ΣEP.[504] (20) Now since it was proved: as twice Γ to Δ, so the <rectangle contained> by TOY to the <square> on ΞO,[505] (21) also the halves of the antecedents; (22) therefore as Γ to Δ, so the <rectangle contained> by ΣEP to the <square> on ΞO, (23) that is to the <square> on EH; (24) for ΞO is equal to EH, (25) through <the fact> that each of them is equal to ΛBE taken together.[506] (26) Now since it is: as ΘAE taken together to MBE taken together, so KAE taken together to ΛBE taken together; (27) for each of the ratios is the same as the <ratio> of AE to EB;[507] (28) therefore

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492 What we call *Conics* II.12.

493 Converse of *Elements* I.43. (As usual, the assumption that NΛMK is a parallelogram is not made explicit. It can be shown on the basis of setting-out, Steps a, m, *Elements* I.30, 33.)

494 What we call *Conics* I.21.

495 Steps b, d, *Elements* I.28, 29, and then I.15 (or I.32), and finally VI.4.

496 *Elements* V.18.

497 Steps d, k, l, *Elements* I.28, VI.2.

498 *Elements* V.22.

499 Successive applications of *Elements* VI.1.

500 *Elements* V.16.

501 Step 15 derives from Step 16 through Steps d, k; *Elements* VI.2, then V.17, then VI.22.

502 Step 16 derives from Step 17 through Step k, *Elements* I.6, 47.

503 Steps e, k, *Elements* I.32.

504 Steps d, k, *Elements* VI.2, and then V.18, and then VI.22.

505 Step 8.

506 Steps q, 17. Also: Steps a, k, l, m together with *Elements* I.29, 34.

507 Step 26 derives from 27 through Steps a, *Elements* I.29, VI.2, and then V.18 (that the angle at A is right is an assumption carried over without mention from the setting-out of the analysis).

the <rectangle contained> by ΘAE taken together and by ΛBE taken together is equal to the <rectangle contained> by KAE taken together and by MBE taken together.[508] (29) But ΘAE taken together is equal to ZE,[509] (30) while ΛBE taken together is equal to EH, (31) and KAE taken together is equal to PE, (32) and MBE taken together is equal to EΣ; (33) therefore the <rectangle contained> by ZEH is equal to the <rectangle contained> by PEΣ. (34) But as Γ to Δ, so the <rectangle contained> by PEΣ to the <square> on EH; (35) therefore also: as Γ to Δ, so the <rectangle contained> by ZEH to the <square> on EH. (36) But as the <rectangle contained> by ZEH to the <square> on EH, so ZE to EH;[510] (37) therefore also: as Γ to Δ, so ZE to EH. (38) And since it is: as MB to BE, so ΘA to AE,[511] (39) and ΘA is equal to ZA, (40) therefore as MB to BE, so ZA to AE. (41) And through the same <arguments> also: as KA to AE, so HB to BE.[512]

Therefore given a line, <namely> AB, and another, <namely> AK, and a ratio, <namely that> of Γ to Δ, a chance point has been taken on AB, <namely> E, and lines have been added, <namely> ZA, HB; and ZE was then to EH in the given ratio, and it is also: as the given <line> MB to BE, so ZA to AE, and as the same given <line> KA[513] to AE, so HB to BE; which it was required to do.

These things proved, it is possible to cut the given sphere according to the given ratio, like this:

For let the diameter of the given sphere be AB, and <let> the given ratio, which the segments of the sphere are required to have to each other, be the <ratio> of Γ to Δ; (a) and let E be center of the sphere; (b) and let a point, Z, be taken on AB, (c) and let HA, ΘB be added so that it is: as Γ to Δ, so HZ to ZΘ, and further yet it is: as HA to AZ, so EB, given, to BZ while as ΘB to BZ, so the same given <line,> EA, to AZ; for it has been proved above that it is possible to do this; (d) and let KZΛ be drawn through Z at right <angles> to AB, (e) and let a plane, produced through KΛ, right to AB, cut the sphere. I say that the segments of the sphere have to each other the ratio of Γ to Δ.

[508] *Elements* VI.16.

[509] Step p. The original syntactic structure is: "But to ΘAE taken together is equal ZE" (and similarly with the following equalities).

