PROCLUS

A COMMENTARY ON THE

FIRST BOOK OF

EUCLID'S ELEMENTS

Translated

with Introduction and Notes by

GLENN R. MORROW

Adam Seybert Professor Emeritus
of Moral and Intellectual Philosophy
University of Pennsylvania

PROPOSITIONS:
PART ONE

NOW THAT we have summed up these matters, it remains[1] for us to examine the propositions that come after the principles. Up to this point we have been dealing with the principles, and it is against them that most critics of geometry have raised objections, endeavoring to show that these parts[2] are not firmly established. Of those in this group whose arguments have become notorious some, such as the Sceptics,[3] would do away with all knowledge, like enemy troops destroying the crops of a foreign country, in this case a country that has produced philosophy, whereas others, like the Epicureans, propose only to discredit the principles of geometry. Another group of critics, however, admit the principles but deny that the propositions coming after the principles can be demonstrated unless they grant something that is not contained in the principles. This method of controversy was followed by Zeno of Sidon,[4] who belonged to the school of Epicurus and against whom Posidonius has written a whole book and shown that his views are thoroughly unsound.

The disputes about the principles have been fairly well disposed of in our preceding exposition, and Zeno's attack will concern us a little later.[5] For the present let us briefly

[1] 199.2   I follow Barocius in reading λοιπόν for λοιπῶν. This section of the text (to 200.18) in Barocius is continuous with the preceding and constitutes the end of the *Principia*. Grynaeus also makes it continuous with the preceding but provides no separate heading for the Propositions that follow.

[2] 199.5   To explain τὰ μέρη in Friedlein's text we must assume that ταῦτα has dropped out just before these words.

[3] 199.9   Ἐφεκτικοί, the followers of Pyrrho of Elis, who advocated withholding judgment in order to avoid falling into error. Diog. Laert. I, 16; IX, 69-70.

[4] 199.15   Zeno of Sidon, an Epicurean of the late second and early first century B.C., noted as a lucid and copious author. Diog. Laert. VII, 35; X, 25; Cicero, *Academica* I, 46.

[5] 200.6   At 214.18ff.

JS:

these matters, it re-
positions that come
e have been dealing
n that most critics of
voring to show that
of those in this group
is some, such as the
owledge, like enemy
country, in this case a
whereas others, like
dit the principles of
ever, admit the prin-
ming after the prin-
ant something that is
ethod of controversy
longed to the school
s has written a whole
ighly unsound.
ive been fairly well
l, and Zeno's attack
present let us briefly

for λοιπῶν. This section
ious with the preceding
aeus also makes it con-
parate heading for the

xt we must assume that

of Elis, who advocated
into error. Diog. Laert.

he late second and early
us author. Diog. Laert.

201

review the definitions of theorem and problem, the distinction between them, the parts of each and the kinds into which they can be divided, and then turn to the exposition of the matters demonstrated by the author of the *Elements*. We shall select the more elegant of the comments made on them by the ancient writers, though we shall cut short their endless loquacity and present only what is most competent and relevant to scientific procedures, giving greater attention to the working out of fundamentals than to the variety of cases and lemmas which, we observe, usually attract the attention of the younger students of the subject.

I.[6] *On a given finite straight line to construct an equilateral triangle.*

Science as a whole has two parts: in one it occupies itself with immediate premises, while in the other it treats systematically the things that can be demonstrated or constructed from these first principles, or in general are consequences of them. Again this second part, in geometry, is divided into the working out of problems and the discovery of theorems. It calls "problems" those propositions whose aim is to produce, bring into view, or construct what in a sense does not exist, and "theorems" those whose purpose is to see, identify, and demonstrate the existence or nonexistence of an attribute. Problems require us to construct a figure, or set it at a place, or apply it to another, or inscribe it in or circumscribe it about another, or fit it upon or bring it into contact with another, and the like; theorems endeavor to grasp firmly and bind fast by demonstration the attributes and inherent prop-

[6] 200.19 There is no point in reproducing Friedlein's separate numbering of problems and theorems, and I have merely assigned numbers to the propositions, as does Heiberg in his edition of the *Elements*. The distinction between theorem and problem is one to which Proclus attaches great methodological importance; but although he usually indicates at the beginning of his commentary on a proposition whether it is a problem or a theorem, yet in later references to it he usually calls it a theorem, or more simply designates it by a number, e.g. "the ninth," or "the fourth."

erties belonging to the objects that are the subject-matter of geometry.

