$$(\Delta E, \Gamma E) = (R, \Gamma\Delta) \qquad\qquad (\Delta E, \Gamma E) = (K\Lambda, R)$$

$$(R, \Gamma\Delta) = (K\Lambda, R)$$

componendo

$$(\Theta\Delta, \Gamma\Delta) = (\Theta K, R)$$

permutando

$$(\Theta\Delta, \Theta K) = (\Gamma\Delta, R)$$

componendo

$$(K\Delta, \Theta K) = (\Theta\Delta, R)$$

$$\text{or } \mathbf{O}(\Theta K, \Theta\Delta) = \mathbf{O}(K\Delta, R) \quad (\alpha)$$

componendo   *componendo*

$$(\Delta\Gamma, \Delta E) = (\Theta\Delta, R) \qquad\qquad (\Delta\Gamma, \Gamma E) = (\Theta K, R)$$

$$[\mathbf{T}(\Delta\Gamma),\ \mathbf{O}(\Delta E, \Gamma E)] = [\mathbf{O}(\Theta\Delta, \Theta K),\ \mathbf{T}(R)]$$ [1]

or, because of (α),

$$[\mathbf{T}(\Delta\Gamma), \mathbf{O}(\Delta E, \Gamma E)] = [\mathbf{O}(K\Delta, R),\ \mathbf{T}(R)] = (K\Delta, R)\,.$$

q. e. d.

Algebraically: If Δ and K are determined successively by

$$\Delta E = h_1 \frac{R+h_2}{h_2} \quad \text{and} \quad KE = h_2 \frac{R+h_1}{h_1},$$

then it has to be proved that

$$\tfrac{1}{3}\pi r^2 . K\Delta = \tfrac{1}{3}\pi d^2 R$$

or

$$\frac{K\Delta}{R} = \frac{d^2}{r^2} = \frac{d^2}{h_1 h_2}.$$

---

[1]) This passage, which we should nowadays perform by multiplication of the corresponding members of the two proportions derived, is motivated in the Greek theory of proportions by writing

$$(\Delta\Gamma, \Delta E) = [\mathbf{T}(\Delta\Gamma),\ \mathbf{O}(\Delta\Gamma, \Delta E)] = [\mathbf{O}(\Theta\Delta, \Theta K),\ \mathbf{O}(R, \Theta K)]$$

$$(\Delta\Gamma, \Gamma E) = [\mathbf{O}(\Delta\Gamma, \Delta E),\ \mathbf{O}(\Gamma E, \Delta E)] = [\mathbf{O}(R, \Theta K), \mathbf{T}(R)]\,,$$

from which the required conclusion follows *ex aequali*.

Now it is true that

$$K\Delta = \Delta E + KE = \frac{h_1{}^2(R+h_2)+h_2{}^2(R+h_1)}{h_1 h_2} = R\frac{(h_1+h_2)^2}{h_1 h_2} = R\frac{d^2}{h_1 h_2}.$$

Note. The property proved is frequently also used, instead of in the form

$$(\Delta E, \Gamma E) = (R+\Delta E, \Delta E)\,,$$

in the form to be derived *separando, viz.*:

$$(\Delta\Gamma, \Gamma E) = (R, \Delta E)\,.$$

Algebraically:

$$\frac{\Delta\Gamma}{h_1} = \frac{R}{h_2}.$$

Proposition 3.

To cut a given sphere by a plane so that the surfaces of the segments may have to each other a ratio which is the same as a given ratio.

From I, 42, 43 it follows readily that for this the plane has to divide the diameter to which it is perpendicular in the given ratio.

This is followed by one of the great problems of Greek geometry:

Proposition 4.

To cut a given sphere ⟨by a plane⟩ so that the segments of the sphere may have to each other a ratio which is the same as a given ratio[1]).

