Magnitudes in *arithmetical progression* are said *to exceed each other by an equal (amount); if there be any number of magnitudes in arithmetical progression* εἴ κα ἔωντι μεγέθεα ὁποσαοῦν τῷ ἴσῳ ἀλλάλων ὑπερέχοντα. The *common difference* is the *excess* ὑπεροχά, and the terms collectively are spoken of as *the magnitudes exceeding by the equal (difference)* τὰ τῷ ἴσῳ ὑπερέχοντα. *The least term* is τὸ ἐλάχιστον, *the greatest term* τὸ μέγιστον. The *sum of the terms* is expressed by πάντα τὰ τῷ ἴσῳ ὑπερέχοντα.

Terms of a *geometrical progression* are simply *in (continued) proportion* ἀνάλογον, the *series* is then ἡ ἀναλογία, the *proportion*, and a term of the series is τὶς τῶν ἐν τᾷ αὐτᾷ ἀναλογίᾳ. *Numbers in geometrical progression beginning from unity* are ἀριθμοὶ ἀνάλογον ἀπὸ μονάδος. *Let the term* Λ *of the progression be taken which is distant the same number of terms from* Θ *as* Δ *is distant from unity* λελάφθω ἐκ τᾶς ἀναλογίας ὁ Λ ἀπέχων ἀπὸ τοῦ Θ τοσούτους, ὅσους ὁ Δ ἀπὸ μονάδος ἀπέχει.

ON THE SPHERE AND CYLINDER.

BOOK I.

"ARCHIMEDES to Dositheus greeting.

On a former occasion I sent you the investigations which I had up to that time completed, including the proofs, showing that any segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base with the segment and equal height. Since then certain theorems not hitherto demonstrated (ἀνελέγκτων) have occurred to me, and I have worked out the proofs of them. They are these: first, that the surface of any sphere is four times its greatest circle (τοῦ μεγίστου κύκλου); next, that the surface of any segment of a sphere is equal to a circle whose radius (ἡ ἐκ τοῦ κέντρου) is equal to the straight line drawn from the vertex (κορυφή) of the segment to the circumference of the circle which is the base of the segment; and, further, that any cylinder having its base equal to the greatest circle of those in the sphere, and height equal to the diameter of the sphere, is itself [*i.e.* in content] half as large again as the sphere, and its surface also [including its bases] is half as large again as the surface of the sphere. Now these properties were all along naturally inherent in the figures referred to (αὐτῇ τῇ φύσει προυπῆρχεν περὶ τὰ εἰρημένα σχήματα), but remained unknown to those who were before my time engaged in the study of geometry. Having, however, now discovered that the properties are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus, and had not been observed by any one. Now, however, it will be open to those who possess the requisite ability to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I should conceive that he would best have been able to grasp them and to pronounce upon them the appropriate verdict; but, as I judge it well to communicate them to those who are conversant with mathematics, I send them to you with the proofs written out, which it will be open to mathematicians to examine. Farewell.

I first set out the axioms* and the assumptions which I have used for the proofs of my propositions.

DEFINITIONS.

1. There are in a plane certain terminated bent lines (καμπύλαι γραμμαὶ πεπερασμέναι)†, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side.

2. I apply the term **concave in the same direction** to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.

* Though the word used is ἀξιώματα, the "axioms" are more of the nature of definitions; and in fact Eutocius in his notes speaks of them as such (ὅροι).

† Under the term *bent line* Archimedes includes not only curved lines of continuous curvature, but lines made up of any number of lines which may be either straight or curved.

3. Similarly also there are certain terminated surfaces, not themselves being in a plane but having their extremities in a plane, and such that they will either be wholly on the same side of the plane containing their extremities, or have no part of them on the other side.

4. I apply the term **concave in the same direction** to surfaces such that, if any two points on them are taken, the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it, but none on the other side.

5. I use the term **solid sector,** when a cone cuts a sphere, and has its apex at the centre of the sphere, to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone.

