ARCHIMEDES

in the Middle Ages

VOLUME I

THE ARABO-LATIN TRADITION

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COMMENTARY

2–3 "Omnis...spera." Cf. the translation by William Moerbeke of this same statement (Note: I include here for convenience the whole passage from the introduction to Book I of the *De sphaera*, MS Vat. Ottob. lat. 1850, 23v): "Primum quidem quod omnis spere superficies quadrupla est maximi circuli eorum qui in spera. Deinde autem quod superficiei omnis portionis spere equalis est circulus, cuius que ex centro est equalis recte ei que a vertice portionis ducitur ad periferiam circuli qui est basis portionis. Ad hoc autem quod omnis cylindrus habens basem quidem equalem maximo circulo eorum qui in spera, altitudinem autem equalem diametro spere, et ipse est emiolius spere, et superficies ipsius superficiei spere.... Quod omnis pyramis est tertia pars prismatis habentis basem eandem pyramidi et altitudinem equalem, et quod omnis conus est tertia pars cylindri habentis basem eandem cono et altitudinem equalem...." Cf. the different wording of this first proposition in the *De curvis superficiebus*, Section 2 below, Proposition VI, lines 43–44. Cf. *Verba filiorum*, Proposition XIV, lines 36–37, and also *De ysoperimetris*, MS Oxford, Bodl. Auct. F.5.28, 106v (see Appendix III, passage 8).

8–12 "Omnis...spere." See the previous comment for the Moerbeke translation of this same statement. Cf. *De curvis superficiebus* (Section 2 below, Proposition VIII, lines 1–5).

13–15 "Omnis...serratilis." Cf. Commentary, lines 2–3, above, for the Moerbeke rendering of this statement. The Adelard-Campanus version of the *Elements* (ed. of Basel [1546], p. 346) defines the *corpus serratile* as follows: "Corpus serratile dicitur, quod quinque superficiebus, quarum tres parallelogrammae sunt, duae vero triangulae, continentur." Apparently in changing from prism to *serratile* the translator has limited the proposition to a pyramid with a parallelogram as a base. If, however, the pyramid and *serratile* were constructed on a triangular face, then the pyramid would be one third of the *serratile* as in the Archimedean proposition.

16–17 "Omnis...eius." Cf. Commentary, lines 2–3, above, for the Moerbeke rendering.

18–20 "Omnis...spere." Moerbeke's translation of these lines from Book II of the *De sphaera* runs (MS. cit., 31r): "...omnis sector solidus [spere] est equalis cono basem quidem habenti circulum equalem superficiei portionis eius que in sectore spere, altitudinem autem equalem diametro ei que ex centro spere." Cf. the *De curvis superficiebus*, Section 2, below, Proposition IX, lines 1–3, and the Commentary to that proposition.

2. The *Liber de curvis superficiebus* of Johannes de Tinemue

Much more significant for the spread of knowledge of the *De sphaera et cylindro* of Archimedes than the brief fragment described in the first section of this chapter was the so-called *Liber de curvis superficiebus* of a mathematician whose name seems to have been Johannes de Tinemue. This treatise was, next to the *De mensura circuli*, the most popular Archimedean work circulating in the Latin West during the thirteenth and fourteenth centuries. This is shown by the fact that at least twelve manuscripts of it are still extant. Furthermore, it was widely quoted by geometers of this period, as we shall see, and although it is probably of Greek origin—as I shall argue—it circulated in the thirteenth century with the tracts of Arabic origin. While I have spoken of it as an "Archimedean" work, I do not mean to imply that it was a work composed by the great Syracusan mathematician but only that it was inspired throughout by the techniques and conclusions found in Book I of the *De sphaera et cylindro* and the *De mensura circuli*.

Before taking up the difficult problem of the origins of this tract and the identity of its author, I should first remark that there was some variation in the title and in the author's name. Among the geometers the work was ordinarily known as *De curvis superficiebus Archimenidis*, and indeed this title appears in some of the manuscripts.[1] Manuscript *B*, our most trustworthy manuscript of the thirteenth century, has what I believe to be the original title given in the hand of the scribe of the main text:

[1] See below, notes 20 through 25, and see the various forms of the title given in the variant readings.

Liber magistri Johannis de Tinemue de curvis superficiebus.[2] This exact title is repeated twice in manuscript *J*, also of the thirteenth century.[3] An alternate title is found in the colophon of the best manuscripts and the title of manuscript *F*: *Commentum Johannis de Tin*ꝰ (*or Tin̄*) *in demonstrationes Archimenidis* (although in *F* the name is spelled out as "Ioannis de Chinemue"). Finally, we can remark that a Latin author of the first half of the thirteenth century, Gerard of Brussels, cited our tract under the title of *De pyramidibus* (presumably abbreviated for *De pyramidibus rotundis*),[4] i.e., *On Cones*, no doubt because of the prominence in the treatise of propositions involving cones.

It will be evident from the titles already mentioned that there was some variation among the manuscripts in the name of the author. MS *B*, in the same hand as is used in the rest of the text, has in the lower lefthand cornor of folio 111r the form "Johannes de Tinemue," as I have indicated above. MS *J* also has the same form. The late manuscript *F* apparently altered the place name slightly to "Chinemue," and incidentally the same altered form appears in another work on quadrature in that manuscript (see Appendix I). MS *A* gives the different form "Tinnenie." In the colophons of the various manuscripts the following forms are found: "Tiṅ," "Tin̄," "Thin̄," "T̄ın," "Thinꝰ."[5] In the colophons of two manuscripts (*A* and *E*) the name "Johannes" is dropped entirely and the name of a Latin scholar is substituted: Gervasius de Essexta (*A*) or Gervasius de Assassia (*E*). But since MS *A*—which I believe to be the beginning of this tradition—has the name of Johannes in the title, it seems clear to me that the Latin scholar Gervasius took the work as it existed in the first tradition of *B* and *C* and made some slight changes to produce the second tradition. One of those changes was to substitute his own name for that of Johannes de Tinemue in the colophon.

This brings us to the problem of when the treatise was originally composed. There seems to be little doubt that it existed in some form in

[2] It should be pointed out that at the beginning of the tract MS *B* has *De curvis superficiebus Archimenides* in what I believe to be a hand later than that in which the tract is written. The title as indicated above is in the left-hand lower corner of the first folio of the work (111v) and is in the same hand as the rest of the text.

[3] See variant readings.

[4] Gerard of Brussels, *Liber de motu*, II. 2 (ed. of M. Clagett, *Osiris*, vol. *12*, 1956, p. 132, lines 7–8): "Patet ergo per primam *de piramidibus* quod triangulus LNP equatur curve superficiei rotunde piramidis *OAE*." (Cf. the variant readings on p. 114, line 74, p. 121, lines 8–10, of the Gerard text, except that the second reference ought also to be to the fourth rather than to the fifth proposition of the *De curvis superficiebus*).

[5] See the variant forms of the colophon as given in the variant readings.

antiquity or the early Byzantine period. This I believe to be confirmed by the fact that Hero of Alexandria quotes a conclusion equivalent to its third proposition in his *Mechanics*[6] and that the author of the *Stereometry*, which is also a part of the Heronian corpus, quotes what is equivalent to its tenth proposition.[7] Furthermore, an examination of the content of the Latin treatise shows that the author had more detailed knowledge of the *De sphaera et cylindro* of Archimedes than would have been possible for a Latin author of the early thirteenth century, when the Latin version first became known.[8] The most probable conclusion is that the tract was translated. Against my earlier opinion[9] and the opinion of Heiberg,[10] I am

[6] Hero in the *Mechanics*, I. 4 (ed. of L. Nix, in *Heronis Alexandrini opera... omnia*, vol. *2* [Leipzig, 1900], p. 11), argues that since the diameter of one circle is twice that of another, the semicircumference of the one is twice the semicircumference of the other. He adds, "Archimedes has already proved this." Archimedes does not directly prove this, either in the *Measurement of the Circle* or in the *Sphere and the Cylinder* (although to be sure the determination of π as a constant could serve as a basis of proof). But the text of the *De curvis superficiebus* includes this as its third proposition.

[7] *Stereometrica*, ed. of J. L. Heiberg in *Heronis Alexandrini opera...omnia*, vol. *5* (Leipzig, 1914), p. 8, lines 6–7; cf. *scholia*, p. 229. In both references it is said that Archimedes proves the equality of eleven cubes of the diameter to twenty-one spheres. But nowhere does Archimedes directly prove this in the extant genuine works. I would suppose, therefore, that this is an indication that the *De curvis superficiebus* (which does include this proposition) was circulating and that even then it was associated in some fashion with Archimedes.

[8] For example, in the course of Proposition VII (lines 179–80), the commentator says that the author (i.e., Archimedes) does not treat the case where the number of sides of half of the regular polygon inscribed in a circle and rotating about the diameter of the circle to make a solid composed of conical segments is an odd number. He offers this omission as an excuse for not investigating it himself. I submit that this is a clear indication that Johannes de Tinemue was following along the text of Archimedes, particularly since the enunciation of Proposition VII seems to come almost directly from the text of *De sphaera et cylindro* and was not available in Latin before 1269 in any of the possible sources of Archmidean material; and the *De curvis superficiebus* was quoted by Gerard of Brussels in his *Liber de motu*, which has to be dated before 1260, and probably should be dated a good deal earlier—see Clagett (cited in note 4 above), pp. 104–107. I suppose it is not impossible to argue that the author of the *De curvis superficiebus* was a Latin Schoolman who was familiar with Greek texts and constructed his Latin text as a kind of paraphrase and commentary on Greek materials, but I think this quite unlikely. At least I know of no other such composition at this time. The only points favorable to an independent Latin composition are that the citations of Euclid seem to be in the form preferred by Latin authors and that Proposition XII. 10 of the Greek text is cited as XII. 9 in the manner of the Adelard and Arabic versions. (To be sure it may well be that these versions ultimately follow a Greek order employed by Hero.)

[9] "The *De curvis superficiebus Archimenidis*," *Osiris*, vol. *11* (1954), pp. 297–99.

[10] J. L. Heiberg, *Archimedis opera omnia*, 2 ed., vol. *3* (Leipzig, 1915), p. XCVIII.

now convinced that the translation was made from the Greek rather than from the Arabic. The Latin text retains no trace of the phrasing so characteristic of works translated from the Arabic (as, for example, the use of *iam* with the perfect tense to translate *qad* with the perfect). On the other hand, lines 18–22 of the first proposition appear to be a translation from the Greek, for it would be surprising if the proper name Pallas (line 19) would survive transmission in unmutilated form from Greek through Arabic to Latin. Further evidence is perhaps provided by the use of the word *discolos* in line 18, which is the accusative plural of *discolus* or, more properly, *dyscolus*, a rare Latin word used to transliterate the Greek word δύσκολος. It is possible that this word was used by the translator of our tract simply because he found the same word used in the Greek text.[11] The continuous use of the term *falsigraphus*,[12] which is commonly used to render the Greek ψευδογράφος, could be another indication of a tract that is Greek in origin. And finally the *explicit* of the treatise including the word *tiphis* (transliterating the Greek τῖφυς) seems to add further weight to the Greek origin of the tract. Nothing can be argued from the geometrical terms, for the terms employed in this text are those employed in translations from both Arabic and Greek.

Assuming that the *De curvis superficiebus* was in its origin a Greek tract, we must then face the problem of identifying the author. Unfortunately, I have not been able to find any trace of a Johannes de Tinemue in antique or Byzantine sources. If the quotation of the third proposition in Hero's *Mechanics* was in the original text of Hero's work and was indeed a quotation of this work, then the treatise cannot be any later than the first century A.D.[13] But even if the quotation in Hero's *Mechanics* is an interpolation in the Arabic text (which I strongly doubt, since it is apparently in all of the Arabic manuscripts), or somewhat more likely an interpolation in the Greek text prior to its translation into Arabic,[14] the antique or Byzantine

[11] I suppose one could argue that since the Latin word *dyscolus* was used in the Vulgate translation of the New Testament (I. Petr. 2. 18) it might have been used by a Latin author who was, in preparing the *De curvis superficiebus*, making an original composition rather than doing a translation, but I have already argued against the text being an original composition.

[12] See Chapter Three, Section 5, Corpus Christi Version. See also Commentary, Section 5, Chapter Three, lines 218–19.

[13] See note 6 above. On the basis of astronomical evidence Hero can rather confidently be dated in the first century. See O. Neugebauer, "Über eine Methode zur Distanzbestimmung Alexandria-Rom bei Heron," *Det Kgl. Danske Videnskabernes Selskab. Historisk-filologiske Meddelelser*, vol. *26*, 2 (1938), p. 23, and his note in *Isis*, vol. *39* (1948), pp. 243–44.

[14] On the question of interpolations in the text of the *Mechanics*, see Nix's intro-

origin of the tract seems to follow from the fact that the tenth proposition was quoted in the *Stereometry*. The *Stereometry*, while probably not by Hero, certainly existed in late antique or early Byzantine times.

I have remarked on the fact that two of the manuscripts include the name of a Latin Schoolman, Gervasius of Essex, who apparently was responsible for that revision of the text which I have called Tradition II. I have not been able to identify any such person from Essex, but there was a Johannes Gervasius of Exter (*Exonia*) who was chancellor of York and who was consecrated Bishop of Winton in the Roman Curia in 1261.[15] A scribal confusion between *Essexta* and *Exonia* is possible. The fact that Johannes Gervasius was in Italy in 1261 and died in Viterbo in 1268 is perhaps an indication that he was a member of that remarkable scientific group which included Witelo, Campanus de Novara, William of Moerbeke, and possibly Aquinas.[16] If so, this would seem to establish his interest in scientific matters. He might well have been the reviser of the *Liber de curvis superficiebus*.

The *De curvis superficiebus* is a treatise of ten propositions with several corollaries dealing for the most part with the surfaces and volumes of cones, cylinders, and spheres. The meaning of the title is evident: On Figures Bounded by Curved Surfaces. In brief, the main objectives of the tract are those of Book I of the *De sphaera et cylindro* of Archimedes. Curtze, Heiberg, and Björnbo tended to identify the two works,[17] and Heiberg published the enunciations of the propositions without proofs. But none of these authors seems to have realized that the *De curvis superficiebus*, while something of a commentary on the *De sphaera et cylindro*, is an independent treatise rather than a fragment or paraphrase of Archimedes' work. Incidentally, they all miss the important point partially recognized in the catalogue of Libri manuscripts[18] that the form and techniques of the *De mensura circuli* influenced the author of the *De curvis*

duction (*op. cit.* in note 6 above), pp. XXVIII–XXXIII.

[15] Th. Tanner, *Bibliotheca Britannico-Hibernica* (London, 1748), p. 313.

[16] For a brief account of this group, see M. Grabmann, *Guglielmo Moerbeke* (Rome, 1946), pp. 56–62.

[17] M. Curtze, "Über eine Handschrift etc.," *Zeitschrift für Mathematik und Physik*, vol. *28* (1883). Hist.-lit. Abt., pp. 1–13. A. A. Björnbo, "Handschriftenbeschreibung etc.," *Avhandlungen zur Geschichte der Mathematischen Wissenschaften*, *26.* Heft (1912), p. 128. Cf. *Bibliotheca mathematica*, Dritte Folge, vol. *9* (1908), p. 152. Heiberg's comments, with the text of the enunciations of the *De curvis superficiebus*, are found in the work cited in note 10 above, *loc. cit.*

[18] *Catalogue of the Extraordinary Collection of Splendid Manuscripts... Formed by M. Guglielmo Libri*, London, 1859, 147 (referring to MS No. 665).

superficiebus. For example, the expression of the enunciation of Proposition I of the *De curvis superficiebus* was influenced by the statement of Proposition I of the *De mensura circuli*. Incidentally, the author several times cites the *De mensura circuli* of Archimedes, sometimes with that title and sometimes as *De quadratura circuli*. Ordinarily it is the first proposition of the *De mensura circuli* that is cited, but the second proposition is referred to twice in the course of Proposition X of the *De curvis superficiebus* (see lines 4, 13), and the third proposition is noted in the course of Proposition III (see lines 34–35). Euclid's *Elements* is the only other work referred to by the author, and he cites it about twenty times, although to be sure many of the citations are to the same propositions, namely, Propositions XII.9 and XII.11. He also borrows from the tradition of the *Elements* the use of terms to label the various parts of a proof: *exemplum* (for the Greek *ekthesis* or setting out), *dispositio* (for the Greek *kataskue* or additional construction needed for the proof), *ratio* (for the Greek *apodeixis* or proof). He also uses the term *improbatio* to label a formal refutation and *dissolutio* for a proof *per impossibile*. I have shown elsewhere how these terms were used by Greek, Latin, and Arabic authors in connection with the *Elements*.[19]

Not only was this work known to Gerard of Brussels by the middle of the thirteenth century, as I have indicated above (note 4), but Roger Bacon speaks of it in his *Communia mathematica*.[20] In the fourteenth century the *De curvis superficiebus* was cited by a number of authors: Thomas Bradwardine,[21]

[19] See my "King Alfred and the *Elements* of Euclid," *Isis*, vol. *45* (1954), pp. 272–73. The terms used in the text of the *De curvis superficiebus* are particularly like those found in the introduction to the version of the *Elements* which I have designated as Adelard III but which is probably to be dated close to 1200. The definitions found in the Adelard III introduction which are pertinent to our present discussion are the following (*ibid*, p. 274): "Exemplum est alicuius figure suppositio. Dispositio est aliarum figurarum ad exemplum applicatio. Ratio est principiorum vel suarum conclusionum inductio, i.e., argumentum.... Instantie dissolutio est cum falsigraphus insistit non sic vel aliter accidere quam geometer affirmat."