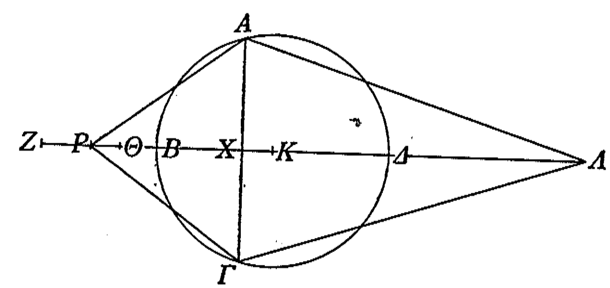

Fig. 82.

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[1]) The original formulation of this proposition (we would remind the reader that we only possess an Attic version of *On the Sphere and Cylinder*) is to be read from its mention in the preface to *On Spirals*: *τὰν δοθεῖσαν σφαῖραν ἐπιπέδῳ τεμεῖν, ὥστε τὰ τμάματα αὐτᾶς ποτ' ἄλλαλα τὸν ταχθέντα λόγον ἔχειν.*

In Fig. 82 let $A\Gamma$ be the intersection of the required plane with the meridian plane of the sphere with centre K perpendicular to it. Apparently we may also require the sphere to have a given ratio to the segment $A\Delta\Gamma$. Let this ratio be equal to

$$(ZB, Z\Theta), \text{ in which } ZB = R.$$

Find according to S.C. II, 2 the points P and Λ so that

$$\text{Cone } PA\Gamma = \text{segment } BA\Gamma$$

$$\text{Cone } \Lambda A\Gamma = \text{segment } \Delta A\Gamma.$$

For this we must have (*vide* Note to II, 2)

$$(PB, BX) = (R, X\Delta) \qquad (1)$$

$$(\Lambda\Delta, \Delta X) = (R, XB). \qquad (2)$$

The problem thus amounts to finding a point X so that

$$(P\Lambda, X\Lambda) = (ZB, Z\Theta), \qquad (3)$$

the positions of the points P and Λ, however, depending according to (1) and (2) on the position of X.

The ratio $(P\Lambda, X\Lambda)$ is compounded of the ratios

$$(P\Lambda, \Delta\Lambda) \quad (\alpha) \quad \text{and} \quad (\Delta\Lambda, X\Lambda) \quad (\beta).$$

These two ratios are first written in a different form:

α. From (1) it follows *permutando* that

$$(PB, R) = (BX, X\Delta)$$

componendo

$$(PK, R) = (B\Delta, X\Delta) \qquad (4)$$

From (2) it follows *permutando* that

$$(\Lambda\Delta, R) = (\Delta X, XB)$$

componendo

$$(\Lambda K, \Lambda\Delta) = (B\Delta, X\Delta) \qquad (5)$$

$$(PK, R) = (\Lambda K, \Lambda\Delta)$$

permutando

$$(PK, K\Lambda) = (R, \Lambda\Delta)$$

componendo

$$(P\Lambda, K\Lambda) = (K\Lambda, \Lambda\Delta).$$

Apparently therefore $(P\Lambda, \Lambda\Delta)$ is the duplicate ratio of $(K\Lambda, \Lambda\Delta)$ and because of (5) also of $(B\Delta, X\Delta)$. Therefore

$$(P\Lambda, \Lambda\Delta) = [\mathbf{T}(B\Delta), \mathbf{T}(X\Delta)].$$

β. From (2) it follows *componendo*

$$(\Lambda X, \Lambda\Delta) = (ZX, ZB) \quad \text{or } \textit{invertendo}$$

$$(\Lambda\Delta, \Lambda X) = (ZB, ZX).$$

The ratio $(P\Lambda, X\Lambda)$ therefore is compounded of the ratios

$$[\mathbf{T}(B\Delta), \mathbf{T}(X\Delta)] \quad \text{and} \quad (ZB, ZX). \qquad (6)$$

The ratio $(ZB, Z\Theta)$ appearing in the 2nd member of (3), however, is compounded of

$$(ZB, ZX) \quad \text{and} \quad (ZX, Z\Theta). \qquad (7)$$

By comparison of the expressions (6) and (7) it follows, because of (3), that

$$(ZX, Z\Theta) = [\mathbf{T}(B\Delta), \mathbf{T}(X\Delta)]. \qquad (8)$$

The question therefore amounts to a given line segment $Z\Delta$ $(3R)$ having to be so divided in a point X that the part ZX shall have to a given line segment $Z\Theta$ the same ratio as the square on another given line segment $B\Delta$ $(2R)$ has to the square on the remaining part $X\Delta$.

