Now these surfaces are respectively equal to circles with radii equal to AB, $A'B$ [I. 42, 43].

Hence the ratio $AB^2 : A'B^2$ is equal to the given ratio, i.e. AM is to MA' in the given ratio.

Accordingly the synthesis proceeds as follows.

If $H : K$ be the given ratio, divide AA' in M so that

$$AM : MA' = H : K.$$

Then $AM : MA' = AB^2 : A'B^2$

$= (\text{circle with radius } AB) : (\text{circle with radius } A'B)$

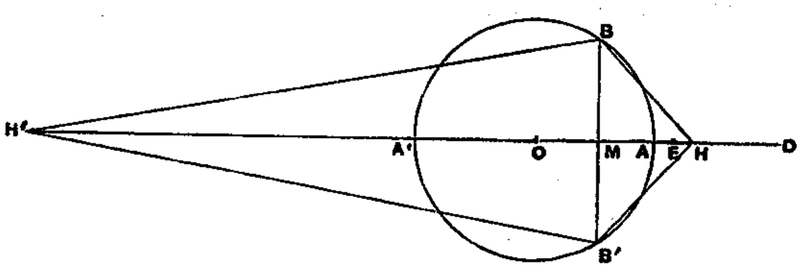
$= (\text{surface of segment } BAB') : (\text{surface of segment } BA'B')$.

Thus the ratio of the surfaces of the segments is equal to the ratio $H : K$.

Proposition 4. (Problem.)

To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio.

Suppose the problem solved, and let the required plane cut the great circle ABA' at right angles in the line BB'. Let AA' be that diameter of the great circle which bisects BB' at right angles (in M), and let O be the centre of the sphere.



Take H on OA produced, and H' on OA' produced, such that

$$OA' + A'M : A'M = HM : MA, \quad\text{.............(1)},$$

and $$OA + AM : AM = H'M : MA' \quad\text{.............(2)}.$$

Join BH, $B'H$, BH', $B'H'$.

Then the cones HBB', $H'BB'$ are respectively equal to the segments BAB', $BA'B'$ of the sphere [Prop. 2].

Hence the ratio of the cones, and therefore of their altitudes, is given, i.e.

$$HM : H'M = \text{the given ratio} \quad\text{..............(3)}.$$

We have now three equations (1), (2), (3), in which there appear three as yet undetermined points M, H, H'; and it is first necessary to find, by means of them, another equation in which only one of these points (M) appears, i.e. we have, so to speak, to *eliminate* H, H'.

Now, from (3), it is clear that $HH' : H'M$ is also a given ratio; and Archimedes' method of elimination is, *first*, to find values for each of the ratios $A'H' : H'M$ and $HH' : H'A'$ which are alike independent of H, H', and then, *secondly*, to equate the ratio compounded of these two ratios to the known value of the ratio $HH' : H'M$.

(*a*) To find such a value for $A'H' : H'M$.

It is at once clear from equation (2) above that

$$A'H' : H'M = OA : OA + AM \quad\text{............(4)}.$$

(*b*) To find such a value for $HH' : A'H'$.

From (1) we derive

$$A'M : MA = OA' + A'M : HM$$
$$= OA' : AH \quad\text{....................(5)};$$

and, from (2), $$A'M : MA = H'M : OA + AM$$
$$= A'H' : OA \quad\text{....................(6)}.$$

Thus $$HA : AO = OA' : A'H',$$

whence $$OH : OA' = OH' : A'H',$$

or $$OH : OH' = OA' : A'H'.$$

It follows that

$$HH' : OH' = OH' : A'H',$$

or $$HH' . H'A' = OH'^2.$$

Therefore $$HH' : H'A' = OH'^2 : H'A'^2$$
$$= AA'^2 : A'M^2, \text{ by means of (6)}$$

(c) To express the ratios $A'H' : H'M$ and $HH' : H'M$ more simply we make the following construction. Produce OA to D so that $OA = AD$. (D will lie beyond H, for $A'M > MA$, and therefore, by (5), $OA > AH$.)

