To express it in modern terms, he excludes the existence of actual infinitesimals; the magnitudes he is going to discuss are to form Eudoxian systems.

At the end of the Lambanomena it is mentioned, by way of conclusion from the second, that the perimeter of a polygon inscribed in a circle is less than the circumference of the circle.

4. *Introductory Propositions* (1–6).

In the first Book of *On the Sphere and Cylinder* the ratio form of the compression method (III; 8, 21) is to be repeatedly applied. The group of the propositions 2–6 serves to prepare the way. It is preceded by the proposition 1, in which it is derived from the second assumption that the perimeter of a polygon circumscribed about a circle is greater than the circumference of the circle.

In the following propositions

C denotes a circle, C_n a regular polygon of n sides circumscribed about this circle, I_n a regular polygon of n sides inscribed in this circle. All three symbols at the same time denote the area of the figures they represent. The sides of the polygons are called successively Z_n and z_n.

Danske Videnskabernes Selskab. Matem.-Fysiske Meddelelser XXV, No 15, København 1950, pp. 4, 5), distinguishes the two axioms in question as the axiom of Eudoxus and the lemma of Archimedes. According to him, the object of the lemma is to establish that, when two magnitudes satisfy the axiom of Eudoxus in respect of each other, their difference also satisfies it in respect of all magnitudes of the same kind with a and b. This view is in agreement with the one defended above; it differs from it only in the motivation: the formulation of the new axiom is considered necessary not for the sake of excluding the method of indivisibles, but to give sense to the difference of two homogeneous magnitudes a and b, *e.g.* in the case where a is a circular arc and b a line segment, or a part of the surface of a sphere and b part of a plane. In the theory of proportions of Euclid this axiom, according to the author's view, was not necessary, because $a-b$ always exists as a magnitude of the same kind with a and b. We are not convinced by this argument. Eudoxus (in Euclid V) merely requires of his magnitudes that they shall satisfy his axiom, and does not say at all what magnitudes they are. It cannot be understood why with him a could not be a circular arc and b a line segment.

The axiom of Archimedes is not therefore required because the scope of the geometrical magnitudes under consideration is widened, but it serves to fill up a gap in the theory of proportions of Euclid V (Euclid, for example, tacitly assumes in V, 8 what the axiom of Archimedes explicitly postulates). In fact, through this gap the indivisibles might slip into geometry again.

Proposition 2.

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the lesser a ratio less than the greater magnitude has to the lesser.

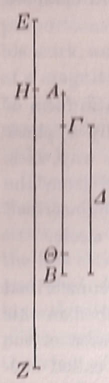

Fig. 52.

Proof: In Fig. 52 let the given magnitudes be AB and Δ ($AB > \Delta$). Measure $B\Gamma = \Delta$. Take any straight line ZH. Measure a multiple $\Lambda\Theta$ of $\Lambda\Gamma$ such that $\Lambda\Theta > \Delta$ (postulate of Eudoxus). Now let HE be the same portion of ZH as $\Lambda\Gamma$ is of $\Lambda\Theta$. Then

$$(EH, HZ) = (\Lambda\Gamma, \Lambda\Theta), \text{ whence, because } \Lambda\Theta > \Delta\,,$$

$$(EH, HZ) < (\Lambda\Gamma, \Delta) = (\Lambda\Gamma, \Gamma B)\,.$$

Componendo (III; 0.42):

$$(EZ, HZ) < (AB, \Gamma B) = (AB, \Delta)\,.$$

Upon superficial examination this proof may appear unduly long. In fact, it might be asked why we do not choose any point Θ between A and B, and then conclude that $(A\Theta, \Delta) < (AB, \Delta)$. In that case, however, it would be forgotten that AB and Δ are indeed represented by line segments, in order to fix our ideas, but that in reality they are geometrical magnitudes (μεγέθη) that are not defined any further, *e.g.* lengths of curves, areas or volumes[1]). EZ and HZ, on the contrary, are real line segments. Although it is not explicitly stated in the proposition, it appears from the given inequality relation and the application of the postulate of Eudoxus that the magnitudes AB and Δ are assumed to be of the same kind.

In modern notation the argument is as follows:

[1]) This is also pointed out by Hjelmslev, l.c. p. 7, who observes in this connection that when at the words "Measure $B\Gamma = \Delta$" the text refers to Euclid I, 2 (Heiberg I 12, line 3), in the first place this should read I, 3, but that it has also to be assumed that this is an interpolation, because the given magnitudes AB and Δ are not line segments, but are only represented by line segments. It might be argued against this that, all the same, Euclidean constructions can be applied to these line segments functioning as symbols. For the rest, the above doubt as to the genuineness of the reference is in itself not unjustified. Archimedes never quotes Euclid anywhere else; why should he do it all at once for this extremely elementary question?

If a and b are two unequal magnitudes of the same kind ($a > b$), find a natural number n such that

$$n(a-b) > b\,.$$

From this it follows that

$$\frac{n+1}{n} < \frac{a}{b}.$$

Proposition 3.

Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to circumscribe another about it so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than the greater magnitude has to the less.

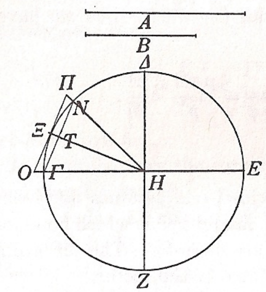

Fig. 53.

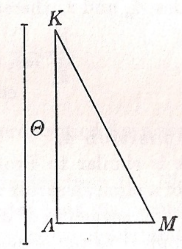

Fig. 54.

Proof (Fig. 53): Assume as given the magnitudes A and B ($A > B$) and the circle H. Find (Prop. 2) two straight lines Θ, $K\Lambda$ ($\Theta > K\Lambda$) such that

$$(\Theta, K\Lambda) < (A, B)\,.$$

Now (Fig. 54) construct a triangle $K\Lambda M$, right-angled in Λ, in which $KM = \Theta$. In the circle H draw two diameters ΓE and ΔZ at right angles. Then apply dichotomy (III; 0.5) to the angle $\Gamma H\Delta$, so that $\angle NH\Gamma < 2\cdot\angle \Lambda KM$. Now $N\Gamma$ is the side of an inscribed equilateral polygon. Let $H\Xi$ be bisector of $\angle \Gamma HN$, $O\Pi$ tangent to the circle at Ξ, then $O\Pi$ is the side of a circumscribed equilateral polygon. If now $H\Xi$ meet $N\Gamma$ in T, then we have

$$\angle NH\Gamma < 2\cdot\angle \Lambda KM, \text{ whence } \angle TH\Gamma < \angle \Lambda KM\,.$$

From this it follows that

$$(\Gamma H, H\Upsilon) < (KM, K\Lambda)$$

or $$(H\Xi, H\Upsilon) < (\Theta, K\Lambda),$$

therefore $$(\Pi O, N\Gamma) < (\Theta, K\Lambda) < (A, B).$$

Modern notation: Find a number p such that

$$\frac{p+1}{p} < \frac{A}{B}.$$

Now construct an angle $\varphi(\Lambda KM)$ such that $\cos\varphi = \dfrac{p}{p+1}$.

By dichotomy find an angle $\alpha = \dfrac{\frac{1}{2}\pi}{2^m}$, so that $\alpha < 2\varphi$.

Now let α be an angle at the centre of a circumscribed and an inscribed regular polygon of n sides ($n = 2^{m+2}$), then we have for the sides Z_n and z_n thereof

$$\frac{Z_n}{z_n} = \frac{1}{\cos\frac{\alpha}{2}} < \frac{1}{\cos\varphi} = \frac{p+1}{p} < \frac{A}{B}.$$

Proposition 4.

This is similar to Prop. 3, provided one assumes as given, instead of a circle, a sector of a circle, in and about which homologous equilateral segments of a polygon are described. The dichotomy is now applied to the angle at the centre of the sector.

Proposition 5.

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another in it, so that the circumscribed polygon may have to the inscribed polygon a ratio less than the greater magnitude has to the less.

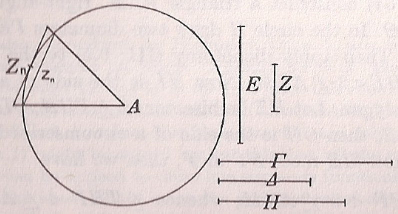

Fig. 55.

Construction: Assume as given (Fig. 55) the circle A and the unequal magnitudes E, Z ($E > Z$). Construct (Prop. 2) two straight lines Γ, Δ ($\Gamma > \Delta$) such that $(\Gamma, \Delta) < (E, Z)$. Find the mean proportional H between Γ and Δ, then $\mathbf{T}(H) = \mathbf{O}(\Gamma, \Delta) < \mathbf{T}(\Gamma)$, whence $H < \Gamma$. Circumscribe (Prop. 3) about A a polygon C_n and inscribe in A a polygon I_n such that the ratio of the sides Z_n and z_n may satisfy the relation

$$(Z_n, z_n) < (\Gamma, H).$$

Then we also have (III; 0.43)

$$\Delta\Lambda(Z_n, z_n) < \Delta\Lambda(\Gamma, H) = (\Gamma, \Delta),$$

therefore $$(C_n, I_n) < (\Gamma, \Delta) < (E, Z).$$

Modern notation:

$$\frac{C_n}{I_n} = \frac{Z_n^2}{z_n^2} < \frac{\Gamma^2}{H^2} = \frac{\Gamma^2}{\Gamma\Delta} = \frac{\Gamma}{\Delta} < \frac{E}{Z}.$$

Proposition 6.

α) In a similar way the corresponding proposition for a sector of a circle may be proved.

β) The reader is reminded of the proposition from the *Elements* (to be found in Euclid XII, 2), which states that upon continued duplication of the number of sides of an inscribed equilateral polygon the sum of the remaining segments of the circle decreases below an assigned area.