Archimedes announces that he will deal with this problem analytically and synthetically at the end of the proposition[1]). This, however, is not done. The solution he promises must already have been lost in the days of Diocles (who gives another solution instead). Eutocius, however, thinks he has found it again[2]), in an apparently mutilated form; he presents a reconstruction which we reproduce as follows: Formulated in a general way (*i.e.* without any relation to the notation and the particular ratios of the line

[1]) *Opera* I, 192. ἐπὶ τέλει ἀναλυθήσεται καὶ συντεθήσεται.

[2]) *Opera* III, 130–132. Eutocius here says that the solution of Archimedes —recognizable by the Dorian dialect and by the old denomination of the conics—, which did not occur in any codex, was traced by him in a book, but that it was obscure and showed many errors in the figures. He himself in his exposition uses the Apollonian terms parabola and hyperbola, an example which we are following in the reproduction of his argument.

segments to each other in II, 4), the problem is as follows: Given (Fig. 83) a line segment AB, a line segment $A\Gamma$, and a surface $\mathbf{\Delta}$, it is required to find on AB a point E such that

$$(AE, A\Gamma) = [\mathbf{\Delta}, \mathbf{T}(BE)] \quad (\gamma)$$

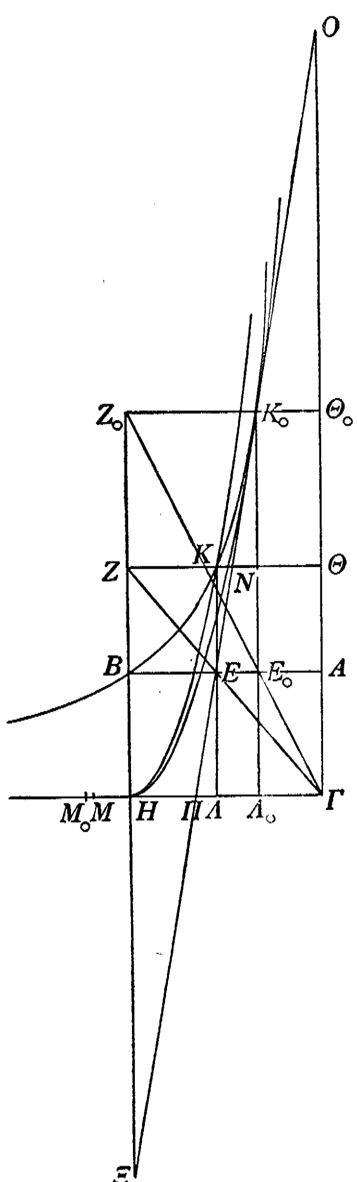

Fig. 83.

This question is treated analytically: Let E be the required point. From the figure, in which $A\Gamma$ is plotted at right angles to AB and the rectangle $HZ\Theta\Gamma$ is completed, it follows that

$$(AE, A\Gamma) = (H\Gamma, HZ)\,. \quad (\delta)$$

Further suppose that $\mathbf{\Delta} = \mathbf{O}(H\Gamma, HM)$, then, because $BE = ZK$, we must have

$$(H\Gamma, HZ) = [\mathbf{O}(H\Gamma, HM), \mathbf{T}(ZK)]$$

or $[\mathbf{O}(H\Gamma, HM), \mathbf{O}(HZ, HM)]$

$$= [\mathbf{O}(H\Gamma, HM), \mathbf{T}(ZK)]\,,$$

therefore

$$\mathbf{T}(ZK) = \mathbf{O}(HZ, HM)\,.$$

From this it follows (III; 2.0) that K lies on a parabola which has H for vertex, HZ for diameter, and HM for *orthia* (*latus rectum*).