Then $$A'H' : H'M = OA : OA + AM$$
$$= AD : DM \quad\text{.....................(7).}$$

Now divide AD at E so that
$$HH' : H'M = AD : DE \quad\text{..................(8).}$$

Thus, using equations (8), (7) and the value of $HH' : H'A'$ above found, we have
$$AD : DE = HH' : H'M$$
$$= (HH' : H'A').(A'H' : H'M)$$
$$= (AA'^2 : A'M^2).(AD : DM).$$

But $$AD : DE = (DM : DE).(AD : DM).$$

Therefore $$MD : DE = AA'^2 : A'M^2 \quad\text{..................(9).}$$

And D is given, since $AD = OA$. Also $AD : DE$ (being equal to $HH' : H'M$) is a given ratio. Therefore DE is given.

Hence the problem reduces itself to the problem of dividing $A'D$ into two parts at M so that

$$MD : (\text{a given length}) = (\text{a given area}) : A'M^2.$$

Archimedes adds: "If the problem is propounded in this general form, it requires a διορισμός [i.e. it is necessary to investigate the limits of possibility], but, if there be added the conditions subsisting in the present case, it does not require a διορισμός."

In the present case the problem is:

Given a straight line $A'A$ produced to D so that $A'A = 2AD$, and given a point E on AD, to cut AA' in a point M so that

$$AA'^2 : A'M^2 = MD : DE.$$

"And the analysis and synthesis of both problems will be given at the end*."

The synthesis of the main problem will be as follows. Let $R : S$ be the given ratio, R being less than S. AA' being a diameter of a great circle, and O the centre, produce OA to D so that $OA = AD$, and divide AD in E so that

$$AE : ED = R : S.$$

Then cut AA' in M so that

$$MD : DE = AA'^2 : A'M^2.$$

Through M erect a plane perpendicular to AA'; this plane will then divide the sphere into segments which will be to one another as R to S.

Take H on $A'A$ produced, and H' on AA' produced, so that

$$OA' + A'M : A'M = HM : MA, \quad\text{..............(1),}$$
$$OA + AM : AM = H'M : MA' \quad\text{............(2).}$$

We have then to show that

$$HM : MH' = R : S, \text{ or } AE : ED.$$

(α) We first find the value of $HH' : H'A'$ as follows.

As was shown in the analysis (b),

$$HH' . H'A' = OH'^2,$$

or $$HH' : H'A' = OH'^2 : H'A'^2$$
$$= AA'^2 : A'M^2$$
$$= MD : DE, \text{ by construction.}$$

(β) Next we have

$$H'A' : H'M = OA : OA + AM$$
$$= AD : DM.$$

Therefore $$HH' : H'M = (HH' : H'A').(H'A' : H'M)$$
$$= (MD : DE).(AD : DM)$$
$$= AD : DE,$$

whence $$HM : MH' = AE : ED$$
$$= R : S.$$ Q. E. D.

Note. The solution of the subsidiary problem to which the original problem of Prop. 4 is reduced, and of which Archimedes promises a discussion, is given in a highly interesting and important note by Eutocius, who introduces the subject with the following explanation.

* See the note following this proposition.

"He [Archimedes] promised to give a solution of this problem at the end, but we do not find the promise kept in any of the copies. Hence we find that Dionysodorus too failed to light upon the promised discussion and, being unable to grapple with the omitted lemma, approached the original problem in a different way, which I shall describe later. Diocles also expressed in his work περὶ πυρίων the opinion that Archimedes made the promise but did not perform it, and tried to supply the omission himself. His attempt I shall also give in its order. It will however be seen to have no relation to the omitted discussion but to give, like Dionysodorus, a construction arrived at by a different method of proof. On the other hand, as the result of unremitting and extensive research, I found in a certain old book some theorems discussed which, although the reverse of clear owing to errors and in many ways faulty as regards the figures, nevertheless gave the substance of what I sought, and moreover to some extent kept to the Doric dialect affected by Archimedes, while they retained the names familiar in old usage, the parabola being called a section of a right-angled cone, and the hyperbola a section of an obtuse-angled cone; whence I was led to consider whether these theorems might not in fact be what he promised he would give at the end. For this reason I paid them the closer attention, and, after finding great difficulty with the actual text owing to the multitude of the mistakes above referred to, I made out the sense gradually and now proceed to set it out, as well as I can, in more familiar and clearer language. And first the theorem will be treated generally, in order that what Archimedes says about the limits of possibility may be made clear; after which there will follow the special application to the conditions stated in his analysis of the problem."