6. I apply the term **solid rhombus,** when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line, to denote the solid figure made up of both the cones.

ASSUMPTIONS.

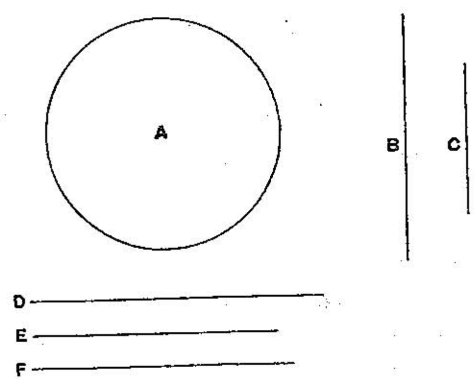
1. *Of all lines which have the same extremities the straight line is the least*.*

* This well-known Archimedean assumption is scarcely, as it stands, a *definition* of a straight line, though Proclus says [p. 110 ed. Friedlein] "Archimedes defined (ὡρίσατο) the straight line as the least of those [lines] which have the same extremities. For because, as Euclid's definition says, ἐξ ἴσου κεῖται τοῖς ἐφ' ἑαυτῆς σημείοις, it is in consequence the least of those which have the same extremities." Proclus had just before [p. 109] explained Euclid's definition, which, as will be seen, is different from the ordinary version given in our text-books; a straight line is not "that which lies evenly between its extreme points," but "that which ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται." The words of Proclus are, "He [Euclid] shows by means of this that the straight line alone [of all lines] occupies a distance (κατέχειν διάστημα) equal to that between the points on it. For, as far as one of its points is removed from another, so great is the length (μέγεθος) of the straight line of which the points are the extremities; and this is the meaning of τὸ ἐξ ἴσου κεῖσθαι τοῖς ἐφ' ἑαυτῆς σημείοις. But, if you take two points on a circumference or any other line, the distance cut off between them along the line is greater than the interval separating them; and this is the case with every line except the straight line." It appears then from this that Euclid's definition should be understood in a sense very like that of

Proposition 5.

Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it, so that the circumscribed polygon may have to the inscribed a ratio less than the greater magnitude has to the less.

Let A be the given circle and B, C the given magnitudes, B being the greater.



Take two unequal straight lines D, E, of which D is the greater, such that $D : E < B : C$ [Prop. 2], and let F be a mean proportional between D, E, so that D is also greater than F.

Describe (in the manner of Prop. 3) one polygon about the circle, and inscribe another in it, so that the side of the former has to the side of the latter a ratio less than the ratio $D : F$.

Thus the duplicate ratio of the side of the former polygon to the side of the latter is less than the ratio $D^2 : F^2$.

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons, since they are similar;

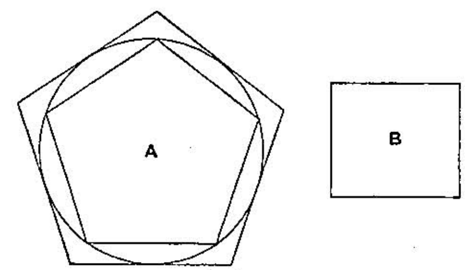
therefore the area of the circumscribed polygon has to the area of the inscribed polygon a ratio less than the ratio $D^2 : F^2$, or $D : E$, and *a fortiori* less than the ratio $B : C$.

Proposition 6.

"Similarly we can show that, *given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less.*

And it is likewise clear that, *if a circle or a sector, as well as a certain area, be given, it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are [together] less than the given area.* For this is proved in the *Elements* [Eucl. XII. 2].

But it is yet to be proved that, *given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area remaining between the circumference and the circumscribed figure is less than the given area.*"



The proof for the circle (which, as Archimedes says, can be equally applied to a sector) is as follows.

Let A be the given circle and B the given area.