Every kind of question that is a possible subject of inquiry is considered by geometry, some of them being referred to problems, others to theorems. Geometry asks the question "What is it?" and that in two senses: it wants either the definition and notion or the actual being of the thing. I mean, for example, when it asks "What is the homoeomeric line?" it wishes to find the definition of such a line, namely, "the homoeomeric line is a line all of whose parts fit upon each other," or to grasp the actual species of homoeomeric lines, that is, "it is either a straight line, a circular line, or a cylindrical helix." In addition, geometry asks "Does the object exist as defined?" This it does most of all in diorismi, examining whether the question proposed is or is not capable of solution, to what extent it is so capable, and in how many ways. And of course geometry asks "What sort of thing is it?" For when it investigates the properties that belong intrinsically to a triangle, or a circle, or to parallel lines, this is clearly an attempt to determine what sort of thing it is.

Many persons have thought that geometry does not investigate the cause, that is, does not ask the question "Why?" Amphinomus is of this opinion, though Aristotle originated it.[7] But you will find this question also included in geometry, says Geminus. For is it not the task of the geometer to inquire why it is that an indefinite number of equilateral polygonal figures can be inscribed in a circle, whereas in a sphere it is not possible to inscribe an indefinite number of polyhedra with equal sides and angles and composed of similar faces? For whose task would it be, if not the geometer's, to ask and find the answer to this question? It is true that, when the reasoning employs reduction to impossibility, geometers are content merely to discover an attribute; and again when they use a previous demonstration to prove a particular conclusion,

202

203

204

[7] 202.11   This reference to Aristotle is difficult to understand. The *Post. Anal.* insists that demonstration is reasoning that establishes the cause (αἰτία or διὰ τί): cf. esp. 85b23ff. And the αἰτία that Aristotle demands appears to be identical with the conception of Geminus, as cited here.

subject-matter of

subject of inquiry
being referred to
asks the question
s either the defini-
thing. I mean, for
oeomeric line?" it
namely, "the ho-
irts fit upon each
omoeomeric lines,
ar line, or a cylin-
"Does the object
1 diorismi, examin-
iot capable of solu-
n how many ways.
of thing is it?" For
long intrinsically to
;, this is clearly an
is.
try does not investi-
e question "Why?"
Aristotle originated
cluded in geometry,
geometer to inquire
quilateral polygonal
eas in a sphere it is
imber of polyhedra
ed of similar faces?
ometer's, to ask and
true that, when the
bility, geometers are
and again when they
particular conclusion,

icult to understand. The
asoning that establishes
And the αἰτία that Aris-
conception of Geminus,

203

204

the cause is not evident. But if the conclusion is universal and applies to all similar cases, the reason why is by that very fact made manifest.

So much for the questions that geometry considers. Every problem and every theorem that is furnished with all its parts should contain the following elements: an enunciation, an exposition, a specification, a construction, a proof, and a conclusion.[8] Of these the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.

So many are the parts of a problem or a theorem. The most essential ones, and those which are always present, are enunciation, proof, and conclusion; for it is alike necessary to know in advance what is being sought, to prove it by middle terms, and to collect what has been proved. It is impossible that any of these three should be lacking; the other parts are often brought in but are often left out when they serve no need. For example, both specification and exposition are omitted in the problem "To construct an isosceles triangle having each of its base angles double the other angle."[9] And in most theorems there is no construction, because the exposition is sufficient, without the addition of anything else, to prove the proposed conclusion from the given. When, then, do we say the exposition is lacking? When the enunciation contains no statement of what is given. For although enunciation in general consists of what is given and what is sought, this is not always so. Sometimes it states only what is sought, that is, what must be known or constructed, as in the problem

---

[8] 203.4f. The Greek terms here are respectively πρότασις, ἔκθεσις, διορισμός, κατασκευή, ἀπόδειξις, συμπέρασμα.
[9] 204.2 Euclid IV. 10.

just mentioned. For that problem does not announce what is the given from which we are to construct an isosceles triangle having each of its equal angles double the remaining angle, but simply that we are to construct such a triangle. At the same time even in this case we understand the proposal on the basis of preexisting knowledge, for as it happens we know the meaning of "isosceles," of "equality," and of "double"; and such preexisting knowledge, Aristotle says,[10] is the characteristic feature of all discursive learning. Nevertheless there is no specific hypothesis, as in other problems—for example, when we are required to divide a given finite straight line into two equal parts.[11] For here a straight line is given, and we are asked to divide it into two parts; so what is given is separate from what is sought. When, therefore, the enunciation contains both these elements, then we find both specification and exposition; but when the given is lacking, so are these others also. For the exposition is dependent on the given and the specification will be identical with the enunciation.[12] For what else could you say in defining the problem mentioned than that we are to construct an isosceles of such-and-such a sort? But this is what the enunciation said. Whenever, therefore, an enunciation does not contain a statement both of what is given and of what is sought, the exposition is silent because there is no given element to expound, and the specification is omitted in order not to repeat the enunciation. You could find many other such problems, particularly in the arithmetical books and in Book X, where we are asked, for example, to find two straight lines commensurate in square that have a mean proportional between them,[13] and many other cases of this sort.