Moreover, because of Euclid I, 43, the rectangle $K\Gamma$ is equal to the rectangle AH, so that K also lies on an orthogonal hyperbola with asymptotes $H\Gamma$ and $\Theta\Gamma$, which passes through B (*Conica* II, 12). K is therefore determined as the point of intersection of two conics.

Before it is possible to proceed to the synthesis, a $\delta\iota o\varrho\iota\sigma\mu\acute{o}\varsigma$[1]) is derived. Indeed, it is evident that for a given value of AB the data $A\Gamma$ and $\mathbf{\Delta}$ are subject to certain limitations, for we have the relation

$$(AE, A\Gamma) = [\mathbf{\Delta}, \mathbf{T}(BE)]$$

which, because of Euclid XI, 34, states that the volume of the parallelepiped with base $\mathbf{T}(BE)$ and height AE is equal to the volume of the parallelepiped whose base has a surface $\mathbf{\Delta}$ and whose height is $A\Gamma$. If we denote the solids in question successively by

$$\Sigma[\mathbf{T}(BE), AE)] \quad \text{and} \quad \Sigma[\mathbf{\Delta}, A\Gamma] \quad (\Sigma = \sigma\tau\varepsilon\varrho\varepsilon\acute{o}\nu)\,,$$

it has to be found out what is the maximum value which can be taken by the volume of the first solid when E varies; the volume of the second solid is then delimited by this value as its upper limit.

It is now proved that the solid with base $\mathbf{T}(BE)$ and height AE attains its maximum volume when $BE = 2.AE$: to see this we construct the above-mentioned curves once more for the point E_0, which satisfies this condition; the corresponding point K_0 lies on a parabola with diameter HZ_0 and *latus rectum* HM_0 so that

$$\mathbf{T}(Z_0K_0) = \mathbf{O}(Z_0H, HM_0)$$

and also on the orthogonal hyperbola already mentioned above. We now proceed to prove that the parabola HK_0 touches this hyperbola in K_0.

To prove this, on Z_0H produced measure $H\Xi = HZ_0$, then by a familiar property of the tangent to a parabola (III; 2.2), ΞK_0 is tangent to the parabola in K_0. If further ΞK_0 meets the asymptote $\Gamma\Theta$ in O, then, because $BE_0 = 2.AE_0$, apparently $K_0\Pi = K_0O$, so that the line ΠO, in view of a property of the tangent to a hyperbola (III; 5.43), touches this curve in K_0. By this the proposition is proved.

The branch $BK_0\ldots$ of the hyperbola in the figure thus lies completely on one side of the parabola $HK\ldots$ If therefore E be any point of AB (in the figure chosen between B and E_0), the perpendicular through E on AB meets the hyperbola in K, while

[1]) $\Delta\iota o\varrho\iota\sigma\mu\acute{o}\varsigma$ is here to be understood in the sense of "condition which the data of a problem have to satisfy, if it is to permit of solutions". *Elements of Euclid* I, 168.

the line ZK, parallel to AB, meets the parabola HK_0 in N, it is found that

$$ZK < ZN\,.$$

Now K lies on the orthogonal hyperbola; from this follows the equality of the rectangles $K\Gamma$ and $B\Gamma$, and consequently also that of the rectangles KA and $B\Delta$; in view of the reverse of Euclid I, 43 the points Γ, E, and Z now lie on one straight line.