The investigation which follows may be thus reproduced. The general problem is:

Given two straight lines AB, AC and an area D, to divide AB at M so that

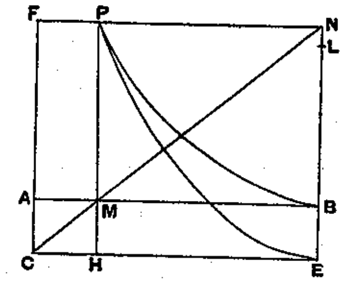
$$AM : AC = D : MB^2.$$

Analysis.

Suppose M found, and suppose AC placed at right angles to AB. Join CM and produce it. Draw EBN through B parallel to AC meeting CM in N, and through C draw CHE parallel to AB meeting EBN in E. Complete the parallelogram $CENF$, and through M draw PMH parallel to AC meeting FN in P.

Measure EL along EN so that

$$CE . EL \text{ (or } AB . EL) = D.$$



Then, by hypothesis,

$$AM : AC = CE . EL : MB^2.$$

And

$$AM : AC = CE : EN,$$

by similar triangles,

$$= CE . EL : EL . EN.$$

It follows that $PN^2 = MB^2 = EL . EN$.

Hence, if a parabola be described with vertex E, axis EN, and parameter equal to EL, it will pass through P; and it will be given in position, since EL is given.

Therefore P lies on a given parabola.

Next, since the rectangles FH, AE are equal,

$$FP . PH = AB . BE.$$

Hence, if a rectangular hyperbola be described with CE, CF as asymptotes and passing through B, it will pass through P. And the hyperbola is given in position.

Therefore P lies on a given hyperbola.

Thus P is determined as the intersection of the parabola and hyperbola. And since P is thus given, M is also given.

διορισμός.

Now, since $AM : AC = D : MB^2$,

$$AM . MB^2 = AC . D.$$

But $AC . D$ is given, and *it will be proved later that the maximum value of $AM . MB^2$ is that which it assumes when $BM = 2AM$.*

Hence *it is a necessary condition of the possibility of a solution that* $AC \cdot D$ *must not be greater than* $\frac{1}{3}AB \cdot (\frac{2}{3}AB)^2$, *or* $\frac{4}{27}AB^2$.

Synthesis.

If O be such a point on AB that $BO = 2AO$, we have seen that, in order that the solution may be possible,

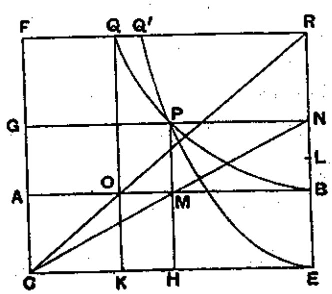
$$AC \cdot D \not> AO \cdot OB^2.$$

Thus $AC \cdot D$ is either equal to, or less than, $AO \cdot OB^2$.

(1) If $AC \cdot D = AO \cdot OB^2$, then the point O itself solves the problem.

(2) Let $AC \cdot D$ be less than $AO \cdot OB^2$.

Place AC at right angles to AB. Join CO, and produce it to R. Draw EBR through B parallel to AC meeting CO in R, and through C draw CE parallel to AB meeting EBR in E. Complete the parallelogram $CERF$, and through O draw QOK parallel to AC meeting FR in Q and CE in K.