Furthermore, everything that is given is given in one of the following ways: in position, in ratio, in magnitude, or in species. A point is given in position only; but the line and

10 204.17   *Post. Anal.* 71a1-2.

11 204.20   X below.

12 205.1   Omitting with Schönberger the period and the γάρ in the following sentence.

13 205.11   Euclid X. 28. The arithmetical books are VII, VIII, and IX.

: not announce what is
ict an isosceles triangle
he remaining angle, but
a triangle. At the same
ie proposal on the basis
happens we know the
and of "double"; and
ays,¹⁰ is the characteris-
Nevertheless there is no
ns—for example, when
te straight line into two
e is given, and we are
that is given is separate
e, the enunciation con-
find both specification
is lacking, so are these
dependent on the given
with the enunciation.¹²
ning the problem men-
in isosceles of such-and-
iciation said. Whenever,
ontain a statement both
t, the exposition is silent
expound, and the speci-
eat the enunciation. You
ims, particularly in the
where we are asked, for
commensurate in square
ween them,¹³ and many

ven is given in one of the
io, in magnitude, or in
a only; but the line and

ie period and the γάρ in the

cal books are VII, VIII, and

206

the other figures may be given in all these ways. When we speak of a given angle [to be bisected] as rectilinear, we declare its kind, that is, show what sort of angle is given, namely, a rectilinear, so that we may not attempt to bisect a curvilinear angle by the same method. When we are required from two given unequal straight lines to cut off from the greater a length equal to the lesser,¹⁴ our given is presented in magnitude; for greater and less, finite and unbounded, are predications peculiar to magnitude. When we say that if four magnitudes are in proportion they will also be in proportion alternately,¹⁵ what is given is an identity of ratios among these four quantities. Whenever we are asked to place at a given point a straight line equal to a given line,¹⁶ then the point is given in position; and since the position may vary, the construction admits of various possibilities. The given point may lie outside the straight line, or on it and at either one of its ends, or on the portion between its extremities. Since, therefore, the given may be understood in these four ways, clearly the exposition may be fourfold in kind. Sometimes two or three of the ways of being given are combined.

What is called "proof" we shall find sometimes has the properties of a demonstration in being able to establish what is sought by means of definitions as middle terms, and this is the perfect form of demonstration; but sometimes it attempts to prove by means of signs.¹⁷ This point should not be overlooked. Although geometrical propositions always derive their necessity from the matter under investigation, they do not always reach their results through demonstrative methods. For example, when from the fact that the exterior angle of a triangle is equal to the two opposite interior angles it is shown that the sum of the interior angles of a triangle is equal to two right angles,¹⁸ how can this be called a demonstration based on the cause? Is not the middle term used here only a sign? For even though there be no exterior

---

¹⁴ 205.21 As in III.   ¹⁵ 206.1 As in V. 16.
¹⁶ 206.4 As in II.
¹⁷ 206.15 τεκμήρια. See Arist. *Prior Anal.* 70b1-3.
¹⁸ 206.22 As in XXXII below.

angle, the interior angles are equal to two right angles; for it is a triangle even if its side is not extended. But when we demonstrate that the triangle constructed by the drawing of circles is equilateral, our approach is from the cause. For we can assert that it is the similarity and equality of the circles that causes the equality of the sides of the triangle.

207

Furthermore, mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort. It is not as having such-and-such a size that the angle before me is bisected, but as being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles; but if I make no use of its rightness and consider only its rectilinear character, the proposition will apply equally to all angles with rectilinear sides.

208

Let us view the things that have been said by applying them to this our first problem. Clearly it is a problem, for it bids us devise a way of constructing an equilateral triangle. In this case the enunciation consists of both what is given and what is sought. What is given is a finite straight line, and what is sought is how to construct an equilateral triangle on it. The statement of the given precedes and the statement of what is sought follows, so that we may weave them together as "If

209

vo right angles; for it
ended. But when we
ed by the drawing of
m the cause. For we
quality of the circles
the triangle.
stomed to draw what
en they have shown
they infer that it is
lar to the universal
se of the particular
ngle or the straight
fore our eyes, they
en angle or straight
similar case. They
n in order that we
ed to the particular
for the demonstra-
agram not as these
others of the same
a size that the
ctilinear and noth-
of the given angle,
feature of all rec-
a right angle. If
, I should not be
lass of rectilinear
less and consider
sition will apply

said by applying
a problem, for it
uilateral triangle.
vhat is given and
ht line, and what
iangle on it. The
ment of what is
together as "If

209

there is a finite straight line, it is possible to construct an equilateral triangle on it." If there were no straight line, no triangle could be produced, for a triangle is bounded by straight lines; nor could it if the line were not finite, for an angle can be constructed only at a definite point, and an unbounded line has no end point.