It had to be proved that

$$\Sigma[\mathbf{T}(BE), AE] < \Sigma[\mathbf{T}(BE_0), AE_0] \qquad (\zeta)$$

or, if we call the particular value of the given surface $\mathbf{\Delta}$, which with $A\Gamma$ given leads to the finding of the point E_0, $\mathbf{\Delta}_0$,

$$\Sigma[\mathbf{\Delta}, A\Gamma] < \Sigma[\mathbf{\Delta}_0, A\Gamma]\,.$$

In this, however,

$$\mathbf{\Delta} = \mathbf{O}(H\Gamma, HM) \quad \text{and} \quad \mathbf{\Delta}_0 = \mathbf{O}(H\Gamma, HM_0)\,,$$

while

$$\mathbf{T}(ZK) = \mathbf{O}(HZ, HM) \quad \text{and} \quad \mathbf{T}(ZN) = \mathbf{O}(HZ, HM_0)\,.$$

Because $ZK < ZN$, we have $HM < HM_0$, therefore $\mathbf{\Delta} < \mathbf{\Delta}_0$, from which the correctness of the proposition becomes evident. For a point E between E_0 and A the proof is analogous.

The $\delta\iota o\varrho\iota\sigma\mu\acute{o}\varsigma$ thus found is then applied as follows in the synthesis.

First find E_0 so that $BE_0 = 2.AE_0$. If now

$$\Sigma[\mathbf{\Delta}, A\Gamma] \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} \Sigma[\mathbf{T}(BE_0), AE_0] \quad \begin{matrix} \text{there is no solution.} \\ E_0 \text{ is the required point.} \\ \text{find the point } M \text{ so that} \end{matrix}$$

$$\mathbf{\Delta} = \mathbf{O}(H\Gamma, HM).$$

Construct the parabola with diameter HZ and *latus rectum* HM, the orthogonal hyperbola through B with asymptotes ΓH and $\Gamma\Theta$, and find the point of intersection K of these two curves, which lies on the same side of E_0K_0 with H. The required point E is then obtained by drawing a perpendicular from K to AB.

The method found may now be applied in the problem posited by Archimedes in Proposition 4. For this, AB from the general discussion is to be replaced by the line segment $Z\Delta = 3R$ of Fig. 82.

$A\Gamma$ is to be replaced by $Z\Theta$ of Fig. 82, and $\mathbf{\Delta}$ is to be taken equal to $\mathbf{T}(B\Delta)$. Then BE_0 becomes equal to $2R$, AE_0 to R. Because $Z\Theta < R$, we now have

$$\Sigma[\mathbf{\Delta}, A\Gamma] = \Sigma[\mathbf{T}(2R), Z\Theta] < \Sigma[\mathbf{T}(2R), R]\,,$$

so that the condition for the solubility of the problem, which has been derived in the general discussion, is satisfied.

We have deliberately reproduced the whole argument in a purely classical form. We will now elucidate it by translating it into algebraic symbolism:

If (again in the general problem) we suppose $AB = a$, $BE = x$, $A\Gamma = p$, $\mathbf{\Delta} = q^2$, the equation of the problem (*vide* (γ)) is:

$$\frac{a-x}{p} = \frac{q^2}{x^2} \text{ or } x^2(a-x) = pq^2 \text{ or } x^3 - ax^2 + pq^2 = 0\,. \qquad (\varepsilon)$$

The Greek solution is equivalent to equating each of the two members of

$$\frac{a-x}{p} = \frac{q^2}{x^2}$$

to $\frac{a}{y}$ in conformity with relation (δ), in which $HZ = y$.

We now have the two equations

$$y(a-x) = ap \quad \text{and} \quad x^2 = \frac{q^2}{a}\cdot y\,,$$

which represent successively the orthogonal hyperbola and the parabola used by Archimedes. The abscissae of the points of intersection of these two curves represent the solutions of the equation (ε). The found values of x, however, can only be used in the geometrical problem II, 4 of Archimedes when they satisfy the relation $0 < x < \frac{2}{3}a$, because x has to be less than $2R$, and a is equal to $3R$.