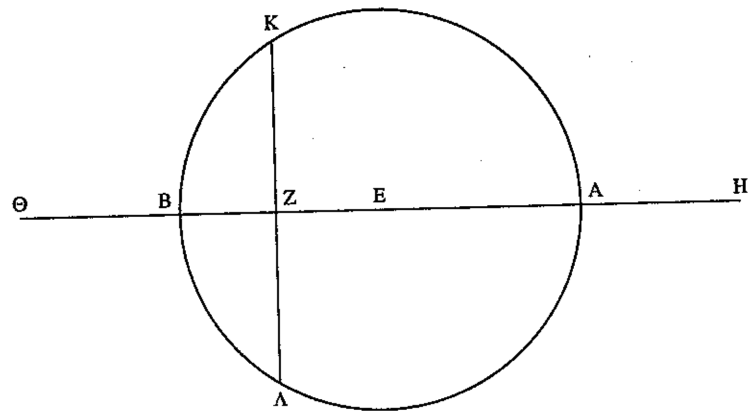
[510] *Elements* VI.1.

[511] Compare the argument for the derivation of Step 26 from Step 27.

[512] Substitute Step q for Step p, in the chain of reasoning, and the argument is indeed the same.

[513] "The same" as MB (equality is taken here for identity: also compare Step c in the synthesis following).

(1) For since it is: as HA to AZ, so EB to BZ,[514] (2) also compoundly;[515] (3) therefore as HZ to ZA, so EB, BZ taken together to BZ; (4) therefore the cone having the circle around the diameter KΛ <as> base, and ZH as height, is equal to the segment of the sphere having the same base and ZA <as> height.[516] (5) Again, since it is: as ΘB to BZ, so EA to AZ, (5) it is also, compoundly: as ΘZ to BZ, so EA, AZ taken together to AZ;[517] (6) therefore the cone having the circle around the diameter KΛ <as> base, and ZΘ as height, is equal to the segment of the sphere having the same base, and BZ <as> height.[518] (7) Now since the said cones, being on the same bases, are to each other as the heights,[519] (8) that is as HZ to ZΘ, (9) that is Γ to Δ, (10) therefore the segments of the sphere, as well, have to each other the given ratio; which it was required to do.



In II.4 Tenth diagram Codex D has AH considerably greater than ΘB.

And we shall prove like this, how one draws a hyperbola through the given point, around the given asymptotes, (as this is not a self-evident outcome of the *Conic Elements*):[520]

Let there be two lines, ΓA, AB, containing a chance angle (that at A), and let some point Δ be given, and let it be put forth: to draw a hyperbola through Δ around ΓA, AB <as> asymptotes.

(a) Let AΔ be joined and produced to E, (b) and let AE be set equal to ΔA, (c) and let ΔZ be drawn through Δ parallel to AB, (d) and let ZΓ be set equal to AZ, (e) and, having joined ΓΔ, let it be produced to B, (f) and let the <rectangle contained> by ΔE, H be equal to the <square> on ΓB,[521] (g) and, producing the <line> AΔ,

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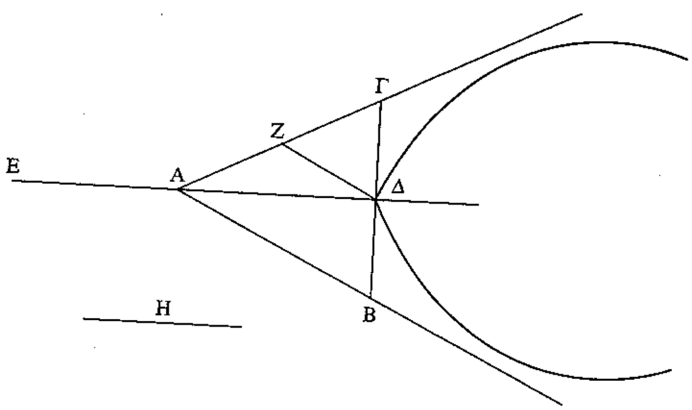
[514] Step c. [515] *Elements* V.18. [516] *SC* II.2.
[517] *Elements* V.18. [518] *SC* II.2. [519] *Elements* XII.14.
[520] As noted by Heiberg, at this point we definitely move from Diocles to Eutocius, who refers to Apollonius' *Conics*.
[521] Step f defined the point E.

let a hyperbola be drawn around it,[522] through the <point> Δ, so that the <lines> drawn down <on the axis> are in square <the rectangles applied> along H, exceeding by <a figure> similar to the <rectangle contained> by ΔE, H.[523] I say that ΓA, AB are asymptotes of the drawn hyperbola.