Now, there being two unequal magnitudes $A + B$ and A, let a polygon (C) be circumscribed about the circle and a polygon (I) inscribed in it [as in Prop. 5], so that

$$C : I < A + B : A \quad \text{.....................(1).}$$

The circumscribed polygon (C) shall be that required.

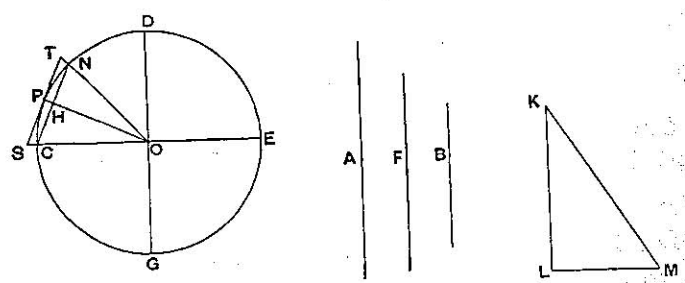
Proposition 3.

Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less.

Let A, B represent the given magnitudes, A being the greater.

Find [Prop. 2] two straight lines F, KL, of which F is the greater, such that

$$F : KL < A : B \text{ ....(1).}$$



Draw LM perpendicular to LK and of such length that $KM = F$.

In the given circle let CE, DG be two diameters at right angles. Then, bisecting the angle DOC, bisecting the half again, and so on, we shall arrive ultimately at an angle (as NOC) less than twice the angle LKM.

Join NC, which (by the construction) will be the side of a regular polygon inscribed in the circle. Let OP be the radius of the circle bisecting the angle NOC (and therefore bisecting NC at right angles, in H, say), and let the tangent at P meet OC, ON produced in S, T respectively.

Now, since $\quad \angle CON < 2 \angle LKM,$

$$\angle HOC < \angle LKM,$$

and the angles at H, L are right;

$$\text{therefore } MK : LK > OC : OH$$
$$> OP : OH.$$

Hence $\quad ST : CN < MK : LK$

$$< F : LK;$$

therefore, *a fortiori*, by (1),

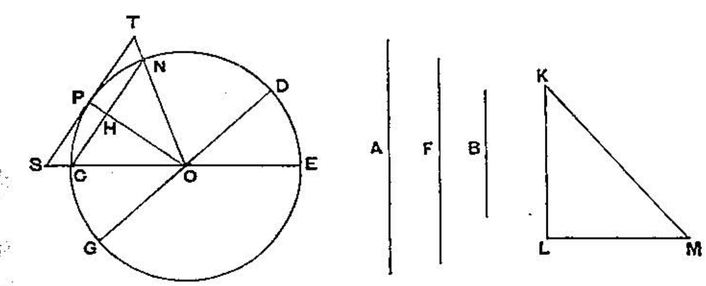
$$ST : CN < A : B.$$

Thus two polygons are found satisfying the given condition.

Proposition 4.

Again, given two unequal magnitudes and a sector, it is possible to describe a polygon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less.

[The "inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is, by construction, some power of 2) subtend equal parts of the arc of the sector; the "circumscribed polygon" is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced.]



In this case we make the same construction as in the last proposition except that we bisect the angle COD of the sector, instead of the right angle between two diameters, then bisect the half again, and so on. The proof is exactly similar to the preceding one.

2. Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other; and that [line] which is included is the lesser [of the two].

3. Similarly, of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area].

4. Of other surfaces with the same extremities, the extremities being in a plane, [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it, or is partly included by, and that [surface] which is partly common with, the other; and that [surface] which is included is the lesser [of the two in area].

5. Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another*.

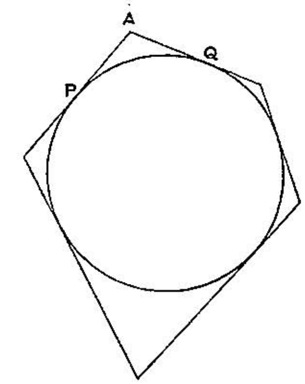
These things being premised, *if a polygon be inscribed in a circle, it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle;* for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it."