Then, since

$$AC \cdot D < AO \cdot OB^2,$$

measure RQ' along RQ so that

$$AC \cdot D = AO \cdot Q'R^2,$$

or

$$AO : AC = D : Q'R^2.$$

Measure EL along ER so that

$$D = CE \cdot EL \text{ (or } AB \cdot EL).$$

Now, since $AO : AC = D : Q'R^2$, by hypothesis,

$$= CE \cdot EL : Q'R^2,$$

and $AO : AC = CE : ER$, by similar triangles,

$$= CE \cdot EL : EL \cdot ER,$$

it follows that

$$Q'R^2 = EL \cdot ER.$$

Describe a parabola with vertex E, axis ER, and parameter equal to EL. This parabola will then pass through Q'.

Again, rect. FK = rect. AE,

or

$$FQ \cdot QK = AB \cdot BE;$$

and, if we describe a rectangular hyperbola with asymptotes CE, CF and passing through B, it will also pass through Q.

Let the parabola and hyperbola intersect at P, and through P draw PMH parallel to AC meeting AB in M and CE in H, and GPN parallel to AB meeting CF in G and ER in N.

Then shall M be the required point of division.

Since $PG \cdot PH = AB \cdot BE$,

rect. GM = rect. ME,

and therefore CMN is a straight line.

Thus $AB \cdot BE = PG \cdot PH = AM \cdot EN$ ....(1).

Again, by the property of the parabola,

$$PN^2 = EL \cdot EN,$$

or $MB^2 = EL \cdot EN$ ....(2).

From (1) and (2)

$$AM : EL = AB \cdot BE : MB^2,$$

or

$$AM \cdot AB : AB \cdot EL = AB \cdot AC : MB^2.$$

Alternately,

$$AM \cdot AB : AB \cdot AC = AB \cdot EL : MB^2,$$

or

$$AM : AC = D : MB^2.$$

Proof of διορισμός.

It remains to be proved that, *if* AB *be divided at* O *so that* $BO = 2AO$, *then* $AO \cdot OB^2$ *is the maximum value of* $AM \cdot MB^2$,

or

$$AO \cdot OB^2 > AM \cdot MB^2,$$

where M *is any point on* AB *other than* O.

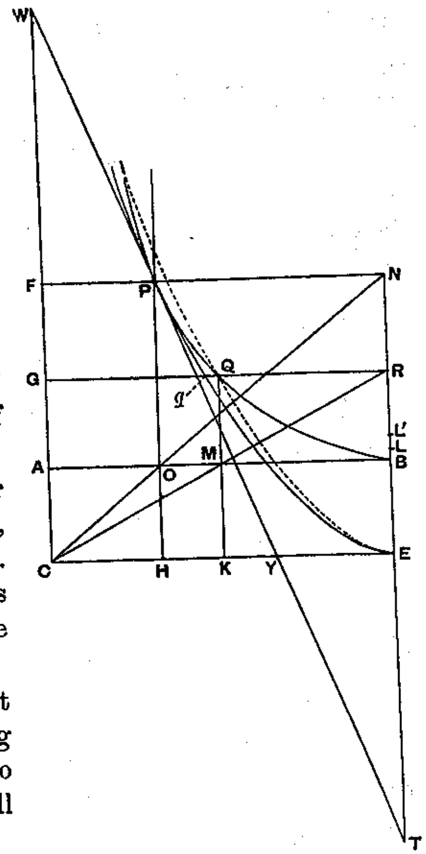
Suppose that $AO : AC = CE . EL' : OB^2$,

so that $AO . OB^2 = CE . EL' . AC$.

Join CO, and produce it to N; draw EBN through B parallel to AC, and complete the parallelogram $CENF$.

Through O draw POH parallel to AC meeting FN in P and CE in H.

With vertex E, axis EN, and parameter EL', describe a parabola. This will pass through P, as shown in the analysis above, and beyond P will meet the diameter CF of the parabola in some point.



Next draw a rectangular hyperbola with asymptotes CE, CF and passing through B. This hyperbola will also pass through P, as shown in the analysis.

Produce NE to T so that $TE = EN$. Join TP meeting CE in Y, and produce it to meet CF in W. Thus TP will touch the parabola at P.

Then, since $BO = 2AO$,

$$TP = 2PW.$$

And $TP = 2PY$.