In order to find out the number of roots, we compute the discriminant

$$\mathbf{D} = -[27pq^2 - 4a^3]pq^2\,,$$

from which the following becomes apparent:

If pq^2 be less than $\frac{4}{27}a^3$, there are three real roots; since their product is $-pq^2$ and is therefore negative, while the sum a is positive, one of these roots is negative and the other two are positive.

If pq^2 be equal to $\frac{4}{27}a^3$, there are two positive roots, which coincide.

If pq^2 be greater than $\frac{4}{27}a^3$, there is only one real root, and this is negative.

Since in the problem discussed by Archimedes only positive roots can be used, it is a prerequisite that pq^2 be less than $\frac{4}{27}a^3$.

This prerequisite is identical with the condition (ζ) derived by Archimedes. In fact,

$$pq^2 = x^2(a-x) ,$$

which latter form reaches its maximum, *viz.* $\frac{4}{27}a^3$, when $x=\frac{2}{3}a$.

Of the two positive roots which satisfy the equation in the case of

$$pq^2 < \tfrac{4}{27}a^3$$

and which correspond to the two points of intersection of hyperbola and parabola, having positive abscissae, in the particular case of II, 4 only one can be used, *viz.* the one which is less than $\frac{2}{3}a$.

That there is always one such root, and only one, is clear upon geometrical consideration: of the two points of intersection of parabola and orthogonal hyperbola, which have positive abscissae, one lies to the left and one to the right of the line K_0E_0.

Algebraically the same is found because the expression

$$x^3 - ax^2 + pq^2$$

appears to have a zero point between $x=0$ and $x=\frac{2}{3}a$, and another between $x=\frac{2}{3}a$, and $x=a$.

Eutocius mentions two more solutions of the proposition **S.C.** II, 4[1]). In the first, which originates from Dionysodorus[2]), the problem is first reduced (by a different method from that of Archimedes) to the form (3). The solution (translated into the notation of II, 4), however, then proceeds as follows:

In Fig. 82 provide for

$$\mathbf{O}(ZX, Z\Theta) = \mathbf{T}(MX) \qquad (\alpha)$$

then by (8) we have:

[1]) *Opera* III, 152 *et seq.* and 160 *et seq.*

[2]) Probably Dionysodorus of Caunus, who was a contemporary of Apollonius. Presumably he was the writer of a treatise περὶ τῆς σπείρας (On the tore), quoted by Heron (*Metrica* II, 13, *Heronis Opera* III, 128). In this he derives an expression for the volume of a tore.

$$[\mathbf{T}(MX), \mathbf{T}(Z\Theta)] = (ZX, Z\Theta) = [\mathbf{T}(B\Delta), \mathbf{T}(X\Delta)] ,$$

therefore

$$(MX, Z\Theta) = (B\Delta, X\Delta) . \qquad (\beta)$$

In the above, (α) represents a parabola with vertex Z, axis $Z\Delta$, and *latus rectum* $Z\Theta$, and (β) an orthogonal hyperbola with centre Δ, the asymptotes of which are ΔZ and the perpendicular erected through Δ on ΔB. X is then determined as the foot of the ordinate of a point of intersection M of these two curves.

Algebraically this means, in the notation adopted above, that for the solution of the equation

$$\frac{a-x}{p} = \frac{q^2}{x^2}$$

we suppose

$$y^2 = p(a-x) ,$$

so that

$$\frac{y}{p} = \frac{q}{x}.$$

We now have to find the points of intersection of the parabola

$$y^2 = p(a-x)$$

and the orthogonal hyperbola

$$xy = pq .$$

In the second solution, which was suggested by Diocles[1]), the problem is generalized even more, as compared with Archimedes. In fact, in Archimedes, when the points B and Δ were given, the points P and Λ were determined by the relations (Fig. 82):

$$(PB, BX) = (R, X\Delta) \quad \text{and} \quad (\Lambda\Delta, \Delta X) = (R, XB),$$

while X then had to satisfy the condition $(PX, \Lambda X)$=given ratio.