γ) Thereafter it is shown that a similar proposition applies fo the sum of the tangent sectors[1]) which lie within a circumscribed equilateral polygon and without the circle.

Proof: Let the assigned area be B and the area of the circle C, then it is possible (Prop. 5) to find a number n such that

$$(C_n, I_n) < (C+B, C), \text{ i.e., because } I_n < C,$$

$$(C_n, C) < (C+B, C),$$

therefore

$$C_n < C+B \quad \text{or} \quad C_n - C < B.$$

[1]) By a tangent sector we understand the figure bounded by the parts of two intersecting tangents to a circle between the point of intersection and the points of contact, and the smaller arc of the circle between the points of contact.

The real meaning of the group of propositions 2–6 becomes clear if we represent the repeatedly recurring ratio of two unequal magnitudes of the same kind by ε (in which therefore $\varepsilon > 1$).

It has then been proved that n can be so chosen that

In Prop. 2 $\quad 1 < \dfrac{n+1}{n} < \varepsilon$, or[1] $\quad \operatorname*{Lim}_{n\to\infty} \dfrac{n+1}{n} = 1$.

In Prop. 3 $\quad 1 < \dfrac{Z_n}{z_n} < \varepsilon$, or $\quad \operatorname*{Lim}_{n\to\infty} \dfrac{Z_n}{z_n} = 1$.

In Prop. 5 $\quad 1 < \dfrac{C_n}{I_n} < \varepsilon$, or $\quad \operatorname*{Lim}_{n\to\infty} \dfrac{C_n}{I_n} = 1$.

In Prop. 6 the reader is reminded that with a given number $\delta > 0$, a number n may be chosen such that

$$0 < C - I_n < \delta, \text{ or } \operatorname*{Lim}_{n\to\infty} (C - I_n) = 0$$

while it is also proved that

$$0 < C_n - C < \delta, \text{ or } \operatorname*{Lim}_{n\to\infty} (C_n - C) = 0 .$$

5. *Curved Surface of Cylinder and Cone. Propositions* 7–20.

The main theorems of this group are the propositions 13 and 14, in which the curved surfaces of a cylinder and a cone respectively are found. The following propositions serve as an introduction:

Props 7 and 8, in which the lateral surfaces of a regular pyramid inscribed in and of one circumscribed about a cone are found.

Props 9 and 10, in which these lateral surfaces are compared with the curved surface of the cone.

Props 11 and 12, in which the lateral surfaces of regular prisms inscribed in and circumscribed about a cylinder are compared with the curved surface of this cylinder.

Proposition 7.

If in an isosceles cone a pyramid be inscribed having an equilateral base, its surface excluding the base is equal to a triangle having its base equal to the perimeter of the base [of the pyramid] and its height equal to the perpendicular drawn from the vertex to one side of the base.

Nowadays this is expressed by saying that the lateral surface of a regular pyramid is equal to half the product of the perimeter of the base and the apothem. This expression, however, is senseless in Greek geometry, because as a rule it will not be possible to represent by numbers the lengths of the line segments multiplied by us. Thus, wherever we use, to denote an area, a product of two factors, the Greek geometer had to introduce a plane figure whose area was equal to that of the figure under consideration. This often makes the argument seem cumbrous to us; it is, however, an essential feature of the Greek point of view which thus becomes manifest.

For the rest, the proof of Prop. 7 is completely identical with the one still commonly used.

In Prop. 8 the corresponding theorem for a circumscribed pyramid is enunciated and proved.

Proposition 9.

If in an isosceles cone a straight line fall within the circle which is the base of the cone, and from its extremities straight lines be drawn to the vertex of the cone, the triangle contained by the chord and the lines drawn to the vertex will be less than the surface of the cone intercepted between the lines drawn to the vertex.

In Fig. 56 let $\Delta.AB\Gamma$ be the given right circular cone, $A\Gamma$ a chord of the circular base. It is required to prove that

$$\triangle \Delta A\Gamma < \text{portion of surface of cone } \Delta A\Gamma .$$

Proof: Let B be the middle point of the arc $A\Gamma$, then

(α) $\triangle \Delta AB + \triangle \Delta B\Gamma > \triangle \Delta A\Gamma$ (*vide* Note).

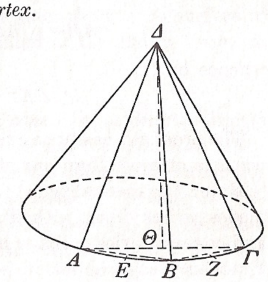

Fig. 56.

$$\text{Suppose } \triangle \Delta AB + \triangle \Delta B\Gamma - \triangle \Delta A\Gamma = \Theta , \qquad (1)$$

then either (I) $\Theta \geqq$ segment of circle AB + segment of circle $B\Gamma$

or (II) $\Theta <$ segment of circle AB + segment of circle $B\Gamma$.

[1]) This word implies that the difference between the propositions of Archimedes and the propositions of the theory of variants mentioned behind them is exclusively a difference of notation.