(1) For since ΔZ is parallel to BA, (2) and ΓZ is equal to ZA, (3) therefore ΓΔ, too, is equal to ΔB:[524] (4) so that the <square> on ΓB is four times the <square> on ΓΔ. (5) And the <square> on ΓB is equal to the <rectangle contained> by ΔE, H; (6) therefore each of the <squares> on ΓΔ, ΔB is a fourth part of the figure <contained> by ΔE, H.[525] (7) Therefore ΓA, AB are asymptotes of the hyperbola, (8) through the first theorem of the second book of Apollonius' *Conic Elements*.



In II.4 Eleventh diagram
Codex D has the hyperbola drawn as a single arc (about a semicircle). (So does the heavily corrected codex B, which also has the asymptotes touch the hyperbola). Codex E does not continue the line EΔ inside the hyperbola itself. Codex A has omitted the line H with its letter; it is reinserted as a late correction in Codex B, at the position printed.

To the synthesis of 4

In the synthesis he produces the diameter of the sphere, ΔB, and sets next to it the line ZB, equal to its half, and he cuts it at Θ by the given ratio, and he takes the <point> X on ΔB in such a way that it is: as

[522] This time "around" means "around it as diameter" (and not, as above, "around as asymptote"). If a hyperbola is around one line, it is around a diameter; if around two lines, it is around asymptotes.

[523] The formulaic way of stating that E is the center, and H the parameter of the hyperbola.

[524] *Elements* VI.2.

[525] In Apollonius' *Conics* II, the expression "the fourth of the figure <contained> by . . ." becomes formulaic, hence the word "figure" here, which refers simply, in this case, to the contained rectangle.

XZ to ΘZ, so the <square> on BΔ to the <square> on ΔX – making
Arch. 205 the same construction as before. He then says that; "let it come to be: as KΔX taken together to ΔX, so PX to XB," and he sets P between the <points> Θ, Z.

It ought to be proved that this is the case.[526] (1) For since it is: as KΔX taken together to ΔX, PX to XB, (2) dividedly: as KΔ to ΔX, PB to XB;[527] (3) alternately: as KΔ to PB, ΔX to BX.[528] (4) But ΔX is greater than XB; (5) therefore KB, too, is greater than BP[529] (6) that is ZB <is greater> than BP; (7) so that P falls inside Z. (8) That it also falls outside Θ shall be proved similarly to the <arguments> in the analysis (as the entire synthesis of the theorem proceeds < = similarly to the analysis>).[530] (9) For it is obtained, that it is: as PX to XΛ, BΘ to ΘZ,[531] (10) so that compoundly, also.[532] And through this the present proof, too, follows in accordance to what was said above.[533]

Arch. 205 "And through the equality in the perturbed proportion." We learned in the *Elements* that "a perturbed proportion is, there being three magnitudes and others equal to them in multitude, when it is: as antecedent to consequent in the first magnitudes, so, in the second magnitudes, antecedent to consequent, while, as consequent to some other <magnitude> in the first, so, in the second, some other <magnitude> to antecedent."[534] Now, it has also been proved here that as antecedent

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526 I.e. that, given the construction, the position of P is indeed as in the diagram, i.e. between the points Θ, Z. As Eutocius will make clear, this is essentially the same as his note to Step 21 of the analysis of this proposition. Nothing in Archimedes' argument relies on the exact position of the point. This is a commentator's, not a mathematician's "ought." The force of the "ought" is that this is an interesting point to comment upon, not that this is a logical lacuna.

527 *Elements* V.17. 528 *Elements* V.16.