In Diocles, R is replaced in the two first-mentioned relations by a line segment of any length, which is no longer conditioned by the length of $B\Delta$. Formulated once again, the problem now is as follows (Fig. 84):

[1]) Diocles probably lived about 100 B.C. He is known to this day for his cissoid. The solution of **S.C.** II, 4 quoted above has been taken from his treatise περὶ πυρίων (on burning-mirrors).

To divide a given line segment AB at E so that, if the relations

$$(ZA, AE) = (AK, BE)$$

$$(HB, BE) = (BM, AE) \quad \text{with } AK = BM$$

apply, E satisfies the condition

$$(ZE, HE) = (\Gamma, \Delta),$$

in which Γ and Δ are given line segments.

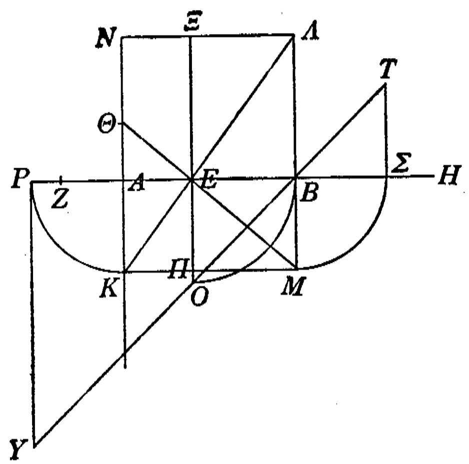

Fig. 84.

This problem is analysed as follows:
Let ME produced meet KA produced in Θ; likewise let KE produced meet MB produced in Λ, then it is proved without difficulty that

$$ZA = \Theta A \quad \text{and} \quad HB = \Lambda B.$$

Now

$$(AE, BE) = (\Theta A, MB) = (KA, \Lambda B) = (\Theta A + AE, MB + BE)$$
$$= (KA + AE, \Lambda B + BE);$$

therefore $\mathbf{O}(\Theta A + AE, \Lambda B + BE) = \mathbf{O}(KA + AE, MB + BE)$

or, if $B\Sigma = BM$, $AP = AK$,

$$\mathbf{O}(ZE, HE) = \mathbf{O}(PE, \Sigma E).$$

From this it follows that

$$(\Gamma, \Delta) = (ZE, HE) = [\mathbf{O}(ZE, HE), \mathbf{T}(HE)] = [\mathbf{O}(PE, \Sigma E), \mathbf{T}(HE)].$$

Now make EO equal to EB and perpendicular to AB, ΣT perpendicular to AB with T on OBproduced. From the similarity of the triangles EOB and ΣTB it now follows that

$(TB, OB) = (\Sigma B, EB)$, therefore *componendo* $(TO, OB) = (\Sigma E, EB)$.

If the perpendicular to BA through P meet BO produced in Y, then we also have

$$(BO, OY) = (BE, EP),$$

from which it follows *ex aequali* that

$$(TO, OY) = (\Sigma E, EP).$$

Thence

$$[\mathbf{O}(TO, OY), \mathbf{T}(OY)] = [\mathbf{O}(\Sigma E, EP), \mathbf{T}(EP)]$$

or

$$[\mathbf{O}(TO, OY), \mathbf{O}(\Sigma E, EP)] = [\mathbf{T}(OY), \mathbf{T}(EP)] = 2:1.$$

Therefore

$$\mathbf{O}(TO, OY) = 2\mathbf{O}(\Sigma E, EP).$$

Above it has been found that

$$[\mathbf{O}(\Sigma E, EP), \mathbf{T}(EH)] = (\Gamma, \Delta),$$

therefore, because $EH = \Xi O$,

$$[\mathbf{O}(TO, OY), \mathbf{T}(O\Xi)] = (2\Gamma, \Delta).$$

Now construct a line segment Φ so that

$$(2\Gamma, \Delta) = (TY, \Phi),$$

then it is found that Ξ lies on an ellipse, produced by elliptical application to Φ with a defect, the ratio of whose sides is (TY, Φ) (III; 1.2). TY is here the diameter conjugated to the direction ΞO.