Next after the enunciation is the exposition: "Let this be the given finite line."[19] You see that the exposition itself mentions only the given, without reference to what is sought. Upon this follows the specification: "It is required to construct an equilateral triangle on the designated finite straight line." In a sense the purpose of the specification is to fix our attention; it makes us more attentive to the proof by announcing what is to be proved, just as the exposition puts us in a better position for learning by producing the given element before our eyes. After the specification comes the construction: "Let a circle be described with center at one extremity of the line and the remainder of the line as distance; again let a circle be described with the other extremity as center and the same distance as before;[20] and then from the point of intersection of the circles let straight lines be joined to the two extremities of the given straight line." You observe that for the construction I make use of the two postulates that a straight line may be drawn from any point to any other and that a circle may be described with [any] center and distance. In general the postulates are contributory to constructions and the axioms to proofs. Next comes the proof: "Since one of the two points on the given straight line is the center of the circle enclosing it, the line drawn to the point of intersection is

[19] 208.17   Euclid's construction is as follows. Since Proclus follows Euclid's proof fairly closely in the commentary on this proposition, it is unnecessary to reproduce Euclid's reasoning here.
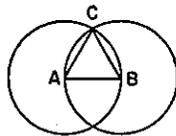
[20] 209.3   This and the following line in Friedlein have obviously been corrupted. Barocius had a better text, and I follow his translation.

equal to the given straight line. For the same reason, since the other point on the given straight line is itself the center of the circle enclosing it, the line drawn from it to the point of intersection is equal to the given straight line." These inferences are suggested to us by the definition of the circle, which says that all the lines drawn from its center are equal. "Each of these lines is therefore equal to the same line; and things equal to the same thing are equal to each other" by the first axiom. "The three lines therefore are equal, and an equilateral triangle [ABC][21] has been constructed on this given straight line." This is the first conclusion following upon the exposition. And then comes the general conclusion: "An equilateral triangle has therefore been constructed upon the given straight line." For even if you make the line double that set forth in the exposition, or triple, or of any other length greater or less than it, the same construction and proof would fit it.

To these propositions he adds: "This is what it was required to do," thus showing that this is the conclusion of a problem; for in the case of a theorem he adds: "This is what was to be demonstrated." For problems announce that something is to be done, theorems that some truth is to be discovered and demonstrated. In general, then, our geometer adds these words to his conclusions to show that what the enunciation stated has been accomplished, joining the end to the beginning in imitation of the Nous that unfolds itself and then returns to its starting-point. But he does not always add the same words: sometimes they are "This is what it was required to do" and sometimes "This is what was to be demonstrated," according to the difference between problems and theorems.

We have thus exercised ourselves and clarified all these distinctions by applying them to a single case, the first problem. The student should do this also for the remaining propositions, asking which of the principal elements are in-

210

211

---

[21] 209.23 Inserted from Euclid's text, which Proclus must have used in his exposition, in order to mark the contrast between the two conclusions drawn.

.Y

 same reason, since
ve is itself the center
 from it to the point
straight line." These
finition of the circle,
 its center are equal.
o the same line; and
to each other" by the
 are equal, and an
constructed on this
usion following upon
ral conclusion: "An
onstructed upon the
rake the line double
r of any other length
struction and proof

 is what it was re-
the conclusion of a
adds: "This is what
announce that some-
 truth is to be dis-
then, our geometer
show that what the
l, joining the end to
at unfolds itself and
does not always add
This is what it was
vhat was to be dem-
tween problems and

d clarified all these
case, the first prob-
for the remaining
bal elements are in-

ch Proclus must have
ntrast between the two

cluded and which are left out, in how many ways the given is formulated, and what are the principles from which we obtain the construction or the proof. For a comprehensive survey of these matters will provide no little exercise and practice in geometrical reasoning.