Therefore $PW = PY$.

Since, then, WY between the asymptotes is bisected at P, the point where it meets the hyperbola,

WY is a tangent to the hyperbola.

Hence the hyperbola and parabola, having a common tangent at P, touch one another at P.

Now take any point M on AB, and through M draw QMK parallel to AC meeting the hyperbola in Q and CE in K. Lastly, draw $GqQR$ through Q parallel to AB meeting CF in G, the parabola in q, and EN in R.

Then, since, by the property of the hyperbola, the rectangles GK, AE are equal, CMR is a straight line.

By the property of the parabola,

$$qR^2 = EL' . ER,$$

so that $QR^2 < EL' . ER$.

Suppose $QR^2 = EL . ER$,

and we have

$$AM : AC = CE : ER$$
$$= CE . EL : EL . ER$$
$$= CE . EL : QR^2$$
$$= CE . EL : MB^2,$$

or $AM . MB^2 = CE . EL . AC$.

Therefore $AM . MB^2 < CE . EL' . AC$

$$< AO . OB^2.$$

If $AC . D < AO . OB^2$, there are two solutions because there will be two points of intersection between the parabola and the hyperbola.

For, if we draw with vertex E and axis EN a parabola whose parameter is equal to EL, the parabola will pass through the point Q (see the last figure); and, since the parabola meets the diameter CF beyond Q, it must meet the hyperbola again (which has CF for its asymptote).

[If we put $AB = a$, $BM = x$, $AC = c$, and $D = b^2$, the proportion

$$AM : AC = D : MB^2$$

is seen to be equivalent to the equation

$$x^2 (a - x) = b^2 c,$$

being a *cubic equation* with the term containing x omitted.

Now suppose EN, EC to be axes of coordinates, EN being the axis of y.

Then the parabola used in the above solution is the parabola

$$x^2 = \frac{b^2}{a} \cdot y,$$

and the rectangular hyperbola is

$$y(a - x) = ac.$$

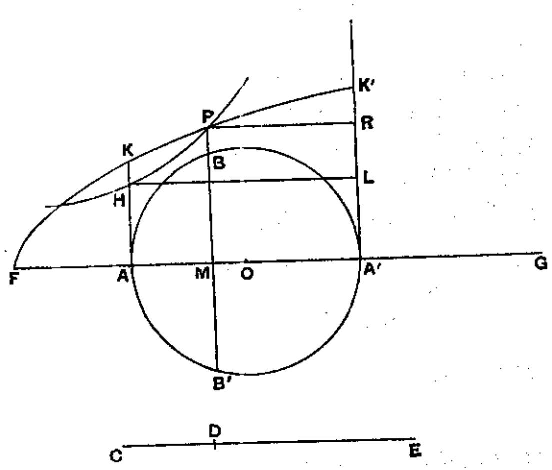
Thus the solution of the cubic equation and the conditions under which there are no positive solutions, or one, or two positive solutions are obtained by the use of the two conics.]

[For the sake of completeness, and for their intrinsic interest, the solutions of the original problem in Prop. 4 given by Dionysodorus and Diocles are here appended.

Dionysodorus' solution.

Let AA' be a diameter of the given sphere. It is required to find a plane cutting AA' at right angles (in a point M, suppose) so that the segments into which the sphere is divided are in a given ratio, as $CD : DE$.

Produce $A'A$ to F so that $AF = OA$, where O is the centre of the sphere.



Draw AH perpendicular to AA' and of such length that

$$FA : AH = CE : ED,$$

and produce AH to K so that

$$AK^2 = FA \cdot AH \quad \text{.....................} (\alpha).$$

With vertex F, axis FA, and parameter equal to AH describe a parabola. This will pass through K, by the equation (α).

Draw $A'K'$ parallel to AK and meeting the parabola in K'; and with $A'F$, $A'K'$ as asymptotes describe a rectangular hyperbola passing through H. This hyperbola will meet the parabola at some point, as P, between K and K'.