But because the rectangles KB and ΠN are equal, Ξ also lies on an orthogonal hyperbola with asymptotes KN and KM, which passes through B. Thus the point Ξ is determined, from which E follows.

Algebraically: If $AB=a$, $AE=x$, $AK=BM=b$, $(\Gamma, \Delta)=m:n$, x has to satisfy the relation

$$\frac{x+\dfrac{b}{a-x}x}{a-x+\dfrac{b}{x}(a-x)} = \frac{m}{n}$$

or

$$\frac{x^2(a-x+b)}{(a-x)^2(x+b)} = \frac{m}{n},$$

which may be reduced to

$$(a-x+b)(x+b) = \frac{m}{n}\left\{a-x+\frac{ab}{x}-b\right\}^2.$$

Now suppose

$$\frac{ab}{x} = y,$$

then the equation reduces to

$$(y+a-x-b)^2 = \frac{n}{m}(a-x+b)(x+b).$$

If we now consider x and y as coordinates in the rectangular system of axes $K(MN)$, we have to intersect the orthogonal hyperbola

$$xy = ab,$$

which has KM and KN for asymptotes and passes through B, by the curve

$$\frac{(y+a-x-b)^2}{(a-x+b)(x+b)} = \frac{n}{m}.$$

Here

$$y+a-x-b = \Xi O,\ a-x+b = \frac{OT}{\sqrt{2}},\ x+b = \frac{OY}{\sqrt{2}}.$$

If we suppose

$$\Xi O = \eta,\ OT = \xi_1,\ OY = \xi_2,$$

we have the equation

$$\frac{\eta^2}{\xi_1\xi_2} = \frac{n}{2m},$$

which stands for an ellipse, written in the two-abscissa form (III;

1.1), which has TY for diameter and the direction of ΞO as conjugate direction.

Proposition 5.

To construct a segment of a sphere similar to a given segment and equal to another given segment of a sphere.

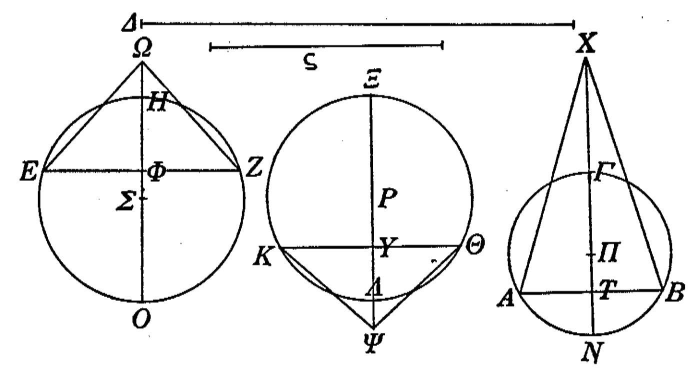

Fig. 85.

In Fig. 85 conceive the required segment $\Lambda K\Theta$ to be equal to the given segment ΓAB and similar to the given segment HEZ[1]). By applying S.C. II, 2 we replace all three segments by cones. Suppose

$(\Psi Y, \Lambda Y) = (P\Xi+\Xi Y, \Xi Y)$, then segment $\Lambda K\Theta$ = cone $\Psi K\Theta$.

$(XT, \Gamma T) = (\Pi N+NT, NT)$, then segment ΓAB = cone XAB.

$(\Omega\Phi, H\Phi) = (\Sigma O+O\Phi, O\Phi)$, then segment HEZ = cone ΩEZ.

Now it must be true that

1) cone XAB = cone $\Psi K\Theta$,

whence

$$[\mathbf{T}(K\Theta,\ \mathbf{T}(AB)] = (XT, \Psi Y) \qquad (\alpha)$$

[1]) Two segments of a sphere are called similar when their heights are in the same ratio as the diameters of their bases, *i.e.* when the meridian sections are similar.