[20] For citation by Bacon, see Chapter IV, division 1, note 3, where our tract is called *de Curvis Superficiebus Archimenidis*.

[21] Thomas of Bradwardine, *Tractatus de proportionibus*, chap. IV (ed. of H. Lamar Crosby, Jr., pp. 124, 126): "Quorumlibet duorum circulorum circumferentiae suis diametris sunt proportionales. (Et hoc est quinta conclusio *De curvis superficiebus*.)... Cuiuslibet spherae superficies aequalis est quandrangulo qui sub lineis aequalibus diametri spherae et circumferentiae maximi circuli continetur. (Et hoc est octava Archimenidis *De curvis superficiebus*.)" Bradwardine makes a similar reference to the so-called fifth conclusion in his *De continuo* (MS Thorn, 4° 2, p. 164). The propositions cited by Bradwardine are actually the third and sixth propositions of the *De curvis superficiebus*. It is obvious that Bradwardine used a text of the tradition of *D* which

Nicole Oresme,[22] Franciscus de Ferraria,[23] an anonymous author of a commentary on the *Liber de ponderibus*,[24] and the anonymous author of the *Liber de inquisicione capacitatis figurarum*.[25] Furthermore, sometime prior to 1328 (see note 21) a Latin author added the two propositions given in Section 3 of this chapter, while another Latin author, probably in the fourteenth century, paraphrased the proofs of the whole tract and added three additional propositions (Section 4 of this chapter), and still a third author (perhaps originally Gervasius) supplied an additional proof to Proposition VII (Section 5 of this chapter).

The text of the *Liber de curvis superficiebus* which I have presented here differs in only a few places from my previous effort,[26] although I have considerably expanded the variant readings. In general, I have preferred the readings of the first tradition. Manuscripts *B* and *C* of that tradition have been employed throughout the text. Manuscript *B* is not the archetype but must be quite a faithful copy. It is only rarely that I have felt it necessary to correct its reading by an appeal to Manuscript *C* or to the second tradition. Manuscript *J*, also of the thirteenth century, may well have been the link between the two traditions, as it seems to have been in the case of the *Liber de motu* of Gerard of Brussels, which I have edited

contained the two extra marginal propositions. Since the *De proportionibus* was composed in 1328, it is evident that the tradition represented by *D* is prior to that date.

[22] Nicole Oresme, *De configurationibus qualitatum*, MS Bibl. Nat. lat. 7371, 227v: "Nunc autem est ita quod proportio circumferentiarum in quantitate est sicut proportio semidiametrorum circulorum quorum sunt circumferentie, ut patet [per] quintam conclusionem Archimenidis *De curvis superficiebus*." Oresme has also used a manuscript of the tradition represented by *D*.

[23] Francischus de Ferraria, *Questio de proportionibus*, MS Oxford, Bodl. Canon. Misc. 226, 58v: "Deinde suppono unam suppositionem Euclidis *De curvis superficiebus*...et est: Qu[or]um angulorum et circulorum circumferentie suis dyametris sunt proportionabiles, igitur qualis est proportio dyametri *a* ad suam circumferentiam talis est proportio dyametri *b* ad suam circumferentiam." Francischus has apparently changed *Archimenidis* to *Euclidis*.

[24] In E. Moody and M. Clagett, *The Medieval Science of Weights* (Madison, 1952), p. 302: "...per quintam Archimedis *De curvis superficiebus*, eo quod eadem est proportio diametrorum vel semidiametrorum vel circumferentiarum." Once more we should note that the author of this tract has used a text in the tradition of manuscript *D*, where the third proposition is numbered as the fifth.

[25] *De inquisicione capacitatis figurarum*, ed. of M. Curtze in *Abhandlungen zur Geschichte der Mathematik*, 8. Heft (1898), p. 39: "Qui cubus per 10^{am} Archimenisis de curvis superficiebus habet se ad sphaeram dati circuli, sicut 21 ad 11....ergo per sextam Archimenidis de curvis superficiebus ipsum parallelogrammum *acdb* est aequale embado sphaerae circuli dati.... [p. 50] et per secundam Archimenidis provenit tota curva superficies columpnae."

[26] Published in the article cited in note 9 above.

elsewhere.[27] For the second tradition I have used only manuscript *A*, its best manuscript, throughout the whole text. Most of the remaining manuscripts of both traditions specified in the Sigla have been collated through line 47 of the first proposition, enough to indicate such groupings as can be detected. Manuscript D_1 was copied from *D* and not very carefully; I have presented here its variant readings only through line 22 of the first proposition (in addition to reporting its variant readings for the colophon). I see no reason to include any more of its careless misreadings. I have also given the variant readings for the poor manuscript *E* only through line 22 of Proposition I. Manuscript *M* is a paraphrase of the text and I have treated it separately in Section 4 of this chapter. Some of the detailed characteristics of each of the manuscript have been included in the Sigla list below. Following my usual custom I have numbered the propositions successively with Roman numerals.[28] The figures are those found in *B*.[29] The marginal folio numbers refer to manuscript *B*.

Sigla of Manuscripts

Tradition I

B = Oxford, Bodleian Library, Auct. F.5.28, 111r–116r, 13c. On the whole this is the most reliable manuscript. Most of its orthographic peculiarities are reflected in the text here published. However, we can here note that it almost always has *ci* before a vowel rather than *ti*, as

[27] See note 4 for my text of the *Liber de motu*. I was able to use the Berlin Manuscript (here designated as *J* but there designated as *B*) in the preparation of that text. As I noted in the introduction to that text, the Berlin manuscript seems to have occupied a middle position between the Oxford and Naples manuscripts (here designated as *B* and *A* but there designated as *O* and *N*); see p. 109 of that text. The identity of *J* and Libri MS 665 is argued on pp. 103–104.

[28] Actually, manuscripts *A* and *D* number the propositions with Arabic numerals in the margin: 1ª, 2ª, and so forth (although *A* omits numbers for Propositions VIII, IX, and X). *B* uses Roman numerals but divides the propositions into two books: I through VI for the first book, and I through IIII for the second book. *J* makes the same division but uses a hybrid system of Roman and Arabic numerals (see variant readings). The only other manuscript to reflect this division into books is *A*, although to be sure it numbers the propositions successively so that Proposition VII is 7ª rather than I.

[29] Inside the figure for Proposition VII in MS *B* we find written in the same hand as that used in the text *mors Hugonis*, i.e., "the death of Hugo." One is tempted to think that reference is to Hugo, the twelfth-century author of a *Practica geometrie*, but this is merely a guess. Or it may be a comment of the scribe of *B*, discouraged by the length and complexity of Proposition VII. If this is so, then the scribe's name could have been Hugo.

in *circumferencia*. I have in my text employed the *ti* form. *B* starts off with the spelling *pyramis* (i.e., with *py* for all forms of this word) but for the major part of the work employs *piramis*. I have used *piramis* throughout. *B* also vacillates between *basem* and *basim*, between *demissa* and *dimissa*, generally preferring the latter of each of these pairs of terms. The form *corellarium* is used, which I have rejected by following other manuscripts. *B* divides the propositions into two books, numbering the propositions of the first book I through VI, and those of the second book I through IV. But this numbering, added as rubrics, may be in a later hand. There are a number of marginal notes in *B* which are entirely explanatory or constitute a geometric elaboration of the text. These have not been included in the variant readings since they appear to be the notes of the scribe of *B* rather than of the author of the text. They are of exactly the same sort as are found in all other works in the codex.

C = Oxford, Bodleian Library, Digby 174, 174v–178r, 13c. This manuscript is very close to *B* and, in fact, may be a copy of that manuscript. Its orthographic peculiarities are evident in the variant readings, but a few of them can be noted here: *teorema* for *theorema* in *B*, *ypothenussalium* for *ypothenusalium* in *B*, *demissa* for *dimissa*, ordinarily *basim* but occasionally *basem*, occasionally *ortogonio* for *orthogonio*. One unusual feature of this copy is the judicious use of commas, a practice very rare indeed among medieval manuscripts, which generally have periods, colons, and dashes used without much attention to rule or discrimination. There are a few interesting notes in a sixteenth-century hand which I have not included in my variant readings. There is a note dated: 1551, 6 May, on folio 175v.

J = Berlin, Deutsche Staatsbibliothek (MS now at Marburg, Westdeutsche Bibliothek), Quarto 150, 90r–94v, 13c. A Gothic hand very much like that of *B*. A copy close to the tradition of *B* and *C*, but not quite as accurate as those manuscripts. Within that tradition *H* and *F* resemble *J* quite closely. I have collected it with the text through Proposition I, line 47, and in the explicit. The forms *piramis*, *ortogonio*, and sometimes *hypothenusa* (for *ypothenusa*) are used. In numbering the propositions and dividing them into two books, it resembles *B*, but interestingly enough it omits lines 26–28 of Proposition I in the same manner as C, although these lines are present in *B* and even in Tradition II. This manuscript is identical with Libri MS 665 described in the Libri sale catalogue of London, 1859, pp. 145–48 (see notes 18 and 27 above).

D = Florence, Biblioteca Nazionale, Conv. Soppr. J.V.30, 1r–4v, 14c. A good copy of this tradition, being quite close to *B* and *C*. It employs *hypothenusse* for *ypothenuse* in *B*, that is, *ss* is found instead of *s* in the various forms of this word in those instances when the word is written out and not abbreviated. We also find in *D* the peculiar form *correlarium*, resembling the *corellarium* of *B* in using *e* as the second vowel. In *D* the letter *y* replaces *i* in the various forms of *diameter*, as, *dyametros*. *D* includes two extra propositions on the lower margins of folios 1r and 1v. These additional propositions appear to be in the same hand (only very much larger in form). They have been edited and are discussed in Section 3 of this chapter. The propositions in this copy are numbered in the margin: 1ª, 2ª, etc. Collation of *D* with the text is through line 47 of Proposition I.

D_1 = Paris, Bibliothèque Nationale, MS lat. 11247, 2r–25v, late 15c or 16c. Copied directly from *D* but includes many misreadings. The propositions are unnumbered and no figure after the first is given. I have included variant readings only through line 22 of the first proposition. Its orthographic peculiarities for the most part duplicate those of *D*.

H = Basel, Univ. Bibl. F.II.33, 151r–153r, 14c. This is another good copy, very close to *B* and *C* in its readings. It was perhaps copied from *J*. I have given variant readings through the first 47 lines of proposition I. The propositions are unnumbered in *H*.

F = Vienna, Nationalbibliothek, cod. 5303, 11r–18v, 15–16c. *F* is of the tradition of *B* and *C*. It has close similarities with *J* and *H*. Orthographic distinctions: *pyramis* for *piramis*, *hypothenuse* for *ypothenuse*. The propositions are unnumbered in *F*. *F* has been collated with the text through line 47 of Proposition I.

I = Dresden, Sächs. Landesbibliothek, Db. 86, 188r–194v, early 14c. The manuscript is badly stained (and often is not readable) but where it is free from staining it is a very good copy and quite close to *B* (e.g., even in the spelling *corellario*, cf. Proposition I, line 104). Occasional orthographic peculiarities: *polligonie* for *poligonie*, *habitum* for *ambitum*. The propositions are unnumbered in *I*. *I* has been collated with the text through line 47 of Proposition I.

M = Florence, Bibl. Naz., Conv. Soppr. J.V.18, 92v–96v (lower pagination). This text consists of a paraphrase but the enunciations appear to be drawn from Tradition I. See Section 4 of this chapter.

Tradition II

A = Naples, Biblioteca Nazionale, VIII.C.22, 57r–60r, 13c. This is the best of three manuscripts of the second tradition, the tradition perhaps inaugurated by Gervasius of Essexta. Notice that in addition to the substitution of his own name for that of Johannes in the colophon, Gervasius eliminated the parenthetical paragraph included in lines 18–22 of Proposition I. The scribe uses the *ci* form as does *B*; he also often employs *maiore* for *maiori*, *basem* for *basim*, *sexcupla* for *sextupla*, *theoreuma* for *theorema*. The propositions are numbered through the seventh: 1ª, 2ª, and so forth.

E = London, British Museum, Harleian 625, 137r–139v, 14c (?). This is a very free copy of the second tradition. I have included variant readings only through line 22 of the first proposition. Note the form Gervasius de Assassia in the colophon. The propositions are unnumbered.

G = Cambridge, University Library, Mm.III.11, 196r–198v, 15c. Corresponds closely to *A* and *E* but cuts off in the middle of Proposition VII. Hence it does not have Gervasius' name. I have included variant readings through line 47 of Proposition I. The propositions are unnumbered.

Incipit Liber Magistri Johannis de Tinemue de curvis superficiebus [Archimenidis]

Liber I

111r I. / CUIUSLIBET ROTUNDE PIRAMIDIS CURVA SUPERFICIES EST EQUALIS TRIANGULO ORTHOGONIO, CUIUS UNUM LATERUM RECTUM ANGULUM CONTINENTIUM EQUATUR YPOTHENUSE PIRAMIDIS, RELIQUUM CIRCUMFERENTIE BASIS.

Sit *ACNP* circulus circa centrum *D* [Fig. 64.]. Et sit *DE* linea cathetus perpendicularis ad *AN*, *CP* diametros, *EA* vero sit ypothenusa dimissa ab *E* in *A*. Stante igitur *DE* immota, circumvolvatur *DEA* triangulus orthogonius ad perficiendam piramidem rotundam per *QPONMCB* donec redeat ad punctum *A* unde ceperat. Sit item

Title Incipit....I *mg.* *BJ** *om.* *IGMD*$_1$ Incipit liber Johannis de tinennie de curvis superficiebus. Liber I *A* De curvis superficiebus Archimenides *manu recentiori* (?) *B* Archimedes (*C manu recentiori*; Archimenides *H*) de curvis superficiebus *CH* Arcimenidis (*ff. 1r, 2r, 3r* Archimenidis, *4r*) de curvis superficiebus (? *2v*) *mg.* *D*. In nomine domini nostri Iesu christi. Incipit commentum Ioannis de Chinemue in demonstrationes Archmiedis (!) *F* De curvis superficiebus *E* liber de curvis superficiebus *M*/[Archimenidis] *supplevi*; *sed cf. var. supra*

1 I *B* I^a *J* 1^a *AD* ((Note: the propositions are numbered only in MSS BJAD; see the Introduction to this chapter, note 28.))

2 est equalis *tr.* *AEG*

3 continentium: continens *D* continens non *D*$_1$

4 *ante* reliquum *add.* *E* et

6 circulus *om.* *AEG* / centrum: diametrum *AEG* / Et *om.* *G* / sit^2 *om.* *H*

7 cathetis *D*$_1$ / AN, CP: AC NP *DD*$_1$ / EA...sit: et vero EA *E* / vero: vero figure (?)*H* vero linea *JF*

8 dimissa *BIF* demissa *JACHGDD*$_1$ protrahata *E* / immota: immoto *F* inmota *IJD*$_1$

9 orthonogius: orthogonius intellectualiter *JFH* / perficiendam: faciendum *H*

9–10 piramidem...per: rotundam superficiem piramidem *A* rotundam pyramidem *EG*

10 *ante* Q- *add.* *A* A- / QP-: APP- *D*$_1$ / -NM-: -MN- *F* / donec: circumferentiam donec *HFJ* / A *om.* *BF* / inceperat *J*

**J* has the title twice on folio 90r, and following the title as given in the upper margin we read *de curvis liber primus*.