Now that we have made these distinctions, let us briefly run through certain things dependent on them, namely, lemma, case, porism, objection, and reduction.[22]

211

The term "lemma" is often used to designate any proposition invoked for the purpose of establishing another, as when we assert that a proof can be made from such-and-such a lemma.[23] But the specific meaning of "lemma" in geometry is "a proposition requiring confirmation." Whenever for a construction or a demonstration we assume something that has not been demonstrated but needs to be proved, in such a case, considering that the assumed proposition, though doubtful, is worthy of inquiry on its own account, we call it a lemma. It differs from a postulate and an axiom in being a matter for demonstration, whereas they are invoked in their own right without demonstration to establish other propositions. The best aid in the discovery of lemmas is a mental aptitude for it. For we can see many persons who are keen at finding solutions but do so without method. Thus Cratistus,[24] in our own day, was expert in arriving at the desired result from first principles, and with the fewest possible; but it was natural ability that led him to his discoveries. Nevertheless there are certain methods that have been handed down, the best being the method of analysis, which traces the desired result back to an acknowledged principle. Plato, it is said, taught this method to Leodamas,[25] who also is reported to

[22] 210.28f. The corresponding Greek terms are λῆμμα, πτῶσις, πόρισμα, ἔνστασις, ἀπαγωγή.

[23] 211.4 Proclus gives an example below (216.1ff.) of the use of a lemma and introduces and establishes a lemma at 319.5f. VII below is also a lemma, he says (264.15), preparatory to the proof of VIII.

[24] 211.16 Nothing more seems to be known of Cratistus.

[25] 211.22 Cf. Diog. Laert. III, 24. That Plato taught the method of analysis need not mean that he discovered it. See Heath I, 291f., and *Euclid* I, 134.

212

have made many discoveries in geometry by means of it. A second is the method of *diaeresis*, which divides into its natural parts the genus proposed for examination and which affords a starting-point for demonstration by eliminating the parts irrelevant for the establishment of what is proposed. This method also Plato praised as an aid in all the sciences.[26] A third is the reduction to impossibility, which does not directly show the thing itself that is wanted but by refuting its contradictory indirectly establishes its truth. Such is the scientific meaning of "lemma."

A "case" announces that there are different ways of making the construction, by changing the position of the points, lines, planes, or solids involved. Variations in case are generally made evident by changes in the diagram, wherefore it is called "case," because it is a transposition in the construction.[27]

"Porism" is a term applied to a certain kind of problem, such as those in the *Porisms* of Euclid.[28] But it is used in its special sense when as a result of what is demonstrated some other theorem comes to light without our propounding it. Such a theorem is therefore called a "porism,"[29] as being a kind of incidental gain resulting from the scientific demonstration.

An "objection" prevents an argument from proceeding on its way by opposing either the construction or the demonstration. Unlike the proposer of a case, who has to show that the proposition is true of it, he who makes an objection does not need to prove anything; rather it is necessary [for his opponent] to refute the objection and show that he who uses it is in error.

[26] 212.1 The method of division (διαίρεσις) is emphasized in almost all of Plato's later dialogues, particularly in the *Phaedrus, Sophist, Politicus,* and *Philebus.*

[27] 212.10 Perhaps because πτῶσις, the noun corresponding to the verb πίπτω, often means a "fall," e.g. of dice, as in Plato's *Rep.* 604c.

[28] 212.13 For further light on Euclid's lost *Porisms* see 301.21-302.13.

[29] 212.16 From πορίζω, "furnish," "provide." For further explanation see 303.5-17.

y means of it. A
.es into its natural
nd which affords
inating the parts
; proposed. This
the sciences.²⁶ A
does not directly
:futing its contra-
ι is the scientific

it ways of making
the points, lines,
ιse are generally
ιrefore it is called
ιe construction.²⁷
kind of problem,
ιt it is used in its
monstrated some
propounding it.
ιm,"²⁹ as being a
scientific demon-

m proceeding on
ιr the demonstra-
. to show that the
bjection does not
sary [for his op-
t he who uses it is

is emphasized in
y in the *Phaedrus,*

orresponding to the
ι Plato's *Rep.* 604c.
²orisms see 301.21-

²or further explana-

213

214

"Reduction" is a transition from a problem or a theorem to another which, if known or constructed, will make the original proposition evident. For example, to solve the problem of doubling the cube geometers shifted their inquiry to another on which this depends, namely, the finding of two mean proportionals; and thenceforth they devoted their efforts to discovering how to find two means in continuous proportion between two given straight lines. They say that the first to effect reduction of difficult constructions was Hippocrates of Chios,³⁰ who also squared the lune and made many other discoveries in geometry, being a man of genius when it came to constructions, if there ever was one.