Draw PM perpendicular to AA' meeting the great circle in B, B', and from H, P draw HL, PR both parallel to AA' and meeting $A'K'$ in L, R respectively.

Then, by the property of the hyperbola,

$$PR \cdot PM = AH \cdot HL,$$

i.e. $$PM \cdot MA' = HA \cdot AA',$$

or $$PM : AH = AA' : A'M,$$

and $$PM^2 : AH^2 = AA'^2 : A'M^2.$$

Also, by the property of the parabola,

$$PM^2 = FM \cdot AH,$$

i.e. $$FM : PM = PM : AH,$$

or $$FM : AH = PM^2 : AH^2$$

$$= AA'^2 : A'M^2, \text{ from above.}$$

Thus, since circles are to one another as the squares of their radii, the cone whose base is the circle with $A'M$ as radius and whose height is equal to FM, and the cone whose base is the circle with AA' as radius and whose height is equal to AH, have their bases and heights reciprocally proportional.

Hence the cones are equal; i.e., if we denote the first cone by the symbol $c(A'M)$, FM, and so on,

$$c(A'M), FM = c(AA'), AH.$$

Now $c(AA'), FA : c(AA'), AH = FA : AH$

$$= CE : ED, \text{ by construction.}$$

Therefore

$$c(AA'), FA : c(A'M), FM = CE : ED \text{ ....}(\beta).$$

But (1) $c(AA'), FA =$ the sphere. [I. 34]

(2) $c(A'M), FM$ can be proved equal to the segment of the sphere whose vertex is A' and height $A'M$.

For take G on AA' produced such that

$$GM : MA' = FM : MA$$
$$= OA + AM : AM.$$

Then the cone GBB' is equal to the segment $A'BB'$ [Prop. 2].

And $FM : MG = AM : MA'$, by hypothesis,

$$= BM^2 : A'M^2.$$

Therefore

(circle with rad. BM) : (circle with rad. $A'M$)

$$= FM : MG,$$

so that $c(A'M), FM = c(BM), MG$

= the segment $A'BB'$.

We have therefore, from the equation (β) above,

(the sphere) : (segmt. $A'BB'$) $= CE : ED$,

whence (segmt. ABB') : (segmt. $A'BB'$) $= CD : DE$.

Diocles' solution.

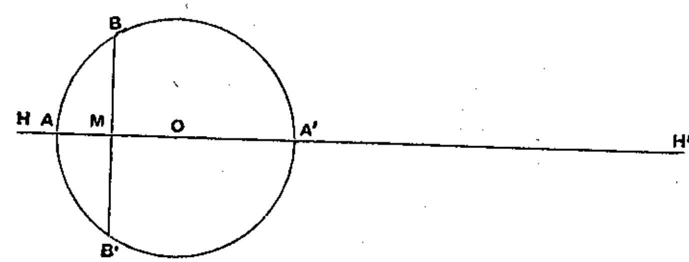
Diocles starts, like Archimedes, from the property, proved in Prop. 2, that, if the plane of section cut a diameter AA' of the sphere at right angles in M, and if H, H' be taken on OA, OA' produced respectively so that

$$OA' + A'M : A'M = HM : MA,$$
$$OA + AM : AM = H'M : MA',$$

then the cones HBB', $H'BB'$ are respectively equal to the segments ABB', $A'BB'$.

Then, drawing the inference that

$$HA : AM = OA' : A'M,$$
$$H'A' : A'M = OA : AM,$$



he proceeds to state the problem in the following form, slightly generalising it by the substitution of *any* given straight line for OA or OA':

Given a straight line AA', its extremities A, A', a ratio $C : D$, and another straight line as AK, to divide AA' at M and to find two points H, H' on $A'A$ and AA' produced respectively so that the following relations may hold simultaneously,

$$\left.\begin{aligned} C : D &= HM : MH' \\ HA : AM &= AK : A'M \\ H'A' : A'M &= AK : AM \end{aligned}\right\} \begin{aligned} &\text{..............}(\alpha), \\ &\text{..............}(\beta), \\ &\text{..............}(\gamma). \end{aligned}$$

Analysis.