Here Begins the Book of Master Johannes of Tinemue on the Curved Surfaces [of Archimedes]

Book I

I. THE LATERAL SURFACE OF ANY [RIGHT CIRCULAR] CONE IS EQUAL TO A RIGHT TRIANGLE, ONE OF WHOSE TWO SIDES CONTAINING THE RIGHT ANGLE IS EQUAL TO THE SLANT HEIGHT OF THE CONE, WHILE THE OTHER [IS EQUAL] TO THE CIRCUMFERENCE OF THE BASE.

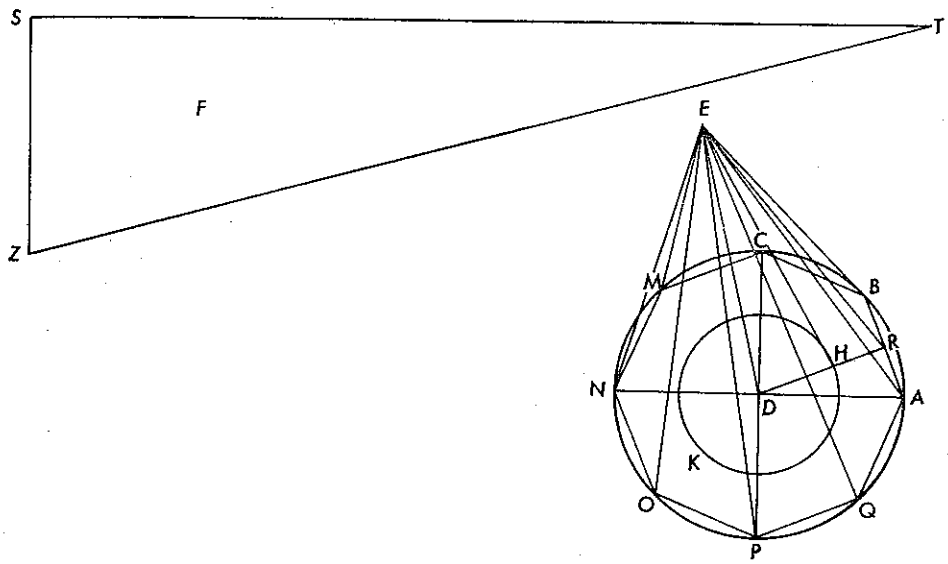

Fig. 64

Let $ACNP$ be a circle with center D [see Fig. 64]. And let line DE be a line perpendicular to diameters AN and CP, while EA is a slant height drawn from E to A. Therefore, with DE as an axis, let right △ DEA be rotated through $QPONMCB$ until it returns to point A from which it began, thereby forming a cone. Also, let slant height EA be

EA ypothenusa equalis linee *SZ* et tota circumferentia *ACNP* equalis *ST*. Et sit *S* angulus rectus.

Dico itaque quod curva superficies rotunde piramidis *DEA* est equalis triangulo orthogonio *F*.

Rationis causa. Presentis demonstrationis ypothesis, et tota sequentium theorematum series, lineam rectam curve et superficiem rectam curve esse equalem sibi postulat admitti.

Unde discolos et obiurgantes non cogit ad scientiam, cum et ipsi sapere non audeant, cum etiam laborent ut nesciant, cum et a pallade ultro vultus detegentes oculos avertant. Hec igitur hiis, qui rerum subtilium fugas, quantitatum miracula, proportionum nexus, nature deposita rimantur, proposuit philiosophus.

Triangulus itaque *F* aut est equalis curve superficiei *DEA* rotunde piramidis aut curve superficiei rotunde piramidis site in minori basi quam *ACNP* circulus, que piramis sit equalis altitudinis piramidi *DEA*, id est, cuius cathetus sit *DE*, aut curve superficiei rotunde

11 *SZ*: scilicet Z DD_1
11–12 et...ST *om.* *AG* et ST equalis circumferentie basis pyramidis *E*
11 equalis[2]: equatur *I* sit equalis *F*
12 *post* rectus *add.* *E* et protrahatur linea TZ ita quod totus ille triangulus sit *F*
13 itaque: ergo DD_1
14 orthogonio *om.* *EF*
15 Rationis causa: ratio *AGE* / demonstrationis *om.* *HF* / *ante* ypothesis *add.* $AGDD_1$ et / hypothenussis D_1 / ypothesis et tota: omnis *E*
15–16 subsequentium *A*
15–16 sequentium...series: series et theorematum sequentium *E* sequencium cathetum series *J*
16 teorematum *C* / series: superficies *scr. et del.* *H et add.* *H mg* series
16–17 lineam...curve *CHFJ* lineam curvam recte et superficiem rectam curve *B* lineam rectam curvam curve et superficiem rectam curve *D* lineam rectam curvam curve et superficiem rectam curve D_1 lineam rectam curve et curvam recte et superficiem rectam curve *I* superficiem curvam recte ut lineam curvam recte (recte *om.* *A suprascr.* *G*) *AG* superficiem recte curvam et lineam curve rectam *E*
17 esse equalem *tr.* *HJF* / equalem sibi *tr.* *D* / admitto *H*
18–22 Unde....philosophus *om.* *AEG*
18 discelos *C* / cum: et cum *I*
19 etiam: ter D_1/ut: nec (?) *H* / cum *om.* *HF* / et: illic (?)*D* autem D_1
20 ultro vultus: vultum ultro *HF* ultimum multum D_1 / detegentes: detegente *CDJ* detegentum *HF* deteguntem D_1 / Hec: hoc D_1
21 subtilians fiiguris *I* / fugas: fugeras *H* fugas *J* fugeras *mg.* *J*
22 rimantur: rimanantur *D* nominantur D_1 / *ante* proposuit *add.* *JHF* physice filiis
23 itaque *om.* *H* / est *om.* *AG* / curve superficiei *tr.* *E* / DEA: dicte *AG*
24 *post* aut *scr. et del.* *C* maioris aut minoris aut / curve superficiei rotunde *om.* *AG* *I* (?) / aut...piramidis *om.* *HF*
25 quam: quam sit *AG* / circulus *om.* *AG* / equalis *om.* *J* / pyramidis *D*
26 aut: aut est equalis *AG*
26–28 aut...DE *om.* *CJ*

equal to line SZ and the whole circumference $ACNP$ be equal to ST; and let S be a right angle.

And so I say that the lateral surface of cone DEA is equal to right △ F.

A supposition necessary for the proof: The hypothesis of the present demonstration as well as the whole series of subsequent theorems postulates that a straight line can be supposed equal to a curved line and a plane surface to a curved surface.

Whence he does not urge to knowledge those who are objectors and are difficult, since they do not venture to learn—for they even work in order not to know—and since, on uncovering their faces, they avert their eyes away from Pallas. Therefore, the philosopher has proposed these things for those who examine the flights of subtle things, the miracles of quantities the connections of proportions, the things committed to nature.

And so △ F either is equal to (1) the lateral surface of cone DEA or to (2) the lateral surface of a cone situated on a base less than circle $ACNP$ but whose altitude is equal to that of cone DEA, namely, to perpendicular DE, or to (3) the lateral surface of a cone situated on a base larger than piramidis site in maiori basi quam sit *ACNP*, que piramis sit equalis altitudinis piramidi *DEA*, id est, cuius cathetus sit *DE*. Cum enim triangulum *F* alicui curve superficiei rotunde piramidis quivis bene sanus admittat, aliter quam diximus nec ratio veritatis indagatrix capit, nec humanus animus sibi consonus aliter esse posse animadvertit. Fere enim ex immediatis trimembris illa connectitur divisio.

Sit itaque, quod mentiatur falsigraphus, *F* non esse equalem curve superficiei rotunde piramidis *DAE*, sed minoris. Esto ergo *F* equalis curve superficiei rotunde piramidis site in minori basi quam *ACNP* circulus. Et basis illius piramidis sit circulus *HK* circa centrum *D*. Et sit cathetus illius piramidis *DE*.

Dispositio. Inscribatur circulo *ACNP* figura poligonia equalium laterum et angulorum et sit *ACNP* ita quod non contingat circulum *HK*. Deinde ab *E* dimittantur ypothenuse singule ad singulos angulos figure poligonie *ABCMNOPQ*. Dimittatur etiam ab *E* linea *ER* perpendicularis ad *AB*. Sit item *EH* ypothenusa dimissa ab *E* ad *H* ubi *DR* linea secat circulum *HK*. Est itaque *DEH* triangulus orthogonius qui intellectualiter circumvolvatur per circumferentiam *HK* donec redeat ad *H* unde ceperat ad perficiendam piramidem rotundam *DEH*, cuius piramidis curve superficiei sit secundum falsigraphum *F* triangulus equalis.

Dissolutio. *ER* est perpendicularis ad *AB*. Ergo quod fit ex ductu *ER* in *RB* est duplum trianguli *ERB* per [41] theorema primi Elemen-

27 base *H* / ACNP: ACNP circulus *HF* / piramis: pyramidis *H*
27–34 site....piramidis *om. G*
27–32 quam....divisio: et cetera *A*
28 cuius *om. HF supra scr. J*
30 admittit *H*
31 consonus: bene constitutus *F* / posse animadvertit: advertit *HF* / Fere *CHDF* Tunc *B*
32 illa *om. D*
33 itaque: ergo *A*
33–34 curve superficiei *tr. D*
34 rotunde *om. JHF* / DAE: DEA *AGD* / ergo: igitur *HF*
35 superficiei: superficiei alicuius *J* / rotunde piramidis *tr. JAG* / minore *AG* / quam *BHF* quam sit *ACIG*
36 circulus:[1] circulus sit *HF* / Et: et sit *HF* / illius: istius *H* alicuius *J* / sit *om. CHFD supra scr. J*
36–37 sit....piramidis *om. I*
38 ACNP *om. AG*
39 et sit ACNP *om. AG* / sit *om. D* / ita quod: in qua *HF*
39 contingant *D* / circulum: minorem circulum AG
40 *post* HK *add. H* per X II[ndi] (?) *et add. F* per *cum lacuna* / ab *E om. AG* / demittantur *HFGD mg. J* / angulos *om. D*
41 demittatur *ACHFGDJ* / linea *om. ADG*
42 demissa *CHGFDJ* / ad: in *AG*
45 redeat: veniat *HF* / piramidem rotundam *tr. HF*
46 DEH: HDE *AG* / curve superficiei *tr. AG*
49–50 in....ER *om. B*
49 [41] *supra scr. C manu recentiori* / theorema: 12 *A*

ACNP but whose altitude is equal to that of cone *DEA*, namely, to perpendicular *DE*. For, since anyone who is quite sound admits that △ *F* is equal to the surface of some cone, no searcher after truth can understand it in any way other than as we have stated it; nor can a sound human mind think that it can be otherwise. For this [logical] division arises almost immediately from the three terms.

And so let it be, as the pseudographer falsely says, that *F* is not equal to the lateral surface of cone *DAE* but [to the surface] of one that is less. Therefore, let *F* be equal to the lateral surface of a cone situated on a base less than circle *ACNP*. And let the base of this cone be circle *HK* with center *D*. And let the altitude of this cone be *DE*.

Disposition*: Let there be inscribed in circle *ACNP* a regular polygon, *ACNP*, in such a way that it does not touch circle *HK*. Then from *E* let slant heights be drawn to each of the angles of polygon *ABCMNOPQ*. Also, from *E* let *ER* be drawn perpendicular to *AB*. Also, let *EH* be the slant height drawn from *E* to *H* where line *DR* cuts circle *HK*. And so right △ *DEH*, which we mentally rotate through circumference *HK* until it returns to *H* from where it began, forms cone *DEH*, to whose lateral surface—according to the pseudographer—△ *F* is equal.

Dissolution**: *ER* is perpendicular to *AB*. Therefore, $ER \cdot RB = 2 \triangle ERB$, by I.41 of the *Elements*. Similarly, $ER \cdot RA = 2 \triangle ERA$.

* That is, "further construction"; see page 444 for the use of this term.

** That is, "the argument by reduction to absurdity"; see page 444.

torum. Similiter id quod fit ex ductu *ER* in *RA* est duplum trianguli *ERA*. Ergo quod fit ex ductu *ER* in *RA* et *RB* est duplum trianguli *EAB*. Ergo per primam secundi id quod fit ex ductu *ER* in *AB* duplum est trianguli *EAB*. Eadem ratione quod fit ex ductu *ER* in *CB* duplum est trianguli *ECB*. Eadem quoque ratione si bene numeraveris, invenies id quod fit ex ductu *ER* in *AB*, *BC*, *CM*, *MN*, *NO*, *OP*, *PQ*, *QA*, latera poligonii, esse duplum omnium triangulorum ypothenusalium, scilicet, *EAB*, *ECB*, *ECM*, et reliquorum ypothenusalium. Sed *R* angulus est rectus. Ergo *EA* maior est quam *ER*; et circumferentia *ACNP* maior est ambitu poligonii, scilicet, *AB*, *BC*, *CM*, *MN*, *NO*, *OP*, *PQ*, *QA*, lineis rectis. Ergo maius est id quod fit ex ductu *EA* in circumferentiam quam id quod fit ex ductu *ER* in ambitum poligonii. Sed id quod fit ex ductu *EA* in totam circumferentiam est duplum trianguli *F*, quia *ST* est equalis circumferentie *ACNP* et *EA* est equalis *SZ*. Et id quod fit ex *ER* in ambitum poligonii est duplum omnium triangulorum ypothenusalium pariter acceptorum. Ergo maius est duplum trianguli *F* / 111v quam duplum omnium triangulorum ypothenusalium pariter acceptorum. Ergo subduplum subduplo maius, scilicet triangulus *F* omnibus triangulis ypothenusalibus pariter sumptis. Sed trianguli ypothenusales pariter sumpti maiores sunt curva superficie *DEH* rotunde piramidis incluse. Ergo triangulus *F* est maior curva superficie rotunde piramidis *DEH*. Non ergo est ei equalis, contra hoc quod predixit falsigraphus. Relinquitur ergo quod non possit esse *F* equalis curve superficiei rotunde piramidis site in minori basi quam *ACNP* circulus, que piramis sit equalis altitudinis piramidis *DAE*, id est cathetus sit *DE*.

Sit item, secundum falsigraphum, *F* equalis curve superficiei rotunde piramidis site in maiori basi quam *ACNP* circulus, cuius pira-

51 RA: ER *A* / RB: RB coniunctim *A* / est duplum *tr.* *AC*
52–54 EAB.... trinaguli *om.* *C*
52–53 Ergo... EAB *om.* *A*
53 quod: id quod *A*
54 duplum est *tr.* *A* / quoque: que *A*
55 ductu *om.* *A*
56 latera poligonii *AC om.* *BI*
56, 57 ypothenussalium *C*
57 scilicet *om.* *A*
57 et... ypothenusalium: et cetera *A*
58 EA...ER: ER est minor EA *CD* / maior: minor *A*
60 NO...QA: et ceteris *A*
61 in[1]: in totam *A*
62 id *om.* *A*
64 id *om.* *A* / fit *om.* *A* / ambitum: habitum *I*
66–67 Ergo... acceptorum *om.* *A*
68 maius *om.* *A* / F *om.* *A*
69 sumptis: acceptis *A*
69–70 pariter sumpti *om.* *A*
72 contra: quod est contra *A* / predixit: dicit *A*
75 id est *BCDI* cuius *A* et *F*
76 item: tunc *A*
77 maiore *A*

Hence, $ER(RA + RB) = 2 \triangle EAB$. Therefore, by II.1 [of the *Elements*], $ER \cdot AB = 2 \triangle EAB$. By the same reasoning, $ER \cdot CB = 2 \triangle ECB$. By the same argument repeated a number of times, $ER \cdot (AB + BC + CM + MN + NO + OP + PQ + QA) = 2 \cdot (\triangle EAB + \triangle ECB + \triangle ECM + \triangle ENM + \triangle ENO + \triangle EPO + \triangle EPQ + \triangle EQA)$, that is, the product of ER and all of the sides of the polygon is equal to double all of the face triangles. But $\angle R$ is a right angle. Hence $EA > ER$, and circumference $ACNP$ is greater than the perimeter of the polygon, that is, the circumference is greater than the sum of the straight lines AB, BC, CM, MN, NO, OP, PQ, QA. Hence, ($EA \cdot$ circumference) $>$ ($ER \cdot$ perimeter of polygon). But ($EA \cdot$ circumference) $= 2 \triangle F$, because ST is equal to circumference $ACNP$ and $EA = SZ$. Furthermore ($ER \cdot$ perimeter of polygon) $= 2 \cdot$ (the sum of all the face triangles). Therefore, $2 \triangle F > 2 \cdot$ (the sum of all the face triangles). Therefore, half of the one is greater than half of the other, that is, $\triangle F >$ (the sum of all the face triangles). But the sum of all the face triangles is greater than the lateral surface of the included cone DEH. Therefore, $\triangle F$ is greater than the lateral surface of cone DEH. Therefore, it is not equal to it, and this is contradictory to that which the pseudographer stated earlier. It remains, therefore, that F could not be equal to the lateral surface of a cone situated on a base less than circle $ACNP$ but with an altitude equal to that of cone DAE, that is, with altitude DE.