So much for these matters. Now let us move on to the problem before us. It is evident to everyone that the equilateral is the most beautiful of triangles and most akin to the circle, which has all its lines from the center equal and a single simple line bounding it³¹ from without. And the enclosing of the triangle by the two circles, by each of them indeed only in part—for it is inscribed in the whole of neither circle but only in the area consisting of segments of both³²— seems to indicate in a likeness how the things that proceed from first principles receive perfection, identity, and equality from these principles. In this way too the things that move in a straight line are carried about in a circle through the eternal world-process, and souls, despite their movements from place to place,³³ are likenesses of the unmovable activity of Nous because of their periodic return to their starting-points. It is said also that the life-giving source of souls is bounded by a twofold Nous. If, then, the circle is the likeness of intelligible being, and the triangle the likeness of the first soul because of the similarity and equality of its angles and its sides, it would seem reasonable to demonstrate it by means of circles as an

³⁰ 213.8   On Hippocrates see note at 66.4 above.
³¹ 213.17   Reading with Barocius αὐτὸν for αὐτὸ in Friedlein.
³² 213.20   Reading with Grynaeus and Barocius ἐκ τῶν for ἔκτον in Friedlein.
³³ 213.26   It is tempting to adopt Friedlein's emendation νοήσεις for Grynaeus' κινήσεις. But see Proclus, *Elements of Theology,* Prop. 198.

equilateral middle area cut off in them. And if, furthermore, every soul proceeds from Nous and returns to Nous and participates in Nous in a twofold fashion, for this reason also it would be proper that the triangle, which is a symbol of the three natures in the constitution of the soul, should take its origin from being comprehended by two circles.[34] Let these remarks, however, be taken only as reminders, through their likenesses, of the nature of things.

Since some persons have raised objections to the construction of the equilateral triangle with the thought that they were refuting the whole of geometry, we shall also briefly answer them. The Zeno whom we mentioned above[35] asserts that, even if we accept the principles of the geometers, the later consequences do not stand unless we allow that two straight lines cannot have a common segment. For if this is not

216
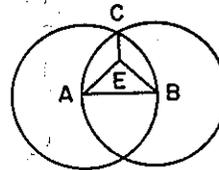
granted, the construction of the equilateral triangle is not demonstrated. Let AB be the straight line, he says, on which we are to construct the equilateral triangle. Let the circles be drawn, and from their point of intersection draw the lines CEA and CEB having CE as a common segment. It then follows that, although the lines from the point of intersection are equal to the given line AB, the sides of the triangle are not equal, two of them being shorter than AB. But if their equality is not established, neither are its consequences. Therefore, says Zeno, even if the principles be granted, the consequences do not follow unless we also presuppose that neither circumferences nor straight lines can have a common segment.

215

[34] 214.13　For understanding the "periodic return," the "life-giving source," the "twofold Nous," and the "three natures in the constitution of the soul" we must turn to Proclus' *Elements of Theology*, particularly Props. 184-211, though Plato's *Timaeus*, one of the chief sources of these doctrines, must always be kept in mind.

[35] 214.18　At 199.15.

To this we must reply first that in a sense it is presupposed in our first principles that two straight lines cannot have a common segment. For the definition of a straight line contains it, if a straight line is a line that lies evenly with all the points on itself. For the fact that the interval between two points is equal to the straight line between them makes the line which joins them one and the shortest; so if any line coincides with it in part, it also coincides with the remainder. For if each of the lines is stretched to the utmost, it must necessarily, because it is the shortest, coincide as a whole with the whole of the other. And, furthermore, this principle is also evidently assumed in the postulates. For the postulate that a finite straight line may be extended in a straight line shows clearly that the extended line is one and that its extension results from a single motion.

But if this be taken as a lemma and we demand that it be proved, let the line AB be, if possible, the common segment of AC and AD, and let a circle ACD be drawn with center at B and AB as distance. Then since ABC is a straight line through the center, AEC is a semicircle; and since ABD is a straight line through the center, AED is a semicircle. Hence AEC and AED are equal to one another, which is impossible.
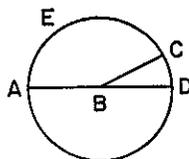

To this demonstration Zeno would reply that the proof we gave[36] that a diameter bisects its circle depends on our previous assumption that two circumferences cannot have a common segment. For we presupposed that one of the two circumferences would coincide with the other or else, not coinciding, fall either inside or outside it. But there is nothing, he says, to prevent its failing to coincide as a whole but coinciding in part. And as long as it has not been proved that the

[36] 216.12 At 157.10ff.