Suppose the problem solved and the points M, H, H' all found.

Place AK at right angles to AA', and draw $A'K'$ parallel and equal to AK. Join $KM, K'M$, and produce them to meet $K'A', KA$ respectively in E, F. Join KK', draw EG through E parallel to $A'A$ meeting KF in G, and through M draw QMN parallel to AK meeting EG in Q and KK' in N.

Now $HA : AM = A'K' : A'M$, by (β),

$= FA : AM$, by similar triangles,

whence $HA = FA$.

Similarly $H'A' = A'E$.

Next,

$FA + AM : A'K' + A'M = AM : A'M$

$= AK + AM : EA' + A'M$, by similar triangles.

Therefore

$$(FA + AM).(EA' + A'M) = (KA + AM).(K'A' + A'M).$$

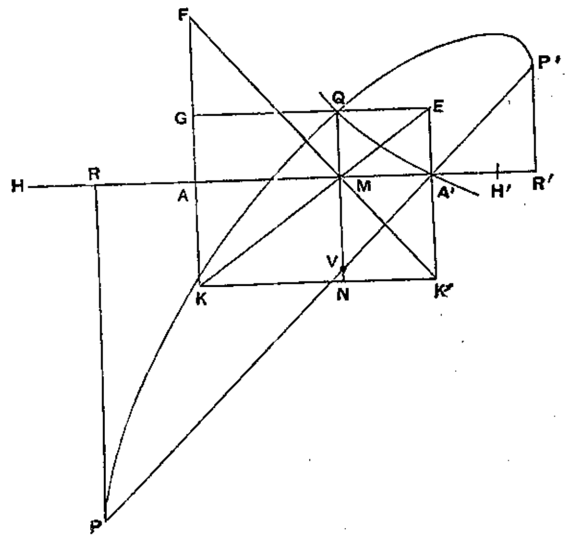
Take AR along AH and $A'R'$ along $A'H'$ such that

$$AR = A'R' = AK.$$

Then, since $FA + AM = HM$, $EA' + A'M = MH'$, we have

$$HM.MH' = RM.MR' \quad\ldots\ldots\ldots\ldots\ldots\ldots(\delta).$$

(Thus, if R falls between A and H, R' falls on the side of H' remote from A', and *vice versa*.)



Now $\quad C : D = HM : MH'$, by hypothesis,

$$= HM.MH' : MH'^2$$

$$= RM.MR' : MH'^2, \text{ by } (\delta).$$

Measure MV along MN so that $MV = A'M$. Join $A'V$ and produce it both ways. Draw RP, $R'P'$ perpendicular to RR' meeting $A'V$ produced in P, P' respectively. Then, the angle $MA'V$ being half a right angle, PP' is given in position, and, since R, R' are given, so are P, P'.

And, by parallels,

$$P'V : PV = R'M : MR.$$

Therefore $\quad PV.P'V : PV^2 = RM.MR' : RM^2.$

But $\quad PV^2 = 2RM^2.$

Therefore $\quad PV.P'V = 2RM.MR'.$

And it was shown that

$$RM.MR' : MH'^2 = C : D.$$

Hence $\quad PV.P'V : MH'^2 = 2C : D.$

But $\quad MH' = A'M + A'E = VM + MQ = QV.$

Therefore $\quad QV^2 : PV.P'V = D : 2C$, a given ratio.

Thus, if we take a line p such that

$$D : 2C = p : PP'^*,$$

and if we describe an ellipse with PP' as a diameter and p as the corresponding parameter [$= DD'^2/PP'$ in the ordinary notation of geometrical conics], and such that the ordinates to PP' are inclined to it at an angle equal to half a right angle, i.e. are parallel to QV or AK, then the ellipse will pass through Q.

Hence Q lies on an ellipse given in position.

Again, since EK is a diagonal of the parallelogram GK',

$$GQ.QN = AA'.A'K'.$$

If therefore a rectangular hyperbola be described with KG, KK' as asymptotes and passing through A', it will also pass through Q.

Hence Q lies on a given rectangular hyperbola.