Following the pseudographer once more, let F be equal to the lateral surface of a cone situated on a base greater than circle $ACNP$ but whose midis cathetus sit *DE*. Brevitatis tamen causa sit *DEH* piramis rotunda proposita, cuius ypothenusa *EH* sit equalis *SZ*. Et circumferentia *KH* sit equalis *ST*. Dicatque falsigraphus *F* non esse equalem curve superficiei piramidis *DEH*, sed alterius piramidis eiusdem altitudinis site in maiori basi. Et sit basis illius piramidis circulus *ACNP*. Piramis vero fundata super *ACNP* sit *DEA*, cuius scilicet *DEA* curve superficiei sit, secundum falsigraphum, *F* equalis.

Improbatio. In hoc triangulo *DEH*, *D* est rectus; maneat enim tota prior linearum dispositio. Ergo *H* est acutus. Ergo *EHR* est obtusus; ergo maior quam *R*. Ergo *ER* est maius *EH*; et omnia latera poligonii pariter sumpta sunt maiora quam circumferentia *HK*. Ergo maius est id quod fit ex *ER* in ambitum poligonii quam id quod fit ex *EH* in *HK*, scilicet, *SZ* in *ST*. Sed quod fit ex *SZ* in *ST* est duplum trianguli *F*. Et quod fit ex *ER* in ambitum poligonii duplum est ad omnes triangulos ypothenusales. Ergo maiores sunt omnes trianguli ypothenusales quam *F*, subduplum subduplo sicut et duplum duplo maius. Sed *F*, secundum falsigraphum, est equalis curve superficiei piramidis *DEA*. Ergo omnes trianguli ypothenusales sunt maiores curva superficie *DEA* rotunde piramidis. Superficies inclusa maior sit superficie includente, quod est impossibile. Relinquitur ergo quod curva superficies *DAE* non sit equalis *F*.

Cum ergo *F* non sit equalis curve superficiei alicuius rotunde piramidis site in maiori basi vel in minori quam in proposite piramidis basi, que sit eiusdem altitudinis proposite piramidi, relinquitur *F* esse equalem curve superficiei proposite piramidis, quod proposui. Hoc itaque theorema satis elegans, satis eleganter propositum, satis eleganti venustatur corollario.

[Corollarium:] Ex hoc manifestum quod proportio curve superficiei

79 EH sit *tr.* *A*
80 KH: HK *A* / phalsigraphus *I*
81 eiusdem altitudinis *om.* *A*
82 maiore base *A*
85 D: D angulus *A* / maneat *B* manet *AC*
88 pariter: insimul *A*
89 ex: ex ductu *A*
89–91 quam....poligonii *om.* *A*
91 duplum est *tr.* *A*
93 F: F triangulus *A* / et *om.* *A*
94 maius *om.* *A* / F: F triangulus *A*, *et tr.* *A post* falsigraphum
96 DEA....piramidis: piramidis DEA *A*
96–97 Superficies...superficie: Et sit superficies inclusa maior *A*
99 alicuius *om.* *A*
100 base *A*
100–101 in[3]...basi *A* piramis proposita *BC*
101 eiusdem: equalis *A*
102 proposui: proposuimus *A*
103 theorema *BI* theoreuma *AH* teorema *C* / satis[2]...satis[3] *om.* *A*
104 corellario *BI* corrollario *F* correlario (?) *D* corollarium *J*
105 quod: est quod *A*

altitude is DE. For the sake of brevity, however, let cone DEH be the proposed cone whose slant height EH is equal to SZ. And let circumference KH be equal to ST. And let the pseudographer say that F is not equal to the lateral surface of cone DEH, but [to that] of another cone of the same altitude but situated on a greater base. Let the base of that cone be circle $ACNP$, while the cone based on $ACNP$ we let be DEA. It is the lateral surface of this cone DEA that the pseudographer supposes F equals.

Refutation: In this $\triangle DEH$, $\angle D$ is a right angle, for the whole arrangement of lines remains as before. Therefore, $\angle H$ is an acute angle. Therefore, $\angle EHR$ is obtuse and thus greater than $\angle R$. Therefore, $ER > EH$ and the perimeter of the polygon is greater than circumference HK. Therefore, ($ER \cdot$ perimeter of polygon) $> (EH \cdot HK)$, and hence ($ER \cdot$ perimeter of polygon) $> (SZ \cdot ST)$. But $(SZ \cdot ST) = 2 \triangle F$, and ($ER \cdot$ perimeter of polygon) $= 2 \cdot$ (the sum of all the face triangles). Hence the sum of all the face triangles is greater than F, the halves being related as their doubles. But F according to the pseudographer is equal to the lateral surface of cone DEA. Therefore, the sum of all the face triangles is greater than the lateral surface of cone DEA, the included surface being greater than the including surface, which is impossible. It remains, therefore, that the lateral surface of DEA is not equal to F.

Since, therefore, F is not equal to the lateral surface of some cone of the same altitude but situated on a base greater than or less than the base of the proposed cone, it remains that F is equal to the lateral surface of the proposed cone, which is the proposition I have put forth. And so this theorem, sufficiently elegant and having been proposed with sufficient elegance, made beautiful by a sufficiently elegant corollary.

[Corollary:] From this it is evident that the ratio of the lateral surface rotunde piramidis ad suam basim est sicut ypothenuse sue ad semidiametrum basis sue. Archimenides enim in quadratura circuli ostendit circulum esse equalem triangulo orthogonio, cuius unum laterum rectum angulum continentium equatur circumferentie circuli, reliquum vero semidiametro. Cum itaque ex ductu circumferentie in semidiametrum fiat superficies dupla circuli et eiusdem circumferentie in ypothenusam fiat superficies dupla curve superficiei piramidis, erit proportio producti ad productum que producentis ad producens. Ergo que est proportio dupli curve superficiei piramidis ad duplum sue basis, eadem est ypothenuse ad semidiametrum. Sed duplorum et subduplorum eadem est proportio. Ergo proportio curve superficiei piramidis ad suam basim que ypothenuse ad semidiametrum basis.

II. CUIUSLIBET COLUMPNE ROTUNDE CURVA SUPERFICIES EQUALIS EST TETRAGONO QUI CONTINETUR SUB LINEIS EQUALIBUS AXI COLUMPNE ET CIRCUMFERENTIE BASIS.

Hoc eodem genere demonstrationis quo et precedens theorema astruitur. Sive enim dicatur quadrangulum equale curve superficiei maioris columpne eiusdem altitudinis, sive minoris columpne quam sit illa cuius axis et circumferentia basis sunt equalia lineis continentibus quadrangulum, probabitur contrarium.

Describatur enim intra basim maiorem poligonium equalium laterum minorem circulum non contingentium. Et erigantur a singulis angulis perpendiculares basi et equales axi, que omnes cadunt in curvam superficiem maioris columpne; alioquin due linee super punctum unum eidem linee orthogonaliter insistent, propter lineam que suo circuitu describit superficiem columpnalem Si igitur capita linearum perpendicularium rectis lineis coniungantur, fient quadranguli rectanguli interiorem columpnam minime contingentes, qui omnes pariter accepti erunt maiores curva superficie minoris columpne incluse et
112r minores mai/oris, et equales ei quod fit ex ductu perpendicularium

106 basim: piramidem *A*
107 Archimedes *F*
111 circuli *C* trianguli circuli *B* ad circulum *A*
112 curve...primidis *A* superficie piramidis *BI* pramidis superficie *C*
116 proportio: que est *A*
117 basem *A* / que: eadem est *A* / *post* basis *add.* *A* et e contrario

1 II *B om.* *C* IIª *J* 2ª *AD*
5 theoreuma *A*
8 sunt: sint *A*
10 basem *A*
14 eidem...insistent: orthogonaliter insistunt uni et eidem linee *A* / insisterunt (?) *C*
16–17 rectianguli *B*

of a cone to its base is the same as that of its slant height to the radius of its base. For Archimedes in the *Quadrature of the Circle* shows that a circle is equal to a right triangle one of whose sides containing the right angle is equal to the circumference of the circle while the other [is equal] to the radius. And so, since from the product of the circumference and radius arises a surface double that of the circle and from the product of the same circumference and the slant height arises a surface double that of the cone, the ratio of product to product will be as the ratio of multiplying factors to multiplying factors. Therefore, [after the elimination of common factors] the ratio of double the lateral surface of a cone to double its base is as the ratio of the slant height to the radius [of the base]. But the ratio of doubles is the same as that of their halves. Therefore, the ratio of the lateral surface of a cone to its base is as that of the slant height to the radius of the base.

II. THE LATERAL SURFACE OF ANY [RIGHT] CYLINDER IS EQUAL TO THE RECTANGLE CONTAINED BY LINES EQUAL [RESPECTIVELY] TO THE AXIS OF THE CYLINDER AND THE CIRCUMFERENCE OF THE BASE.

This is provided with the same kind of demonstration as the preceding theorem. For if it be said that the rectangle is equal to the lateral surface of a cylinder with the same axis but which is either on a greater or lesser base than that whose axis and base circumference are equal to the lines containing the rectangle, the contrary will be proved.

For within such a larger base let there be described a regular polygon whose sides do not touch the smaller circle [which is the base of the proposed cylinder]. And let there be erected from each of the angles [of the polygon] a line perpendicular to the base and equal to the axis, all of which lines fall in the lateral surface of the greater cylinder. Moreover, two [such] lines [opposite to one another] will be perpendicular to the same line [each] at one [terminal] point. When this line [with its perpendiculars] is rotated, it will generate a cylindrical surface. If, therefore, the terminal points [of each adjacent pair] of perpendicular lines are joined by straight lines, there will be produced [a set of] rectangles which do not touch the interior cylinder at all. All of these rectangles taken together will be greater than the lateral surface of the lesser, included cylinder and will be less than [the surface of] the greater [cylinder], and they will be equal to the product of [one of] the perpendiculars, or the axis, and the sive axis in totum ambitum poligonii, qui maior est circumferentia basis minoris et minor maioris. Observanti ergo ordinem precedentis demonstrationis facile patebit propositum.

[Corollarium:] Ex hoc igitur manifestum quod proportio curve superficiei columpne ad curvam ⟨superficiem⟩ sue ⟨rotunde⟩ piramidis est tanquam proportio axis columpne ad medietatem ypothenuse piramidis. Erit etiam proportio curve superficiei columpne ad suam basim sicut axis columpne ad quartam partem diametri basis.

Quia ex ductu circumferentie circuli in medietatem ypothenuse fit curva superficies piramidis per proximam et ex ductu eiusdem circumferentie in axem columpne fit curva superficies columpne per istam, ergo cum productorum et producentium eadem sit proportio, erit proportio curve superficiei columpne ad curvam piramidis tanquam axis columpne ad medietatem ypothenuse piramidis.

Amplius, ex ductu circumferentie circuli in semidiametri medietatem provenit area circuli, et ex ductu eiusdem circumferentie in axem columpne provenit curva superficies columpne; ergo cum productorum et producentium eadem sit proportio, erit proportio curve superficiei columpne ad suam basim que est axis columpne ad quartam partem diametri basis. Et sic duplex patet corollarium suo iunctum theoremati.

III. QUORUMLIBET DUORUM CIRCULORUM CIRCUMFERENTIE SUIS DIAMETRIS SUNT PROPORTIONALES.

Describantur duo circuli *ABC*, *EFG* circa centra *D*, *H* [Fig. 65]. Dico quod proportio diametri *CDA* ad diametrum *GHE* que circumferentie *ABC* ad circumferentiam *EFG*. Si enim ita non fuerit, erit proportio *CA* diametri ad *GE* diametrum que circumferentie *ABC* ad minorem vel maiorem circumferentiam quam *EFG*.

Et sit primo ad minorem, scilicet circumferentiam *IK*. Describatur ergo intra circulum *EFG* poligonium equalium laterum et angulorum

20 sive: sue *A*
21 minoris: maioris minoris *A*
23 igitur: etiam *A* / quod: est quod *A*
26 basem *A*
28 Quia: quoniam *A*
31 *ante* proportio *del.* *A* productio
33 piramidis *om.* *B*
34 circuli: basis *A* / semidiametri medietatem: medietatem sui semidiametri *A*
39–40 Et...theoremati: Sic ergo patet duplex corollarium premissi theoreumatis *A*
39 duplex: dupliciter*B* / corellarium *B*
40 teorematì *C*
1 III *B om.* *C* III^a^ *J* 3^a^ *AD* / circulorum circumferentie *tr.* *A*
4 quod *A* quoniam *BC* / diametri CDA *tr.* *A* / que: est que *A*
5 ita *AB supra scr.* *C*
7 minorem vel maiorem: maiorem vel minorem *A* / EFG: sit EFG *A*
8 scilicet *tr.* *A ante* IK

perimeter of the polygon which is [itself] greater than the circumference of the lesser base and less than that of the greater. Therefore, that which was proposed will be easily evident to one observing the order of the preceding demonstration.

[Corollary:] From this it is evident, therefore, that the ratio of the lateral surface of a [right] cylinder to the lateral surface of its cone [constructed on the same base] is as the ratio of the axis of the cylinder to one half the slant height of the cone. Furthermore, the ratio of the lateral surface of the cylinder to its base will be as that of the axis of the cylinder to one foutrh the diameter of the base.

Since from the product of the circumference of the [base] circle and one half the slant height arises the lateral surface of a cone, by the preceding [theorem], and from the product of the same circumference and the axis of the cylinder arises the lateral surface of the cylinder, by this [theorem], therefore, the products and the factors producing the products being of the same ratio, the ratio of the lateral surface of the cylinder to the lateral surface of the cone will be as that of the axis of the cylinder to one half the slant height of the cone.

Further, from the product of the circumference of the [base] circle and one half of its radius arises the area of the circle, and from the product of the same circumference and the axis of the cylinder arises the lateral surface of the cylinder. Therefore, since the ratio of the products is the same as that of the factors producing the products, the ratio of the lateral surface of the cylinder to its base will be as that of the axis of the cylinder to one fourth the diameter of its base. And thus the two-part corollary joined to its theorem is evident.

III. THE CIRCUMFERENCES OF ANY TWO CIRCLES ARE PROPORTIONAL TO THEIR DIAMETERS.

Let the two circles *ABC*, *EFG* be described about centers *D* and *H* [see Fig. 65]. I say that the ratio of diameter *CDA* to diameter *GHE* is as that of circumference *ABC* to circumference *EFG*. For if it were not thus, the ratio of diameter *CA* to diameter *GE* will be as the ratio of circumference *ABC* to a circumference which is either greater than or less than *EFG*.

Now at first let [the ratio of diameters] be as that [of *ABC*] to a lesser circumference, namely, *IK*. Therefore, let there be described within circle *EFG* a regular polygon whose sides do not touch circle *IK* at all. And minime contingens *IK* circulum. Et describatur simile poligonium intra circulum *ABC* et sit *IK* circa centrum *H*. Ratio. Age, que est proportio *CA* diametri ad *GE* eadem est *ABC* circumferentie ad *IK* circumferentiam ex ypothesi falsigraphi; et que est *CA* ad *EG* eadem est ambitus poligonii ad ambitum poligonii, per primam duodecimi. Ergo que est *ABC* circumferentie ad *IK* circumferentiam eadem est ambitus poligonii ad ambitum poligonii. Ergo permutatim que est proportio *ABC* circumferentie ad ambitum poligonii *ABC* eadem est circumferentie *IK* ad ambitum poligonii *EFG*. Sed *ABC* est maior ambitu sui poligonii. Ergo *IK* circumferentia maior est ambitu poligonii *EFG*, quod est falsum. Relinquitur ergo quod non sit proportio *ABC* circumferentie ad minorem circumferentiam quam *EFG* que diametri *CA* ad diametrum *EG*.