Left margin column:

r
nd if, furthermore,
s to Nous and par-
this reason also it
is a symbol of the
oul, should take its
circles.[34] Let these
iders, through their

ons to the construc-
ught that they were
also briefly answer
bove[35] asserts that,
geometers, the later
w that two straight
For if this is not

eral triangle is not
, he says, on which
e. Let the circles be
tion draw the lines
on segment. It then
point of intersection
f the triangle are not
. But if their equality
quences. Therefore,
d, the consequences
that neither circum-
mon segment.

return," the "life-giving
tures in the constitution
s of Theology, particu-
one of the chief sources
nd.

diameter bisects its circle, the proposition before us cannot be demonstrated. To this Posidonius gave the right answer when he made fun of the shrewd Epicurean[37] for not realizing that the proof is valid even though the circumferences coincide only in part. At the part where they do not coincide one circumference is inside, the other outside, and the same absurd consequences result when we draw a straight line from the center to the outer circumference. For the lines, because they are drawn from the center, will be equal, both that to the outer circumference, which is longer, and that to the inner circumference, which is shorter. Then either they completely coincide and are equal to one another, or one will coincide with the other in part and diverge in part, or no part of one will coincide with any part of the other; and if this last is the case, the one circumference will lie either outside or inside the other. All these alternatives are refuted in the same way. So much for this argument.

Zeno has also constructed another proof, as follows, which he tries to discredit. Let there be two straight lines, AC and AD, having a common segment AB, and let BE be drawn at right angles to AC. The angle EBC will then be a right angle. If, then, the angle EBD is a right angle, they will be equal,
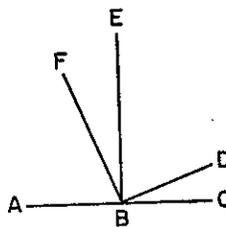
which is impossible; and if it is not a right angle, let FB be drawn at right angles to AD. Angle FBA is then a right angle; but angle EBA was also a right angle; therefore they are equal to each other, which is impossible. This is the proof; he attacks it as presupposing something that is established later,

[37] 216.21 Reading, apparently with ver Eecke, 'Επικούρειον for 'Επίκουρον.

RY

n before us cannot be
he right answer when
for not realizing that
cumferences coincide
do not coincide one
le, and the same ab-
/ a straight line from
'or the lines, because
qual, both that to the
ind that to the inner
ither they completely
or one will coincide
rt, or no part of one
and if this last is the
: outside or inside the
in the same way. So

of, as follows, which
raight lines, AC and
l let BE be drawn at
:hen be a right angle.
, they will be equal,

3ht angle, let FB be
BA is then a right
le; therefore they are
This is the proof; he
: is established later,

3ecke, 'Επικούρειον for

219

PROPOSITIONS: PART ONE

that one can draw a line at right angles to a given straight line from a given point. Posidonius says that such a proof has never appeared in an elementary treatise and that Zeno is slandering the geometers of his time in accusing them of using a shabby proof. Nevertheless, he says, there is something to be said for this proof, since one of two straight lines can clearly be at right angles to the other; that is, any two straight lines can make a right angle. This indeed we presupposed in defining a right angle; for it is by virtue of this particular inclination alone that we construct the right angle. So let it be this one that we have by chance erected. Besides, he adds, Epicurus himself, and all other philosophers, admit that they have proposed many possible as well as many impossible hypotheses for the sake of examining their consequences.

So much for the equilateral triangle. We must also construct the others, and first the isosceles. Let the line AB be that on which an isosceles is to be constructed. Let circles be drawn as they were for the equilateral triangle, and let the line AB be prolonged in both directions to the points C and D. CB is then equal to AD. With B as center and distance CB
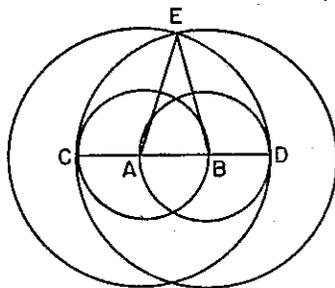
let the circle CE be drawn; and again with A as center and distance DA let the circle DE be drawn. From E, the point of intersection of the circles, let lines EA and EB be joined to the points A and B. Then since EA is equal to AD, and EB to BC,[38] and AD to BC, EA is also equal to EB. But they are

[38] 218.25 Inserting τῇ before βγ in Friedlein.

— 171 —

also longer than AB. The triangle ABE is therefore isosceles; and this is what we were required to construct.

Now let it be further required to construct a scalene triangle upon the given straight line AB. Let circles be drawn with centers and distances as before. Let a point C be taken on the circle whose center is A, and let the connecting line AC be drawn; upon this let a point D be taken and the line DB
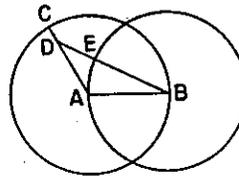
be drawn. Then since the center is A and AB is equal to AC, AB is longer than AD. B also is a center, and therefore EB is equal to AB. DB thus is longer than AB, and AB is longer than AD. The three sides DB, BA, and AD are therefore unequal. Hence the triangle is scalene, so that we have constructed the three kinds of triangle.