Archimedes' assumption, and we might perhaps translate as follows, "A straight line is that which extends equally (ἐξ ἴσου κεῖται) with the points on it," or, to follow Proclus' interpretation more closely, "A straight line is that which represents equal extension with [the distances separating] the points on it."

* With regard to this assumption compare the Introduction, chapter III. § 2.

Proposition 1.

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the perimeter of the circle.



Let any two adjacent sides, meeting in A, touch the circle at P, Q respectively.

Then [*Assumptions*, 2]

$$PA + AQ > (\text{arc } PQ).$$

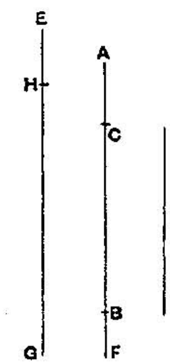
A similar inequality holds for each angle of the polygon; and, by addition, the required result follows.

Proposition 2.

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let AB, D represent the two unequal magnitudes, AB being the greater.

Suppose BC measured along BA equal to D, and let GH be any straight line.



Then, if CA be added to itself a sufficient number of times, the sum will exceed D. Let AF be this sum, and take E on GH produced such that GH is the same multiple of HE that AF is of AC.

Thus $\quad EH : HG = AC : AF.$

But, since $AF > D$ (or CB),

$$AC : AF < AC : CB.$$

Therefore, *componendo*,

$$EG : GH < AB : D.$$

Hence EG, GH are two lines satisfying the given condition.

For the circle (A) is greater than the inscribed polygon (I).

Therefore, from (1), *a fortiori*,

$$C : A < A + B : A,$$

whence
$$C < A + B,$$

or
$$C - A < B.$$

Proposition 7.

If in an isosceles cone [i.e. *a right circular cone*] *a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base.*

Since the sides of the base of the pyramid are equal, it follows that the perpendiculars from the apex to all the sides of the base are equal; and the proof of the proposition is obvious.

Proposition 8.

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side [i.e. *a generator*] *of the cone.*

The base of the pyramid is a polygon circumscribed about the circular base of the cone, and the line joining the apex of the cone or pyramid to the point of contact of any side of the polygon is perpendicular to that side. Also all these perpendiculars, being generators of the cone, are equal; whence the proposition follows immediately.

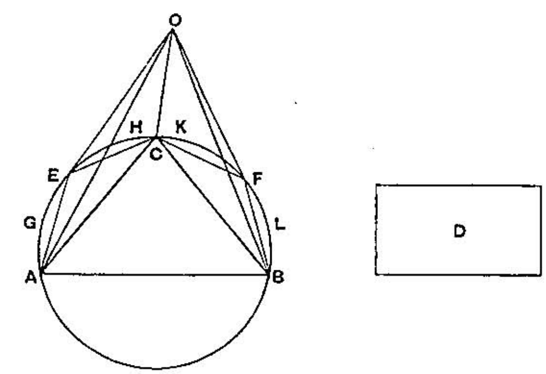
Proposition 9.

If in the circular base of an isosceles cone a chord be placed, and from its extremities straight lines be drawn to the apex of the cone, the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex.

Let ABC be the circular base of the cone, and O its apex.

Draw a chord AB in the circle, and join OA, OB. Bisect the arc ACB in C, and join AC, BC, OC.

Then
$$\triangle OAC + \triangle OBC > \triangle OAB.$$



Let the excess of the sum of the first two triangles over the third be equal to the area D.

Then D is either less than the sum of the segments AEC, CFB, or not less.

I. Let D be not less than the sum of the segments referred to.

We have now two surfaces

(1) that consisting of the portion $OAEC$ of the surface of the cone together with the segment AEC, and

(2) the triangle OAC;

and, since the two surfaces have the same extremities (the perimeter of the triangle OAC), the former surface is greater than the latter, which is *included* by it [*Assumptions*, 3 or 4].