Sit itaque proportio *ABC* ad maiorem circumferentiam quam *EFG* que diametri ad diametrum; sitque *MS*. Ergo e contrario proportio *EG* diametri ad *AC* que *MS* ad *ABC*. Est itaque proportio *EG* ad *AC* que *EFG* circumferentie ad aliquam circumferentiam; sitque illa *H*. Que est proportio *EG* ad *AC* eadem est tam *MS* circumferentie ad *ABC* circumferentiam quam *EFG* circumferentie ad *H*. Ergo que est proportio *MS* ad *ABC* eadem est *EFG* ad *H*. Ergo permutatim que est *MS* ad *EFG* eadem est *ABC* ad *H*. Sed *MS* est maior *EFG*. Ergo *ABC* est maior *H*. Est itaque proportio *EG* diametri ad *AC*

11 Ratio *AC om. B*
12 CA: EA *A*
14–16 Ergo...poligonii *om. A*
17 ad *A* in *BC* / poligonii ABC: sui poligonii *A*
18 ABC: EFG *AC*
19 *post* poligonii *add. A* vel ABC
20 est falsum *tr. A* / ergo *om. C*
21 que: que est *A*
23 itaque: ergo *A*
27 *post* H *add. A* ratio
28 circumferentie *om. A*
29 est proportio *om. B* / proportio *om. C, sed supra scr. C in manu recentiori*

let there be described a similar polygon within circle *ABC*; let the center of *IK* be *H*. Proceed with the demonstration as follows: the ratio of diameter *CA* to diameter *GE* is as that of circumference *ABC* to circumference *IK*, by the hypothesis of the pseudographer, and *CA* is to *EG* as the perimeter of [one] polygon is to the perimeter of the [other] polygon, by XII.1 [of the *Elements*]. Therefore, circumference *ABC* is to circumference *IK* as the perimeter of the one polygon [within *ABC*] is to the perimeter of the polygon [within *EFG*]. Therefore, permutatively, circumference *ABC* is to the perimeter of the polygon within *ABC* as circumference *IK* is to the perimeter of the polygon within *EFG*. But [circumference] *ABC* is greater than the perimeter of its [inscribed] polygon. Therefore, circumference *IK* is greater than the perimeter of

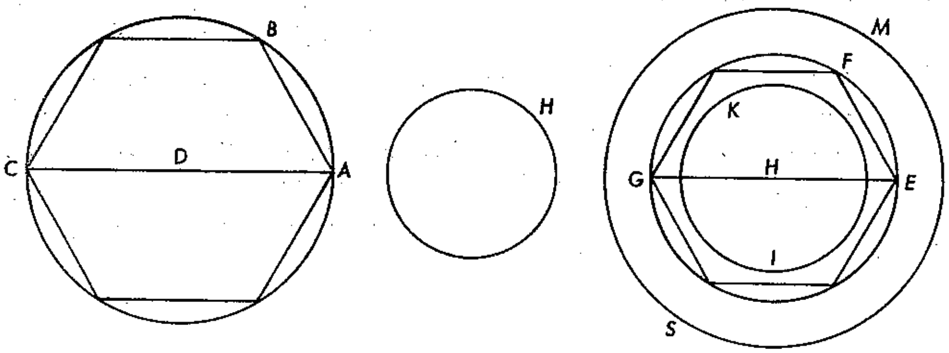

Fig. 65

polygon *EFG*, which is false. It remains, therefore, that the ratio of circumference *ABC* to a circumference less than *EFG* is not as that of diameter *CA* to diameter *EG*.

And so let the ratio of *ABC* to a circumference greater than *EFG* be as diameter to diameter. And this [greater circumference] we let be *MS*. Therefore, by inversion the ratio of diameter *EG* to [diameter] *AC* is as that of [circumference] *MS* to [circumference] *ABC*. And so the ratio of *EG* to *AC* is that of circumference *EFG* to some circumference, and that [latter] circumference we let be *H*. [Thus] the ratio of *EG* to *AC* is as the ratio of circumference *MS* to circumference *ABC* and as the ratio of circumference *EFG* to [circumference] *H*. Therefore, [by the equality of ratios] the ratio of *MS* to *ABC* is as that of *EFG* to *H*. Therefore, by the alternation [of ratios] *MS* is to *EFG* as *ABC* is to *H*. But [circumference] *MS* is greater than [circumference] *EFG*. Therefore, [circumference] *ABC* is greater than [circumference] *H*. And so the ratio of diameter *EG* que est *EFG* ad circumferentiam *H* minorem *ABC*, quod prius improbatum est. Relinquitur ergo propositum inconcussum.

Idem aliter posset demonstrari per tertiam Archimenidis de mensura circuli. Sed non est adeo sufficienter.

[Corollarium.] Ex hoc liquet theoremate quod id quod fit ex ductu cuiuslibet circumferentie in diametrum alterius circuli est equale ei quod fit ex ductu secunde circumferentie in diametrum alterius circuli.

IV. QUARUMLIBET DUARUM PIRAMIDUM ROTUNDARUM INEQUALIUM ET SIMILIUM CURVE SUPERFICIES HABENT DIFFERENTIAM EQUALEM EI QUOD FIT EX DUCTU DIFFERENTIE YPOTHENUSARUM IN DIMIDIAS CIRCUMFERENTIAS SUARUM BASIUM.

Esto exemplum *OEC* piramis rotunda [Fig. 66], cuius basis circulus *EQB*, cathetus *OC*, ypothenusa *CE*, curva superficies *R*, et minor piramis *DIM* similis piramidi *OEC*, cuius basis *ANM*, cathetus *DI*, ypothenusa *IM*, curva superficies *L*. Dico ergo quod differentia aug-

32 EFG: EFG circumferentie *A*
33 inconcussum *om. A*
35 non *om. A* / est *om. C*
36 [corollarium] *mg. C*
36–38 Ex...circuli *om. F*
38 circuli: circuli scilicet prioris *A*
1 IV: IIII *B om. C* IIII[a] *J* 4[a] *AD*
2 curve *om. A*
7 EQB: ECB *A* / OC: OE *A*

to [diameter] *AC* is as that of [circumference] *EFG* to circumference *H* where *H* is less than [circumference] *ABC*, which [proportionality] was refuted earlier. Therefore, that which was proposed remains firm.

The same thing can be demonstrated in another way by [using Proposition] III of Archimedes' *On the Measurement of the Circle*, but not so adequately.

[Corollary:] From this theorem it is clear that the surface arising from the product of any circumference and the diameter of another circle is equal to that which arises from the product of the second circumference and the diameter of the first circle.

IV. THE DIFFERENCE BETWEEN THE LATERAL SURFACES OF ANY TWO UNEQUAL BUT SIMILAR CONES IS EQUAL TO THAT WHICH ARISES FROM THE MULTIPLICATION OF THE DIFFERENCE BETWEEN THE SLANT HEIGHTS BY HALF [THE SUM OF] THE CIRCUMFERENCES OF THEIR BASES.

For example, let *OEC* be a cone [see Fig. 66], whose base circle is *EQB*, altitude is *OC*, slant height is *CE*, and whose lateral surface is *R*. And [let there also be] a lesser cone *DIM* similar to cone *OEC*. Its base we let be *ANM*, its altitude *DI*, its slant height *IM*, and its lateral surface

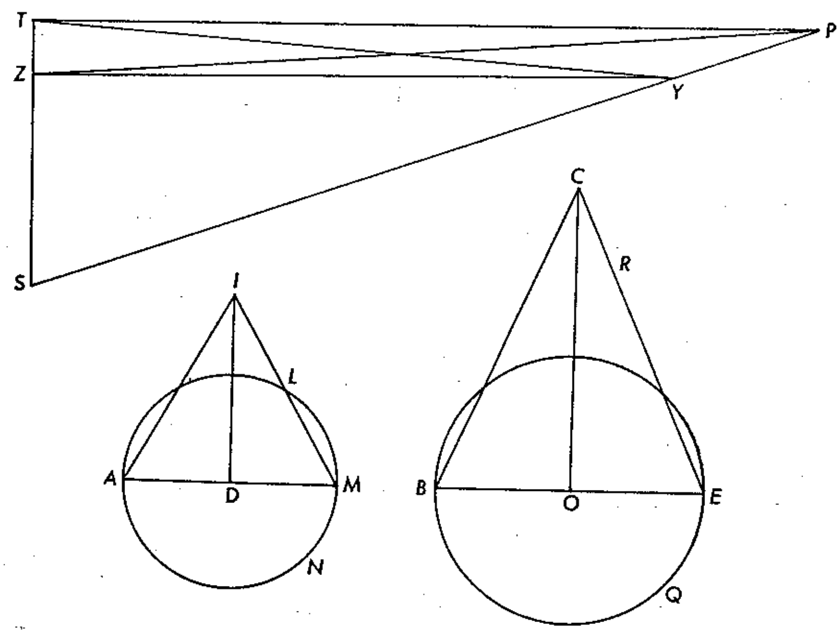

Fig. 66

Note: I have added the prime sign to *O′* here and in the text.

menti *R* superficiei curve ad superficiem *L* est id quod fit ex ductu
112v augmenti *CE* ypothenuse super *IM* in medietates / circumferentiarum, *QBE*, *ANM*.

Sit enim trigonus orthogonius *STP*, cuius latus *ST* sit equale *CE* et *TP* sit equalis circumferentie *EQB*. Est enim trigonus *STP* equalis *R*, per primam huius. Resecetur, ab *ST*, *ZS* equalis *IM*, *ZY* equidistante demissa a *Z*.

Ratio. Age. Tam *Z* quam *T* est rectus, et *S* est communis, et *Y*, *P* sunt equales propter *ZY*, *PT* equidistantes. Ergo trigonus *STP* est similis trigono *SZY*. Ergo proportio *ST* ad *TP* que *SZ* ad *ZY*. Item que est proportio diametri *EB* ad diametrum *MA* eadem est circumferentie *EQB* ad circumferentiam *ANM*, per proximam. Et que est *EB* ad *AM* eadem est *CE* ypothenuse ad *IM* ypothenusam, per diffinitonem similium piramidum. Ergo proportio *CE* ad *IM* que *EQB* ad *ANM*. Sed *ST*, *CE*; et *EQB*, *TP*; et *IM*, *SZ* sunt equales. Ergo proportio *ST* ad *SZ* que *TP* ad *ANM*. Sed que est *ST* ad *SZ* eadem est *TP* ad *ZY*, propter *STP*, *SYZ* triangulos similes. Ergo *ANM*, *ZY* sunt equales, cum *TP* ad illa eadem sit proportio. Sit ergo trigonus *SYZ* equalis *L*, per primam huius. Patet ergo quod differentia *R* ad *L* est superficies *ZYPT*. Ducatur ergo linea *ZP* a *Z* in *P* et *YT* a *Y* in *T*.

Ratio. *ZTY*, *YPZ* sunt trianguli super *ZY* basim inter *ZY*, *TP* equidistantes. Ergo sunt equales. Ergo id quod est ex *ZT* in *ZY* est duplum ad utrumque, et sic ad *ZPY*. Ergo id quod fit ex *TZ* in medietatem *ZY* est equale *ZPY*. Similiter id quod fit ex *TZ* in *TP* est duplum *TPZ*. Ergo quod fit ex *ZT* in medietatem *TP* est equale *TPZ*. Et sic *ZTPY* superficies, que est differentia *R* ad *L*, fit ex ductu differentie *CE* ad *IM*, que est linea *ZT*, in medietatem circumferentie *EQB* et medietatem *ANM*, mediantibus *TP*, *ZY* quod proposuimus.

V. SI IN CIRCULO DESCRIPTI POLIGONII EQUILATERI ET EQUIANGULI MEDIETAS AD TERMINOS DIAMETRI TERMINATA, DIAMETRO STANTI, CIRCUMDUCATUR, E-

12 QBE: scilicet ABC *A*
14 EQB: EOB *A*
16 Z: T *A*
19 que: que est *A* / ZY: ZP *A*
24 TP: SP *A* / TP *corr. C ex* SP / IM, SZ: IMS, ZY *A* / equales: equalia *A* equa *C*
26 SYZ: SZY *A* / triangulos similes *tr. A*
27 Sit: Est *AC* / SYZ: STY *A*
28 primam: proximam *A*
29 ZYPT *B* YZTP *A* ZYTP *C* ((with signs to reverse *T* and *P* in C))
30 sunt *om. C* / basem *A* / TP: et TP lineas *A*
31 id quod *AC* quod *B* / ZT in ZY *B* TZ in TY *A* TZ in ZY *C*
32 id *om. C*
34 TPZ[1] *A* ZTP *BC* / TPZ[2]: ZTP *A*
1 V *B om. C* V[a] *J* 5[a] *AD*
3 stanti: circumstanti *A*

L. I say, therefore, that

(latsurf R — latsurf L) = (CE — IM)·(1/2 circum QBE + 1/2 circum ANM). For let there be a right △ STP, whose side ST is equal to CE and TP is equal to circumference EQB, for △ $STP = R$, by the first [proposition] of this [work]. Let line ZS, equal to line IM, be cut from ST, with line ZY drawn from Z parallel [to TP].

Now proceed with the proof: Both ∠ Z and ∠ T are right angles, ∠ S is common, and ∠ Y = ∠ P since ZY and PT are parallel. Therefore, △ STP is similar to △ SZY. Therefore, $ST/TP = SZ/ZY$. Also, diameter EB is to diameter MA as circumference EQB is to circumference ANM, by the preceding [proposition]. And EB is to AM as the slant height CE is to the slant height IM, by the definition of similar cones. Therefore, $CE/IM = EQB/ANM$. But $ST = CE$, $EQB = TP$, and $IM = SZ$. Therefore, $ST/SZ = TP/ANM$. But $ST/SZ = TP/ZY$, since △ STP and △ SYZ are similar. Therefore, $ANM = ZY$, since TP has the same ratio to [each of] them. Therefore, let △ $SYZ = L$, by the first [proposition] of this [work]. It is evident, therefore, that $(R - L)$ = surface $ZYPT$. Then let line ZP be drawn from Z to P and YT from Y to T.

[Further] argument. ZYT and YPZ are triangles [constructed] on base ZY and between parallel lines ZY and TP. Therefore the triangles are equal. Therefore $(ZT \cdot ZY) = 2 \cdot$ (each triangle) = 2 △ ZPY. Therefore $(TZ \cdot 1/2\ ZY)$ = △ ZPY. Similarly $(TZ \cdot TP)$ = 2 △ TPZ. Therefore $(ZT \cdot \frac{1}{2} TP)$ = △ TPZ. And thus

surf $ZTPY = (R - L) = (CE - IM) \cdot (\frac{1}{2}$ circum $EQB + \frac{1}{2}$ circum $ANM)$, since $ZTPY = ZT \cdot (\frac{1}{2}ZY + \frac{1}{2}TP)$, and $(CE - IM) = ZT$. This is what we proposed.

V. IF HALF OF A REGULAR POLYGON—TERMINATED AT THE ENDS OF A DIAMETER AND DESCRIBED IN A CIRCLE—IS ROTATED WHILE THE DIAMETER REMAINS FIXED, THE

RUNT CONICE SUPERFICIES TOTIUS CORPORIS EQUALES EI QUOD FIT EX DUCTU LATERIS CIRCUMDUCTI IN OMNES CIRCUMFERENTIAS DESCRIPTAS AB ANGULIS POLIGONII, SIVE EI QUOD FIT EX DUCTU CIRCUMFERENTIE CIRCULI CONTINGENTIS POLIGONIUM IN LINEAM QUE CUM DIAMETRO EIUSDEM CIRCULI ET LATERE POLIGONII IN EODEM CIRCULO CONSTITUIT TRIANGULUM ORTHOGONIUM.

Esto exemplum circulus *AHID* et poligonium equilaterum ei inscriptum [Fig. 67].

Dispostio. A singulis angulis poligonii ad oppositos ducantur linee, et sint *NM*, *FG*, *HD*, *LE*, *BC*, secantes diametrum *AI* perpendiculariter. Ne tamen minus diligens lector scrupulum, quo progrediens pedem offendat, possit reperire, sic probetur illas perpendiculariter secare *AI*.

Ratio. Age. *ADC*, *AHB* arcus sunt equales. Ergo *CIY*, *BIY* anguli cadentes in illos sunt equales. Item *YBI*, *YCI* in *BI*, *CI* arcus equales sunt equales. Ergo cum in *BIY* triangulo, *B*, *I*, *Y* anguli sunt equales duobus rectis, similiter et in triangulo *YIC*, *Y*, *I*, *C* anguli sunt equales duobus rectis, gemini *Y* sunt equales, et sic uterque rectus. Ergo *BC* est perpendicularis ad *AI*. Simili ratione *NM* perpendicularis ad *AI*.

12 ei: illi *A*
14 angulis *om.* *A*
15 NM: MN *A*
19 AHB *om.* *C*

CONICAL SURFACES OF THE WHOLE BODY [FORMED] WILL BE EQUAL [IN SUM] TO THAT WHICH ARISES FROM THE PRODUCT OF A SIDE OF THE ROTATED [POLYGON] AND THE SUM OF THE CIRCUMFERENCES DESCRIBED BY THE ANGLES OF THE POLYGON, OR [IT IS EQUAL] TO THAT WHICH ARISES FROM THE PRODUCT OF THE CIRCUMFERENCE OF THE CIRCLE TOUCHING THE POLYGON AND THE LINE WHICH FORMS A RIGHT TRIANGLE ALONG WITH THE DIAMETER OF THE SAME CIRCLE AND A SIDE OF THE POLYGON IN THE SAME CIRCLE.