Thus Q is determined as the intersection of a given ellipse

* There is a mistake in the Greek text here which seems to have escaped the notice of all the editors up to the present. The words are ἐὰν ἄρα ποιήσωμεν, ὡς τὴν Δ πρὸς τὴν διπλασίαν τῆς Γ, οὕτως τὴν ΤΤ πρὸς ἄλλην τινὰ ὡς τὴν Φ, i.e. (with the lettering above) "If we take a length p such that $D : 2C = PP' : p$." This cannot be right, because we should then have

$$QV^2 : PV.P'V = PP' : p,$$

whereas the two latter terms should be reversed, the correct property of the ellipse being

$$QV^2 : PV.P'V = p : PP'. \qquad \text{[Apollonius I. 21]}$$

The mistake would appear to have originated as far back as Eutocius, but I think that Eutocius is more likely to have made the slip than Diocles himself, because any intelligent mathematician would be more likely to make such a slip in writing out another man's work than to overlook it if made by another.

and a given hyperbola, and is therefore given. Thus M is given, and H, H' can at once be found.

Synthesis.

Place AA', AK at right angles, draw $A'K'$ parallel and equal to AK, and join KK'.

Make AR (measured along $A'A$ produced) and $A'R'$ (measured along AA' produced) each equal to AK, and through R, R' draw perpendiculars to RR'.

Then through A' draw PP' making an angle ($AA'P$) with AA' equal to half a right angle and meeting the perpendiculars just drawn in P, P' respectively.

Take a length p such that

$$D : 2C = p : PP'*,$$

and with PP' as diameter and p as the corresponding parameter describe an ellipse such that the ordinates to PP' are inclined to it at an angle equal to $AA'P$, i.e. are parallel to AK.

With asymptotes KA, KK' draw a rectangular hyperbola passing through A'.

Let the hyperbola and ellipse meet in Q, and from Q draw $QMVN$ perpendicular to AA' meeting AA' in M, PP' in V and KK' in N. Also draw GQE parallel to AA' meeting AK, $A'K'$ respectively in G, E.

Produce KA, $K'M$ to meet in F.

Then, from the property of the hyperbola,

$$GQ . QN = AA' . A'K',$$

and, since these rectangles are equal, KME is a straight line.

Measure AH along AR equal to AF, and $A'H'$ along $A'R'$ equal to $A'E$.

From the property of the ellipse,

$$QV^2 : PV . P'V = p : PP'$$
$$= D : 2C.$$

* Here too the Greek text repeats the same error as that noted on p. 77.

And, by parallels,

$$PV : P'V = RM : R'M,$$

or $$PV . P'V : P'V^2 = RM . MR' : R'M^2,$$

while $P'V^2 = 2R'M^2$, since the angle $RA'P$ is half a right angle.

Therefore $$PV . P'V = 2RM . MR',$$

whence $$QV^2 : 2RM . MR' = D : 2C.$$

But $$QV = EA' + A'M = MH'.$$

Therefore $$RM . MR' : MH'^2 = C : D.$$

Again, by similar triangles,

$$FA + AM : K'A' + A'M = AM : A'M$$
$$= KA + AM : EA' + A'M.$$

Therefore

$$(FA + AM) . (EA' + A'M) = (KA + AM) . (K'A' + A'M)$$

or $$HM . MH' = RM . MR'.$$

It follows that

$$HM . MH' : MH'^2 = C : D,$$

or $$HM : MH' = C : D \text{ .... } (\alpha).$$

Also $$HA : AM = FA : AM,$$
$$= A'K' : A'M, \text{ by similar triangles} \ldots (\beta),$$

and $$H'A' : A'M = EA' : A'M$$
$$= AK : AM \text{ .... } (\gamma).$$

Hence the points M, H, H' satisfy the three given relations.]

Proposition 5. (Problem.)

To construct a segment of a sphere similar to one segment and equal in volume to another.

Let ABB' be one segment whose vertex is A and whose base is the circle on BB' as diameter; and let DEF be another segment whose vertex is D and whose base is the circle on EF