These matters are common knowledge. What is elegant in these constructions is that the equilateral triangle, which is equal on every side, can be constructed in only one way, whereas the isosceles, which has only two of its sides equal, can be constructed in two ways; for the given straight line is either shorter than either of the two equal sides, as in the triangle we constructed, or longer than both. And the scalene, having all its sides unequal, can be constructed in three ways; for the given straight line is either the longest or the shortest of the three or longer than one and shorter than the other. The reader can practise himself by examining at length or briefly each of these three hypotheses. For us what has been presented must suffice.

In general we shall see that some problems have a unique solution, others more than one, and some an indefinite number. We call "ordered," to use Amphinomus' term, those that have only one solution, "intermediate" those that have more than one but a finite number and "unordered" those

220

221

!R Y

∃ is therefore isosceles;
construct.

⊃ construct a scalene
3. Let circles be drawn
_et a point C be taken
the connecting line AC
taken and the line DB

)
/

nd AB is equal to AC,
er, and therefore EB is
AB, and AB is longer
1 AD are therefore un-
so that we have con-

lge. What is elegant in
teral triangle, which is
cted in only one way,
two of its sides equal,
1e given straight line is
equal sides, as in the
both. And the scalene,
1structed in three ways;
longest or the shortest
shorter than the other.
examining at length or
. For us what has been

roblems have a unique
ome an indefinite num-
phinomus' term, those
diate" those that have
and "unordered" those

221

having an indefinite variety of solutions. How problems are handled that are capable of one or more than one solution is clear with regard to the triangles considered; the equilateral triangle is constructed in a single way, and of the others one has two solutions and the other three. Problems admitting of an indefinite number of solutions would be such as the following: "To divide a given straight line into three parts in continued proportion." If the line be divided in a ratio of two to one, and if then the square on the shorter length be applied[39] to the longer so as to fall short by a square figure, it will have been divided into three equal parts. But if the greater segment be more than double, say, triple the lesser and an area equal to the square on the lesser be applied to the greater in such a way as to fall short by a square, the line will have been divided into three unequal parts in continued proportion. Since there are an indefinite number of ways in which the line can be divided into two parts of which the greater is more than double or triple the lesser—for the series of multiples proceeds to infinity—there are consequently an indefinite number of ways in which the line may be divided into three parts in continued proportion.

We must also recognize that "problem" is used in several senses. Anything propounded may be called a problem, whether it be put forward for the purpose of instruction or of construction. But its special use in mathematics is to denote something proposed for theoretical construction, since the constructing carried out in mathematics is done for the purpose of theory. Frequently things incapable of solution are called problems; but more characteristically we use this designation for what is capable of solution and is neither excessive nor deficient. A problem such as the following is excessive: "To construct an equilateral triangle having its vertical angle two-thirds of a right angle." For this brings in an unneeded addition, since this property belongs to every equilateral triangle. Of excessive problems those that exceed by containing

[39] 220.19 On the "application of areas" used in this example see 419.15ff. and note at 420.23. The algebraic solution of the problem here discussed is neatly given by Heath, *Euclid* I, 128.

— 173 —

inconsistent or unreal conditions are called "impossibles," while those that contain realizable conditions are called "more than problems." A problem is deficient and is called "less than a problem" when it needs to have something added to make it definite and bring it into order and scientific deter-minateness, such as "To construct an isosceles-triangle." This is insufficiently determinate and requires an addition specify-ing what sort of isosceles is wanted, whether one having its base greater or less than each of the equal sides, or one having its vertical angle double each of those at the base, like the half-square, or one having each of the base angles double the angle at the vertex, or one having these angles in some other ratio, triple or quadruple. One could vary the possibilities endlessly. These examples show that problems properly so-called aim at avoiding the indeterminateness that renders them capable of an indefinite number of solutions; neverthe-less even those that are deficient are called problems, for the term is ambiguous. Clearly the very first problem in the *Elements* is in this respect superior in that it is neither exces-sive nor deficient nor indeterminate and thus having indef-initely many solutions; for such should be the character of what is to be an "element" of the others.

II. *At a given point to place a straight line equal to*
a given straight line.

Some problems have no cases, while others have many; and the same is true of theorems. A proposition is said to have cases when it has the same force in a variety of diagrams, that is, can be demonstrated in the same way despite changes in position, whereas one that succeeds only with a single posi-tion and a single construction is without cases. For the pres-ence of cases, whether in a theorem or a problem, generally shows itself in the constructions. Now our second problem has many cases. In it the point is given in position, and given only in this way; but the line is given both in species (for it is not simply a line, but this kind of line) and in position. We want to place a straight line equal to this straight line with its ex-tremity at the given point, wherever the point may lie. It is

222

223

224