For example, let there be circle *AHID* and a regular polygon inscribed in it [see Fig. 67].

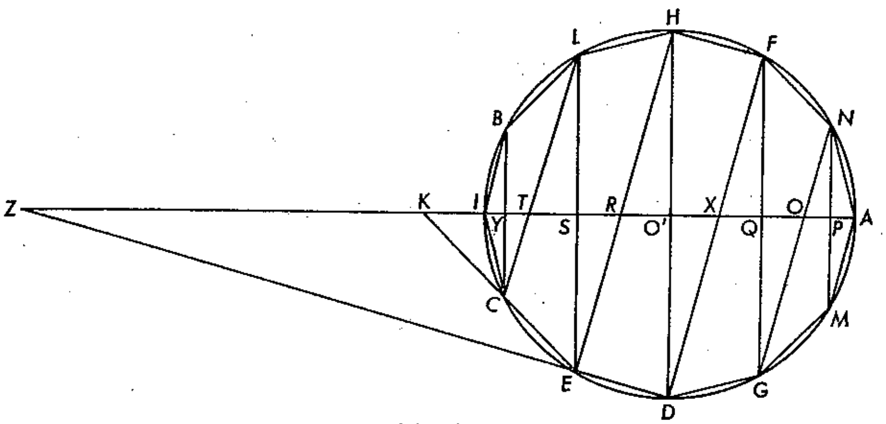

Fig. 67

Disposition: From the individual angles of the polygon let lines be drawn to the opposite angles, namely, lines *NM*, *FG*, *HD*, *LE*, and *BC*, all these lines cutting diameter *AI* perpendicularly. Lest, however, the less attentive reader be able to find an obstacle against which he strikes his foot as he goes forth, let it be proved as follows that these lines cut *AI* perpendicularly.

Proceed with the proof: Arcs *ADC* and *AHB* are equal. Therefore, angles *CIY* and *BIY* falling in those arcs are equal. Also, [angles] *YBI* and *YCI* in equal arcs *BI* and *CI* are equal. Therefore, since angles *B*, *I*, *Y* in △ *BIY* are equal to two right angles and similarly angles *Y*, *I*, *C* in △ *YIC* are equal to two right angles, the pair of angles at *Y* are equal, and thus each is equal to a right angle. Therefore, *BC* is perpendicular to *AI*. By a similar argument *NM* is perpendicular to *AI*. Also, *LC* intersects

Item *LC* secat *LE*, *BC*, et facit *L*, *C* angulos cadentes in *LB*, *CE* equos arcus equales et coalternos. Ergo *LE*, *BC* sunt equidistantes, et eadem ratione, *LE*, *HD*, et *FG*, et *NM*.

Protrahuntur deinde oblique secantes diametrum *AI* linee *NG*, *FD*, *HE*, *LC*. Protrahatur deinde *AI* in occursum *EC* et *ED* extra circulum, singulis angulis et punctis rationi necessariis per equalia notatis. Erunt ergo *NM*, *FG*, *HD*, *LE*, *BC* diametri circulorum quos describunt anguli semipoligonii *ADI* circumducti. Dico itaque quod conica superficies corporis poligonii fit ex ductu unius lateris poligonii, scilicet *DE*, in omnes circumferentias [circulorum] quorum diametri sunt *MN*, *FG*, *HD*, *LE*, *BC*; et etiam est equalis ei quod fit ex *HE* in *ADIH* circumferentiam.

Rationis causa, superficiem piramidis *YIC* describit ypothenusa *IC*. Et est axis illius piramidis *IY*, diameter basis linea *BC*. Ergo per primam huius superficies curva piramidis *YIC*, quam describit *IC* circumvoluta, est superficies que fit /
113r
ex ductu *IC* in medietatem circumferentie *BC* diametri. Item in *SEK*, *YCK* triangulis, et *Y* et *S* est rectus et *K* communis. Ergo *SEK*, *YCK* trianguli sunt similes. Item superficies curva quam continent *LB*, *CE* et circumferentie diametrorum *LE*, *BC*, scilicet quam describit linea *CE* circumvoluta, est differentia duarum piramidum inequalium et similium, quarum minoris axis est *YK*, ypothenusa *KC*, diameter basis *BC*; maioris piramidis axis *KS*, ypothenusa *KE*, diameter basis *LE*. Ergo per proximam illa superficies, quam describit *CE* circumvoluta, fit ex ductu *CE* in circumferentie *BC* medietatem et medietatem circumferentie *LE*. Ergo curva superficies piramidis *YIC*, quam describit *IC*, et curva superficies quam describit *CE*, pariter sumpte, fiunt ex ductu *CE* in circumferentiam *BC* et medietatem circumferentie *LE*, quoniam *IC*, *CE* latera poligonii sunt equalia. Item curva superficies quam describit *DE* circumvoluta est differentia piramidum similium et inequalium *O'DZ*, *SEZ*. Ergo illa superficies fit ex ductu *ED* in medietatem circumferen-

25 LE *AC* IC (?) *B* / LB, CE: LC LB *A*
26 equos...equales: arcus equales equos *A*
29 Protrahantur *B*
31 ergo: igitur *A*
32 itaque *om. A*
35 ei: illi *A*
36 ADIH: AHI *A*
39 superficies curva *tr. A*
41 triangulis *A om. BC*
42 K: K est *A*
42–43 superficies curva *tr. A*
43 circumferentie diametrorum *A* circumferentia diametrum *B* circumferenti[a] diametrorum *C*
46 piramidis *A om. BC*
47 LE: LEI *C*, *sed supra scr. C* LE
48–51 circumvoluta...CE[1] *AB mg. C*
52 LE *om. A* / *post* LE *del. C* et medie[ta]tem circumferentie DH
55 ED: DE *C*

LE and *BC* and makes equal alternate angles *L* and *C* falling in equal arcs *LB* and *CE*. Therefore, *LE* and *BC* are parallel, and by the same argument *LE*, *HD*, *FG*, and *NM* are parallel.

Then let the oblique lines *NG*, *FD*, *HE*, and *LC*, all cutting the diameter *AI*, be drawn. Then let *AI* be extended to meet *EC* and *ED* outside of the circle, with the separate angles and points necessary for the argument marked off by equal lines. Therefore, *NM*, *FG*, *HD*, *LE* and *BC* will be the diameters of the circles which the angles of the half polygon *ADI* describe in rotation. And so I say that the conical surface of the polygonal body arises from the product of one side of the polygon, namely, *DE*, and the sum of the circumferences [of the circles] whose diameters are *MN*, *FG*, *HD*, *LE*, and *BC*; and also that it is equal to the product of *HE* and circumference *ADIH*.

For the sake of the argument, [let us say that] hypotenuse *IC* describes the surface of cone *YIC*. And the axis of this cone is *IY*; the diameter of its base is line *BC*. Therefore, by the first [proposition] of this [work], the lateral surface of cone *YIC*, which *IC* describes in rotation, is the surface arising from the product of *IC* and one half the circumference of which *BC* is the diameter. Also in triangles *SEK*, and *YCK*, *Y* and *S* are right angles and $\angle K$ is common. Therefore, triangles *SEK* and *YCK* are similar. Further, the lateral surface contained by *LB*, *CE* and the circumferences of diameters *LE* and *BC*, i.e., the lateral surface which line *CE* describes in its revolution, is [equal to] the difference between the two unequal but similar cones, the smaller of which is that one with axis *YK*, slant height *KC*, and base diameter *BC*, and the larger of which is that one with axis *KS*, slant height *KE*, and base diameter *LE*. Therefore, by the preceding [proposition], that surface which *CE* describes in its revolution is equal to $CE \cdot (1/2 \text{ circum } BC + 1/2 \text{ circum } LE)$. Therefore, the lateral surface *YIC* described by *IC* and the lateral surface described by *CE* taken together are equal to the product of (1) *CE* and (2) the sum of circumference *BC* and one half the circumference *LE*, since *IC* and *CE*, as sides of the [regular] polygon, are equal. Further, the lateral surface described by *DE* in revolution is [equal to] the difference of the similar and unequal cones *O'DZ* and *SEZ*. Therefore, that surface is equal to the product of (1) *ED* and (2) the sum of 1/2 circumference tie *LE* et medietatem circumferentie *DH*. Et sic superficies curva quam describit *IC* et quam describit *CE* et quam describit *DE* sunt equales ei quod fit ex ductu *DE* in circumferentias *BC* et *LE* et medietatem circumferentie *HD*. Simili ratione invenies curvam superficiem quam describit *AM* et quam describit *MG* et quam describit *GD*, scilicet medietatem totius conice superficiei poligonii corporis, esse id quod fit ex *GD*, sive *DE* illi equali, in circumferentiam *NM* et *FG* et medietatem circumferentie *HD*. Ergo si sufficienter enumeres, reperies totam conicam superficiem fieri ex uno latere poligonii, scilicet *DE* in omnes circumferentias quas describunt anguli poligonii, quarum diametri sunt *NM*, *FG*, *HD*, *LE*, et *BC*. Et sic prior pars constat propositi.

Rusus, *Y* undique est rectus, et *C*, *B* anguli cadentes in *IB*, *IC* equos arcus sunt equales. Ergo et *T*, *I* anguli sunt equales. Ergo *BYI*, *TYC* sunt trianguli similes. Simili quoque ratione *TYC*, *TSL*, *SRE*, *RO'H*, *O'XD*, *XQF*, *QOG*, *OPN*, *PAM* sunt similes. Ergo que est proportio *IY* ad *BY* eadem est *TY*, *YC*; et *ST*, *SL*; et *RS*, *SE*; et *RO'*, *O'H*; et *XO'*, *O'D*; et *XQ*, *QF*; et *OQ*, *QG*; et *OP*, *PN*; et *AP*, *PM*. Ergo que est proportio *IY* ad *BY* eadem est totius *AI* ad omnes rectas lineas, *BC*, *LE*, *HD*, *FG*, *NM*. Sed que est *AI* ad *NM* et *FG* et *HD* et *LE* et *BC*, eadem est *ADIH* circumferentie ad omnes circumferentias quas describunt anguli poligonii, quarum diametri sunt *NM*, *FG*, *HD*, *LE*, et *BC*, per tertiam huius et penultimam quinti. Ergo que est proportio *IY* ad *BY* eadem est *ADIH* ad quinque circumferentias diametrorum *NM*, *FG*, *HD*, *LE*, *BC* [,quas describunt anguli poligonii. Sed *HED* angulus est rectus cadens in semicirculum; ergo est equalis *Y*. Et *H*, *B* sunt equales, ut qui cadunt in *DE*, *IC* arcus equales. Et sic *BYI*, *DEH* trianguli sunt similes. Ergo que est proportio *IY* ad *BY* eadem est *DE* ad *EH*. Ergo que est *DE* ad *EH* eadem est *ADIH* circumferentie ad quinque circumferentias]. Ergo quod fit ex ductu primi in ultimum, scilicet *ED* in

56 *post* LE *del.* *C* et medietatem circumferentie LE
62 NM: AM *AC*
64 reperies: invenies *A*
65 scilicet...poligonii *om.* *A*
67 constat *tr.* *A* *ante* prior *in linea 66*
68 IB: LB *C*
69 equos arcus *tr.* *A*
69 anguli *A* *om.* *BC*
70 TYC: TIC *A* / trianguli *A* *om.* *BC* / quoque: que *A* / TSL: TIL *A*
71 *RO'H*: HRO *A* / OPN: ONP *A*
72 IY: TY *A*
73 O'H: CH *A* / OQ: CQ *A*
78 et[1] *om.* *A*
79 IY: QY *C* / ADIH: totius ADIH *A*
80–86 [quas....cricumferentias] *supplev* *sed cf. mg.* *B* *ubi scr. post* IY ad BY *in linea 79* DE ad EH cum triangul DEH, BIY sunt similes

LE and 1/2 circumference DH. And thus the lateral surfaces described by IC, CE, and DE are [together] equal to the product of (1) DE and (2) the sum of circumference BC, circumference LE, and one half the circumference HD. By a similar argument you will find that the lateral surfaces described by AM, MG, and CD, i.e., 1/2 the whole conical surface of the polygonal body, is equal to the product of (1) GD, or its equal DE, and (2) the sum of circumference NM, circumference FG, and 1/2 circumference HD. Therefore, if you make the complete enumeration, you will find that the whole conical surface arises from the product of (1) one side of the polygon and (2) the sum of the circumferences described by the angles of the polygon, i.e., it is equal to $DE \cdot$ (circum of NM + circum of FG + circum of HD + circum of LE + circum of BC). And so the first part of the proposition is evident.

Again, every angle at Y is a right angle and angles C and B falling in equal arcs IB and IC are equal. Therefore, angles T and I are equal. Therefore, $\triangle BYI$ is similar to $\triangle TYC$. By a similar argument, triangles TYC, TSL, SRE, $RO'H$, $O'XD$, XQF, QOG, OPN, and PAM are similar. Therefore, $IY/BY = TY/YC = ST/SL = RS/SE = RO'/O'H = XO'/O'D = XQ/QF = OQ/QG = OP/PN = AP/PM$. Therefore, $IY/BY = AI/(BC + LE + HD + FG + NM)$ [by Euclid V.1]. But $AI/(NM + FG + HD + LE + BC) =$ circum $ADIH$/(circum of NM + circum of FG + circum of HD + circum of LE + circum of BC), by [Proposition] III of this [work] and the penultimate [proposition] of [Book] V [of the *Elements*]. Hence $IY/BY =$ circum $ADIH$/(circum of NM + circum of FG + circum of HD + circum of LE + circum of BC). [But $\angle HED$ is a right angle, falling as it does in a semicircle. Hence $\angle HED = \angle Y$. And $\angle H = \angle B$, as they fall in equal arcs DE and IC. Thus triangles BYI and DEH are similar. Hence $IY/BY = DE/EH$. Hence $DE/EH =$ circum $ADIH$/the 5 circumferences.*] Hence ($ED \cdot$ the 5 circumferences) = ($EH \cdot$ circum $ADIH$). But ($ED \cdot$ the 5 circumferen-

* That is, (circum of NM + circum of FG + circum of HD + circum of LE + circum of BC).

omnes circumferentias, est equale ei quod fit ex *EH* in *ADIH*. Sed ex *DE* in quinque circumferentias fit tota conica superficies corporis poligonii, ut preostensum est. Ergo tota superficies corporis poligonii equatur ei quod fit ex *EH* in *ADIH* circumferentiam. Et sic utraque propositi pars plene constat

Eodem modo poterit probari propositum si fuerint latera semipoligonii numero imparia. Descripta tamen superficies a medio latere columpnalis erit. Esto enim figura X equalium angulorum inscripta circulo *ADIH* et equalium laterum [Fig. 68]. Ductis lineis a singulis angulis ad oppositos angulos, sicut prius, perpendiculariter ad *AI*, et [duce] alias oblique secantes *AI*, sicut in dispositione proxima, deinde ab *O* centro ducatur *OD* perpendicularis ad *GE*, secans *GE* in duo equa, et *OH* perpendicularis ad *FL*, secans *FL* in duo equa. Deinde ab *O* in *F* et in *E* ducantur due linee *OF*, *OE*.

Ratio. Age. *HF*, *GD* arcus sunt equales, quia sunt medietates *GE*, *FL* equalium. Et *FA* et *GA* sunt equales. Ergo *AH*, *AD* sunt equales arcus. Sed utraque est quarta totius circumferentie. Ergo *O* gemini centrales cadentes in illos sunt recti. Ergo *HOD* est una recta linea. Similiter cum *FAE* sit semicirculus, erit *FOE* una linea recta. Item cum *HD* sit perpendicularis ad *GE* et *FL* ex dispositione, erunt *GE*, *FL* equidistantes. Et sunt equales ex ypothesi. Ergo *FGEL* est parallelogramum rectangulum. Cum itaque *FG* et *LE* sunt equales et equidistantes, similiter *FL* et *GE*, et stanti *M'P*, circumvolvatur superficies *FE*, et fiet columpna, cuius basis erit circulus quem describit *LE*; axis vero *M'P*. Fit autem curva superficies illius columpne

87 ei: illi *A* / ADIH: ADIH circumferentiam *A* / Set *A*
90 EH: HE *A*
91 propositi...constat: pars constat propositi *A*
92 Eodem: eodem autem *A*
94 columpnalis erit *tr.* *A*
98–99 ad...perpendicularis *om.* *A*
99 secans *om.* *B*
101 arcus...equales: sunt equales arcus *A*
102 equales: equalia *A*
104 cadentes...recti: sunt equales cadentes in illos arcus quare recta *A*
105 recta *A om.* *BC*
106 GE: G *A*
107 Et *A* Sed et *BC*
108 et *A om.* *BC*
109 et[3] *om.* *A*
110 erit: est *A*
111 curva superficies: illa superficies curva *A*

ces) is equal to the whole conical surface of the polygonal body, as was just demonstrated. Therefore, the whole surface of the polygonal body is equal to the product of *EH* and circumference *ADIH*. And thus each part of the proposition is fully evident.