529 KΔ, KB are taken to be interchangeable (both radii).

530 The original grammar is very compressed; perhaps some words have been lost? The point is clear: the synthesis is the same as the analysis, even with the same labeling of the diagram, hence precisely the same arguments would apply without any change including, presumably, Eutocius' comment to Step 21.

531 Step 16 in the synthesis (inverted).

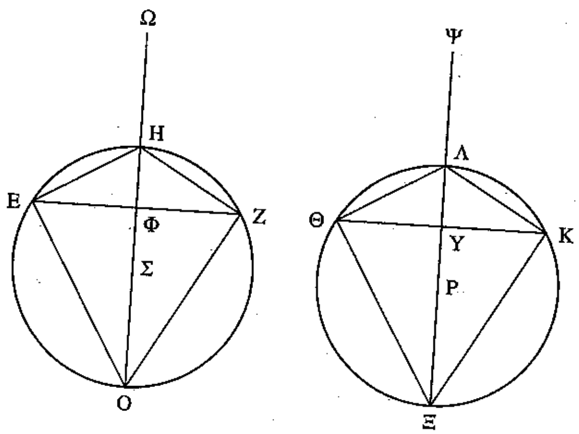
532 *Elements* V. 18. The implicit claim is: PΛ:XΛ::BZ:ΘZ.

533 What Eutocius has done is to show that the construction of Step h in the analysis holds in the synthesis as well (although it is not made explicit in the synthesis). Having shown that, he is justified in simply pointing backwards to his argument on that Step h.

534 Euclid's formulation at *Elements* V Def. 18 (essentially unchanged by Eutocius) suffers from the difficulty of marking out, without lettering, an object which is arbitrary and yet fixed. The strange bare nouns, "antecedent in the first magnitudes," "consequent in the first magnitudes" etc. are just this: one of the terms in the proportion, no matter which, but the same throughout the definition. Heath's lettered and typographic transcription is "a *perturbed proportion* is an expression for the case when, there being three magnitudes a, b, c and three others A, B, C, a is to b as B is to C, b is to c as A is to B."

PΛ to consequent ΛΔ, so antecedent XZ to consequent ZΘ, and as consequent ΔΛ to some other <magnitude>, the <line> ΛX, so some other <magnitude>, the <line> BZ, to antecedent XZ. Therefore, as proved in the fifth book of the *Elements*,[535] it follows through the equality, as well: as PΛ to ΛX, so BZ to ZΘ.[536]

To 5



In II.5
Codex D has the lines EZ, ΘK slanted (E somewhat higher than Z, Θ somewhat higher than K). Codices E4 have Z instead of Ξ, while codices DE4 omit Y. (Codex A probably had Ξ and, if it had the Y, it must have been very inconspicuous.)

"And since the segment EZH is similar to the segment ΘKΛ, therefore
Arch. 209 the cone EZΩ, as well, is similar to the cone ΨΘK." (a) For let the diagrams be imagined set apart,[537] (b) and the <lines> EH, HZ, EO, OZ, ΘΛ, ΛK, ΘΞ, ΞK joined. (1) Now since the segments EZH, ΘKΛ are similar,[538] (2) the angles <contained> by EHZ, ΘΛK are equal, too;[539] (3) so that their halves, too. (4) And the <angles> at Φ, Y are right;[540] (5) therefore the remaining, as well, is equal to the remaining.[541] (6) Therefore the triangle HΦZ is equiangular to the <triangle> ΛYK, (7) and it is: as HΦ to ΦZ, ΛY to YK.[542] (8) So, through the same (the triangles ΦZO, YKΞ being equiangular) (9) as ZΦ to ΦO, KY to YΞ; (10) therefore through the equality: as HΦ to ΦO, ΛY to YΞ.[543]

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[535] *Elements* V.23.

[536] This is the first time we see a comment whose sole function is to show how Euclid's *Elements directly* validate an Archimedean move. This is extremely interesting in showing how the various components of the *Elements* need not be all equally accessible for all readers. Was Eutocius puzzled by *Elements* V.23?

[537] I.e. we concentrate on just the two circles. [538] Step a of Archimedes' proof.

[539] *Elements* III Def. 11. [540] Step c of Archimedes' proof.

[541] *Elements* I.32. [542] *Elements* VI.4. [543] *Elements* V.22.