By the same method the proposition could be proved if the number of the sides of the half polygon was an odd number. However, the surface described by the middle side will be cylindrical. For let there be described in circle *ADIH* a regular polygon of ten sides [see Fig. 68]. Having drawn lines connecting each angle with the angle opposite to it, as before, these lines being perpendicular to *AI*, [draw] other [lines] cutting *AI* obliquely, as in the previous construction. Then from center *O* let

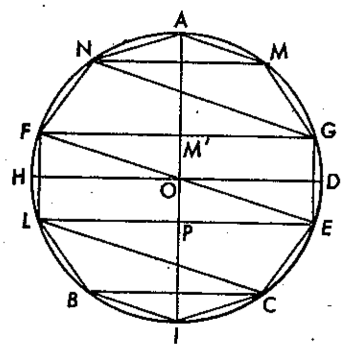

Fig. 68
Note: I have added the prime sign to *M'*.

OD be drawn perpendicular to *GE*, bisecting *GE*, and *OH* perpendicular to *FL*, bisecting *FL*. Then from *O* let there be drawn two lines *OF* and *OE* to *F* and *E* [respectively].

Proceed with the argument: *HF* and *GD* are equal arcs because they are halves of equal arcs *GE* and *FL*, and *FA* and *GA* are equal. Therefore *AH* and *AD* are equal arcs. But each is a quadrant of the whole circumference. Therefore, both the central angles *O* falling in those arc are right angles. Therefore, *HOD* is a straight line. Similarly, since *FAE* is a semicircle, *FOE* will be a straight line. Also, since *HD* is perpendicular to *GE* and to *FL* by construction, *GE* and *FL* are parallel, and they are equal by hypothesis. Therefore *FGEL* is a rectangle. And so, since *FG* is equal and parallel to *LE*, and similarly *FL* to *GE*, with *M'P* fixed, surface *FE* is rotated and a cylinder is produced whose base circle will be the circle described by *LE* and whose axis is *M'P*. Moreover, the lateral surface of this cylinder is equal to the product of (1) the axis ex ductu axis $M'P$, sive FL, in circumferentiam diametri LE, per secundam huius. Ratiocinando ergo ut superius, proba quod conica superficies poligonii corporis fit ex FL in circumferentias quatuor diametrorum, NM, FG, LE, BC. Similiter etiam ut supra probetur conicas illas superficies fieri ex ductu FG in $ADIH$ circumferentiam. Et sic omni modo constat propositum.

113v / VI. CUIUSLIBET SPERE SUPERFICIES EST EQUALIS QUADRANGULO RECTANGULO QUI SUB LINEIS EQUALIBUS DIAMETRO SPERE ET CIRCUMFERENTIE MAXIMI CIRCULI CONTINETUR.

Esto exemplum $ACBD$ circumferentia, diameter cuius sit AB [Fig. 69]. Stanti ergo AB, circumducatur ACB, et fiat spera. Dico quod id quod fit ex ductu diametri in $ACBD$ circumferentiam est equale superficiei spere $ACBD$.

Sin autem, sit equalis superficiei minoris spere vel maioris; et primo minoris spere, scilicet spere quam describit SNH semicirculus circumvolutus circa O centrum, constituitur.

Dispositio. Inscribatur circulo $ACBD$ poligonium equalium laterum et angulorum, circulum SNH minime contingentium. Deinde ab O centro ducatur OI perpendicularis ad AF et secans AF in duo equa et alia a B in F.

Ratio. Age. AB diameter est maior BF. Ergo id quod fit ex AB

114 corporis *om.* *A*
114–15 quatuor diametrorum *tr.* *A*
115 etiam: et *A*
116 ADIH / AHDI *ABC*
117 omni modo: omnino *A*
1 VI *B om.* *C* VI^a / 6^a *AD* / est equalis *tr.* *A*
2 rectangulo *om.* *A*
5 Esto *A om.* *BC* / cuius sit: eius *A*
7 ACBD *A* / ABCD *BC*
8 spere *om.* *A*
13 SNH minime *B* SH non *A* SH minime *C*

M'P, or [its equal] *FL*, and (2) the circumference of diameter *LE*, by the second [proposition] of this [work]. Therefore, by reasoning as above, prove that the conical surface of the polygonal body is equal to the product of (1) *FL* and (2) the circumferences of the four diameters *NM*, *FG*, *LE*, and *BC*. Also let it be proved in the same way as above that those conical surfaces [making up the surface of the whole polygonal body] are equal to the product of *FG* and circumference *ADIH*. And so this proposition is evident in every way.

VI. THE SURFACE OF ANY SPHERE IS EQUAL TO THE RECTANGLE CONTAINED BY LINES EQUAL TO THE DIAMETER OF THE SPHERE AND THE CIRCUMFERENCE OF THE GREATEST CIRCLE [OF THE SPHERE].

For example, let *ACBD* be a circumference whose diameter is *AB* [see Fig. 69]. With *AB* fixed, let *ACB* be rotated, producing a sphere.

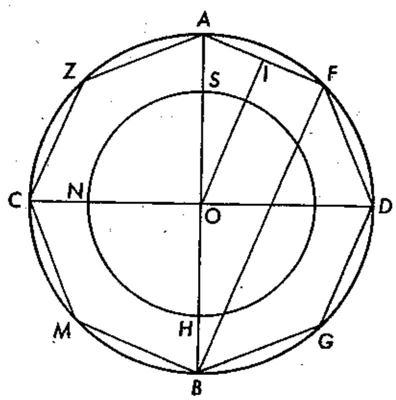

Fig. 69

I say that the product of (1) the diameter and (2) circumference *ACBD* is equal to the surface of sphere *ACBD*.

But if not, let it be equal to the surface of sphere that is less or greater than [*ACBD*]. In the first place let it be [equal to the surface] of a lesser sphere, i.e., of the sphere which semicircle *SNH* describes in rotation about center *O*.

Disposition: Let there be inscribed in circle *ACBD* a regular polygon whose sides do not touch circle *SNH* at all. Then let *OI* be drawn from center *O* perpendicular to *AF* and bisecting *AF*. Let another line be drawn from *B* to *F*.

Proceed with the proof: Diameter *AB* is greater than *BF*. Therefore, in *ACBD* circumferentiam maius est quam id quod fit ex *BF* in *ACBD*. Sed id quod fit ex *BF* in *ACBD* est equale conicis superficiebus corporis poligonii per proximam. Et conice superficies corporis poligonii sunt maiores superficie spere *SH* interioris. Ergo id quod fit ex ductu *AB* in *ACBD* circumferentiam maius est superficie spere *SH*; contra preconcessa. Relinquitur ergo id quod fit ex ductu *AB* in *ACBD* non est equale superficiei minoris spere quam *ACBD*.

Sit itaque si fieri potest equale superficiei spere maioris. Brevitatis tamen causa, prior observetur figura, et sit *SH* spera proposita et diameter *SH*. Dico quod fit ex *HS* diametro in *HS* circumferentiam non est equale superficiei maioris spere quam *SH*. Sin autem, sit equalis superficiei spere *ACBD*.

Ratio. Age. *F* cadens in semicirculum *ADB* est rectus. Similiter et *I* rectus ex dispostitione; et *A* communis. Ergo *AIO*, *AFB* trianguli sunt similes. Ergo que est proportio *AF* ad *AI* eadem est *BF* ad *OI*. Sed *AF* est dupla ad *AI*. Ergo *BF* est dupla *OI*. Sed *OI* est maior quam *OS*, que subdupla est ad *HS* diametrum. Ergo *BF* est maior *HS*; et *ACBD* circumferentia maior *SNH* circumferentia. Ergo id quod fit ex *BF* in *ACBD* est maius eo quod fit ex *HS* in *SNH* circumferentiam. Sed quod fit ex *BF* in *ACBD* est equale conicis superficiebus corporis poligonii per proximam. Et quod fit ex *HS* diametro in *SNH* circumferentiam est equale superficiei spere *ACBD* secundum falsigraphum. Ergo conice superficies corporis poligonii sunt maiores superficie *ACBD* spere, inclusum includente, quod est impossibile. Relinquitur propositum.

[Corollaria:] Ex hoc ergo manifestum quoniam superficies spere est quadrupla maximo circulo eiusdem spere, et equalis curve superficiei columpne, cuius tam axis quam diameter basis equatur diametro

18 BF: FB *C* / ACBD[2]: ADBC *A* / equales *A*
19–20 corporis poligonii *tr. A*
20 spere SH *tr. A*
21–23 circumferentiam...ACBD[1] *AB mg. C*
22 ergo: ergo quod *A*
23 ACBD[1]: ABCD *C* / minoris spere *tr. A* / ACBD[2]: ABCD *C*
24 itaque: ergo *A*
24 equale...spere *A om. BC*
25 prior: prius posita *A* / sit *A om. BC*
26 HS diametro: ductu HS diametri *A*
29 ADB: ABD *A*
29–30 Similiter et I: Et similieter I est *A*
30 *post* A *add. A* est / AIO, AFB: AIO AFD *A*
32 Ergo...OI[1] *om. A*
33 que: quia *A*
35 ex[1]: ex ductu *A* / ACBD: ACBD circumferentiam *A*
36 BF: ductu *A* / ACBD: ACBD circumferentiam *A*
38 HS: HH *B*
41 propositum: ergo propositum *A*
42 ergo *om. A* / manifestum: manifestum est *A*

(*AB* · circum *ACBD*) > (*BF* · circum *ACBD*). But (*BF* · circum *ACBD*) is equal to the conical surfaces of the polygonal body, by the preceding [proposition]. Further, the conical surfaces of the polygonal body are greater than the surface of the interior sphere *SH*. Therefore, (*AB* · circum *ACBD*) > surface of sphere *SH*, which is contradictory to what was conceded earlier. It remains, therefore, that the product of *AB* and *ACBD* is not equal to the surface of a sphere less than *ACBD*.

And so, if possible, let it be made equal to the surface of a sphere greater [than sphere *ACBD*]. For the sake of brevity, however, let the prior figure [Fig. 69] be consulted, and let sphere *SH* be the proposed sphere with diameter *SH*. I say that the product of diameter *HS* and the circumference of *HS* is not equal to the surface of a sphere greater than *SH*. But if this is not [equal to the surface of sphere *SH*], let it be equal to the surface of sphere *ACBD*.

Proceed with the proof: *F*, falling in semicircle *ADB*, is a right angle. And similarly *I* is a right angle by construction, and ∠ *A* is common. Therefore, triangles *AIO* and *AFB* are similar. Therefore, $AF/AI = BF/OI$. But $AF = 2\ AI$. Therefore, $BF = 2\ OI$. But *OI* is greater than *OS*, which is one half diameter *HS*. Therefore, *BF* is greater than *HS*, and circumference *ACBD* is greater than circumference *SNH*. Therefore, (*BF* · circum *ACBD*) > (*HS* · circum *SNH*). But (*BF* · circum *ACBD*) is equal to the conical surfaces of the polygonal body, by the preceding [proposition]. And the product of diameter *HS* and circumference *SNH* is equal to the surface of sphere *ACBD*, according to the pseudographer. Therefore, the conical surfaces of the polygonal body are greater than the surface of sphere *ACBD*, i.e., the "included" [is greater] than the "including." which is impossible. [Hence] the proposition remains [as the only true possibility].

[Corollaries:] From this, therefore, it is evident that the surface of a sphere is quadruple [the area of] the greatest circle of the same sphere, and [it is further evident that] it is equal to the lateral surface of a cylinder spere. Tota quoque huius columpne superficies superficiei spere sexquialtera est.

Ut enim in quadratura circuli ostensum est quod fit ex ductu quarte partis diametri in circumferentiam est equale circulo. Ergo quod fit ex diametro in circumferentiam, scilicet superficies spere, est quadrupla ad circulum. Item ex ductu diametri in circumferentiam maximi circuli fit superficies spere, et ex ductu axis in circumferentiam basis fit curva superficies columpne. Si ergo et axis et diameter basis columpne sit equalis diametro spere, erit superficies spere superficiei illius columpne equalis. Ex hoc liquet quod curva superficies columpne cum duobus circulis qui sunt extremitates columpne est sexquialtera ad superficiem spere. Et hoc erat probandum.

Liber secundus

[Descriptio:] Omne corpus erectum super basim supereminenti superficiei equalem et similem et equidistantem sub basi et altitudine dicitur contineri.

VII. OMNE SOLIDUM CONICARUM SUPERFICIERUM INSCRIPTIBILE ET CIRCUMSCRIPTIBILE SPERE EQUUM EST PIRAMIDI CUIUS BASIS SIT EQUALIS SUPERFICIEI SOLIDI ET ALTITUDO SEMIDIAMETRO SPERE INSCRIPTE SOLIDO.

Esto solidum conicarum superficierum *CAQB* inscriptibile et circumscriptibile spere [Fig. 70]. Dico quod corpus *ACBQ* est equale piramidi cuius basis est equalis conicis superficiebus *ACBQ* corporis propositi et altitudo equalis semidiametro spere inscripte solido *ACBQ*.

Rationis causa. Protrahantur a centro *O* ad tria latera superficie poligonie, que circumvoluta facit solidum, tres perpendiculares. E sint *OI*, *OT*, *OZ*. / Protrahantur item ad quatuor angulos *C*, *F*, *N*
114r

46 est *om. C*
48 *ante* diametri *del. B* circumferentie / diametri: circumferentie diametri *C*
49 diametro: ductu diametri *A*
52 et[1] *om. B*
54 equalis *tr. A post* spere / equale *C*
56 erat probandum *A* est *BC*, *sed post* est *add. C manu recentiori* quod ostendere voluimus / *in mg. habet B manu recentiori* Explicit primus liber
1 Liber secundus *mg. B mg. A* secundus liber Archimenidis de curvis superficiebus *mg. B manu recentiori* secundus *mg.* (*f. 93r*)
2 [Descriptio] *mg. H om. alii MSS* diffinitio *mg. D*
5 VII: 7ª *AD* I *B* Iª *J* / solidum *BC* supra scr. *A* corpus *in textu A*
10 ACBQ *corr. ex* ACQB *in BCA*
11 ACBQ *C* ACQB *BA*
13 Protrahatur *C*
14 poligonii *A*
15 OT, OZ: OZ, OT *C*

whose axis and base diameter are each equal to the diameter of the sphere. Further, the whole surface of this cylinder is three halves the surface of the sphere.

For in the *Quadrature of the Circle* it was demonstrated that the product of (1) one quarter of the diameter and (2) the circumference is equal to the circle. Therefore, that which arises from the product of the diameter and the circumference—that is, the surface of the sphere—is quadruple the circle. Also [if] from the product of the diameter and the circumference of the greatest circle arises the surface of a sphere, and if from the product of the axis and the base circumference the lateral surface of a cylinder is produced, then, when the axis and the base diameter of the cylinder are each equal to the diameter of the sphere, the surface of the sphere will be equal to the [lateral] surface of the cylinder. From this it is evident that the lateral surface of the cylinder together with the two circles which are the bases of the cylinder is three halves the surface of the sphere. And this is what was to be proved.

Book II

[Description:] Every body erected on a base which is parallel, equal, and similar to a surface rising directly above [the base] is said to be contained by the base and the altitude.

VII. EVERY SOLID CONSISTING OF CONICAL SURFACES AND INSCRIBABLE OR CIRCUMSCRIBABLE IN A SPHERE IS EQUAL TO A CONE WHOSE BASE IS EQUAL TO THE SURFACE OF THE SOLID AND WHOSE ALTITUDE [IS EQUAL] TO THE RADIUS OF THE SPHERE INSCRIBED IN THE SOLID.

Let there be a solid *CAQB* (*ACBQ*) having conical surfaces, which solid is inscribable and circumscribable in a sphere [see Fig. 70]. I say that body *ACBQ* is equal to a cone whose base is equal to the conical surfaces of the proposed body *ACBQ* and whose altitude is equal to the radius of the sphere inscribed in solid *ACBQ*.

[Construction] needed for the proof: Let there be drawn three perpendiculars from center *O* to three of the sides of the polygonal surface whose rotation makes the solid. Let these perpendiculars be *OI*, *OT*, and *OZ*.

A a centro quatuor linee *OC*, *OF*, *ON*, *OA*. Protrahantur itaque *AN*, *NF* in occursus linee *OC*. Ab *F* item ducatur perpendicularis *FP* ad *OC*, et ab *N* perpendicularis *NV* ad *OC*.

Ratio. Age. Triangulus *OFC*, stanti *OC*, circumductus facit duplicem piramidem *OFC*, equalem piramidibus *OPF*, *PFC*. Sed columpna que fit ex ductu *PC* in circulum quem describit *FP* circumvoluta est tripla ad piramidem *PFC* per IX duodecimi Elementorum. Et columpna que fit ex ductu *PO* axis in circulum quem describit *FP* est tripla ad piramidem *FPO* per eandem. Quod ex ductu axis in basim fiat columpna sumat demonstrator pro rato per precendentem descriptionem. Quod ergo fit ex ductu totius *OC* in eundem circulum triplum est duplicis piramidis *OFC* per primam secundi. Sed tam *I* quam *P* est rectus et *C* communis. Ergo *OIC*, *CPF* trianguli sunt similes. Ergo que est porportio linee *OC* ad *OI* eadem est *CF* ad *FP*. Et que est proportio *CF* ad *FP* eadem est curve superficiei piramidis *FCP*, quam describit linea *FC*, ad circulum quem describit *FP* per corollarium prime huius. Ergo quod fit ex ductu *OC* linee in circulum descriptum ab *FP*, quod, scilicet, est aliqua columpna per descriptionem precedentem, est equale ei quod fit ex *OI* linea in curvam superficiem piramidis *FPC*, quam describit linea *FC*. Sed quod fit ex ductu *OC* in circulum quem describit *FP* circumvoluta est columpna tripla duplici piramidi *OPF*, *PFC*, per IX duodecimi Elementorum et

16 itaque: que *C*
17 *ante* NF *del.* *C* AF
17, 21, 23 FP: SP *A*
21 circumvoluta *om.* *B*
22 IX: XI *A*
23 PO: PC *A*
24 FPO: SPC *A*
25 sumat *B* sumit *A* summit *C*
28 CPF: PFC *A*
29, 30 CF ad FP: OF ad SP *A*
30 Et: Sed *A* / proportio *om.* *A*
31 FCP: FOP *A* / FP: SP *A*
32 corellarium *B*
33 FP: SP *A*
34 ei: illi *A* / ex: ex ductu *A*
33–34 descriptionem precedentem *tr.* *A*
34 linee *A*
36 FP: SP *A*
37 IX: XI *A*

Also let the four lines *OC*, *OF*, *ON*, and *OA* be drawn from the center to the four angles *C*, *F*, *N*, and *A*. And then let *AN* and *NF* be extended until they meet line *OC* [extended]. Also, from *F* let *FP* be drawn perpendicular to *OC*, and from *N*, *NV* perpendicular to *OC*.

Proceed with the proof: With *OC* fixed, △ *OFC* by rotation produces the double cone *OFC*, equal to cones *OPF* and *PFC*. But the cylinder which arises from the product of *PC* and the circle described by *FP* in rotation is three times the cone *PFC*, by [Proposition] XII.9* of the *Elements* And the cone which arises from the product of axis *PO* and the circle described by *FP* is three times the cone *FPO*, by the same [proposition]. The demonstrator assumes that the cylinder arises from the product of

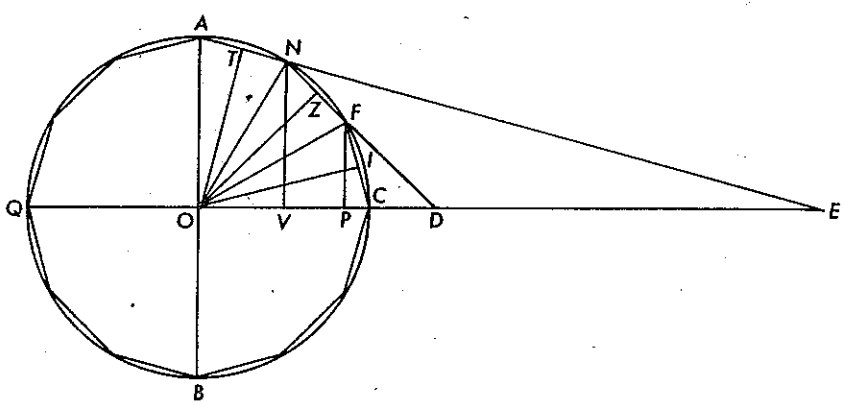

Fig. 70

the axis and the base, as is confirmed by the preceding description [given just prior to Proposition VII]. Therefore, the product of the whole *OC* and the same circle is three times the double cone *OFC*, by [Proposition] II.1 [of the *Elements*]. But both ∠ *I* and ∠ *P* are right angles, and ∠ *C* is common. Therefore, triangles *OIC* and *CPF* are similar. Therefore, *OC*/*OI* = *CF*/*FP*. But the ratio of *CF* to *FP* is the same as that of the lateral surface of cone *FCP* described by line *FC* to the circle described by *FP*, by the corollary of [Proposition] I of this [work.] Therefore, the product of line *OC* and the circle described by *FP*, which product evidently is equivalent to some cylinder by the preceding description, is equal to the product of line *OI* and the lateral surface of cone *FPC* described by line *FC*. But the product of *OC* and the circle described by *FP* in rotation is [equal to] a cylinder which is [in volume] three times the double cone *OPF*, *PFC*, by [Proposition] XII.9 of the *Elements* and by the preceding

* Greek text XII.10 throughout this proposition.

precedentem descriptionem. Ergo solidum quod fit ex ductu linee *OI* in curvam superficiem quam describit *FC* in *FCP* piramidem est equale illi columpne, et ita est triplum ad duplicem piramidem *OPF*, *PFC*. Sed illius eiusdem columpne, que fit ex *OI* in curvam superficiem que sit equalis curve superficiei piramidis *FPC*, quam superficiem describit *FC* circumvoluta, subtripla est piramis cuius altitudo *OI*, basis vero equalis curve superficiei quam describit *FC*, per IX duodecimi Elementorum. Ergo duplex piramis *OFP*, *FPC* est equalis piramidi cuius altitudo *OI*, basis vero circulus equalis curve superficiei quam describit *FC*. Sit ergo illa piramis rotunda piramis *M*.

Amplius, ex ductu *OV* axis in circulum descriptum ab *NV* fit columpna tripla piramidi *NVO* et ex ductu *VD* in eundem circulum fit columpna tripla piramidi *NDV*. Ergo per primam secundi quod fit ex *OD* in eundem circulum est triplum duplici piramidi *DNV*, *NVO*. Item tam *V* quam *Z* est recus et *D* est communis. Ergo *NVD*, *DZO* trianguli sunt similes. Ergo que est proportio *OD* ad *ZO* eadem est *DN* ad *NV*. Sed que est *DN* ad *NV* eadem est curve superficiei piramidis *DNV*, quam describit *DN* circumvoluta, ad circulum cuius semidiameter est *NV*, per corollarium prime huius. Ergo que est proportio linee *OD* ad *OZ* eadem est curve superficiei quam describit *ND* ad circulum quem describit *NV*. Ergo quod fit ex ductu linee *OD* in circulum descriptum ab *NV*, primi, scilicet, in ultimum, equum est ei quod fit ex ductu linee *OZ* in curvam superficiem quam describit *ND*, scilicet unius medii in reliquum. Sed ex ductu *OD* in circulum quem describit *NV* fit columpna, per precedentem descriptionem, tripla duplici piramidi *NVO*, *NVD*, per IX duodecimi Elementorum. Ergo quod fit ex ductu linee *OZ* in curvam superficiem piramidis *NDV*, quam describit linea *ND*, est equale columpne triple ad duplicem piramidem *NVO*, *NVD*. Ergo piramis cuius altitudo est *OZ*, basis vero circulus equalis curve superficiei piramidis *NVD*, quam

38–39 linee OI *tr. A*
39 FCP: FOP *A*
42 superficiei *om. A* / FPC: SPC *A*
43 piramidis *A*
45 FPC: SPC *A*
46 equalis *om. A*
52 NVD: NOD *A*
53 est *om. C*
54 NV[1]: NO *C*
55 DN: linea DN *A*
56 semi- *supra scr. B* / corellarium *B* / Ergo *om. A*
59 OD *om. C*
60 linee OZ *tr. A*
61 scilicet *A om. BC*
63 tripla: est tripla *A* / piramidi *om. A* / IX: XI *A*
65 equale columpne *B* equalis *A* equalis columpne *C*
66 est *A om. BC*
67 equalis *A om. BC*

description. Therefore, the solid arising from the product of line *OI* and the lateral surface of cone *FCP* described by *FC* is equal to that cylinder, and thus is triple the double cone *OPF*, *PFC*. But the cone whose altitude is *OI* and whose base is equal to the lateral surface which *FC* describes is one third that same cylinder which arises from the product of *OI* and the lateral surface of cone *FPC* described by *FC* in rotation; this is by [Proposition] XII.9 of the *Elements*. Therefore, the double cone *OFP*, *FPC* is equal to the cone whose altitude is *OI* and whose base circle is equal to the lateral surface described by *FC*. Therefore, let this cone be cone *M*.

Further, from the product of axis *OV* and the circle described by *NV* arises a cylinder which is three times the cone *NVO*, and from the product of *VD* and the same circle arises a cylinder which is three times the cone *NDV*. Therefore, by [Proposition] II.1 [of the *Elements*], the product of *OD* and the same circle is three times the double cone *DNV*, *NVO*. Also, *V* as well as *Z* is a right angle, and $\angle D$ is common. Therefore, triangles *NVD* and *DZO* are similar. Therefore, $OD/ZO = DN/NV$. But as *DN* is to *NV*, so the lateral surface of cone *DNV* described by *DN* in rotation is to the circle whose radius is *NV*, by the corollary of the first [proposition] of this [work]. Therefore, the ratio of *OD* to *OZ* is the same as that of the lateral surface described by *ND* to the circle described by *NV*. Therefore, the product of line *OD* and the circle described by *NV*, i.e., the product of the first and the last term, is equal to the product of line *OZ* and the lateral surface described by *ND*, i.e., the product of the two middle terms. But the product of *OD* and the circle described by *NV* equals, by the preceding description, a cylinder which is three times the double cone *NVO*, *NVD*, by [Proposition] XII.9 of the *Elements*. Therefore, the product of line *OZ* and the lateral surface of cone *NDV* described by line *ND* is equal to a cylinder which is three times the double cone *NVO*, *NVD*. Therefore, the cone whose altitude is *OZ* and whose base is a circle equal to the lateral surface of cone *NVD* described by *ND* is describit *ND*, est equalis duplici piramidi *NVO*, *NVD*, cum earum utraque sit ad idem subtripla. Item tam *P* quam *Z* est rectus et *D* communis. Ergo *DFP*, *DOZ* trianguli sunt similes. Ergo proportio *OD* ad *OZ* que *FD* ad *FP*. Sed que *FD* ad *FP* eadem est curve superficiei piramidis *FDP*, quam describit *FD*, ad circulum cuius semidiameter est linea *FP* per corollarium prime huius. Ergo proportio *OD* ad *OZ* que curve superficiei quam describit *FD* ad circulum quem describit *FP*. Ergo quod fit ex ductu linee *OD* in circulum quem describit *FP* est equale ei quod fit ex ductu *OZ* in curvam superficiem piramidis *FDP*, quam superficiem describit linea *DF*. Sed ex ductu *OD* in circulum quem describit *FP* est columpna tripla duplici piramidi *FDP*, *FOP*. Et quod fit ex ductu *OZ* in circulum equalem curve superficiei piramidis *FDP*, quam superficiem describit *FD*, est equale columpne triple ad piramidem cuius *OZ* altitudo, basis vero circulus equalis curve superficiei piramidis *FDP*, quam superficiem describit *FD*, per precedentem descriptionem et IX duodecimi Elementorum. Ergo piramis cuius altitudo *OZ*, basis vero circulus equalis curve superficiei quam describit *FD* in piramidem *FDP* est equalis duplici piramidi *FDP*, *FPO*; subtriplum subtriplo equale sicut tri- 114v plum / triplo. Sed piramis cuius altitudo *OZ*, basis vero circulus equalis curve superficiei piramidis *NVD*, quam superficiem describit linea *ND* circumvoluta, *VD* stanti, preostensa est esse equalis duplici piramidi *NVO*, *NVD*. Ergo piramis cuius altitudo *OZ*, basis vero circulus equalis curve superficiei quam describit *NF* circumvoluta, *PV* stanti, est equalis differentie duplicium piramidum *DVN*, *NVO* et *DPF*, *FPO*, scilicet corpori quod describit triangulus *NFO*, dependens a linea *ND*, circumvolutus, *DO* stanti. Illa piramis sit *S*.

Regula enim hec firmissima est: Quarumlibet duarum pirimidum inequalium [eiusdem altitudinis] differentia equatur piramidi eiusdem altitudinis cuius basis equatur differentie basium illarum piramidum. Quod sic probatur. Sit piramis *DBO*, cuius basis circulus *AB*, cathetus *OD*, ypothenusa *DB*, et minor piramis *ODE* eiusdem altitudinis

68 NVO, NVD: NVD, NVO *C*
69 D: D est *A*
72–74 cuius...circulum *om.* *A*
73 corellarium *B*
77 piramidis *A* piramidum *BC*
78 OD *om.* *A* / columpna: conica *A*
79 equale *C*
81 columpne: conice superficiei *A* / OZ altitudo *tr.* *A*
83 IX: XI *A*
86 equale: est equale *A*
93 FPO: PF *A*
94 Illa piramis *om.* *A*
95–96 piramidum inequalium *tr.* *A*
96 [eiusdem altitudinis] *supplevi*
98 AB: IB *A*

equal to the double cone *NVO*, *NVD*, since each of them is one third of the same quantity. Also, *P* as well as *Z* is a right angle, and $\angle D$ is common. Therefore, triangles *DFP* and *DOZ* are similar. Therefore, $OD/OZ = FD/FP$. But the ratio of *FD* to *FP* is the same as that of the lateral surface of cone *FDP* described by *FD* to the circle whose radius is line *FP*, by the corollary of the first [proposition] of this [work]. Therefore,

$$\frac{OD}{OZ} = \frac{\text{lateral surface described by } FD}{\text{circle described by } FP}.$$

Therefore, the product of line *OD* and the circle described by *FP* is equal to the product of *OZ* and the lateral surface of cone *FDP* described by line *DF*. But from the product of *OD* and the circle described by *FP* arises a cylinder which is three times the double cone *FDP*, *FOP*. And the product of *OZ* and the circle equal to the lateral surface of cone *FDP* described by *FD* is equal to a cylinder which is three times the cone whose altitude is *OZ* and whose base is equal to the lateral surface of cone *FDP* described by *FD*, by the preceding description and [Proposition] XII.9 of the *Elements*. Therefore, the cone whose altitude is *OZ* and whose base is a circle equal to the lateral surface which *FD* describes in cone *FDP* is equal to the double cone *FDP*, *FPO*, for one third is to one third as triple is to triple. But the cone whose altitude is *OZ* and whose base is a circle equal to the lateral surface of cone *NVD* described by line *ND* in rotation, with *VD* fixed, has previously been shown to be equal to the double cone *NVO*, *NVD*. Therefore, the cone whose altitude is *OZ* and whose base is a circle equal to the lateral surface described by *NF* in rotation, with *PV* fixed, is equal to the difference between the double cones *DVN*, *NVO* and *DPF*, *FPO*, i.e., to the body described by $\triangle$ *NFO* in rotation, where line *ND* is revolved with *DO* remaining fixed. Let this cone be *S*.

For the following rule is firmly established: The difference [in volume] between two unequal [right circular] cones [of the same altitude] is equal to a [third right circular] cone of the same altitude whose base is equal to the difference [in area] between the bases of those cones. Proof: Let there be cone *DBO*, whose base circle is *AB*, altitude *OD*, and slant cuius basis circulus *GE*, ypothenusa *DE* [Fig. 71]. Differentia *AB* circuli ad *GE* circulum vocetur *K*. Sitque circulus *M* equalis *K*, et piramis *MS* fundata super *M* circulum, cuius altitudo equalis *OD*.

Ratio. Age. Que est proportio *DBO* piramidis ad *DOE* piramidem eadem est *AB* circuli basis ad *GE* circulum basim, per XI duodecimi. Ergo que est proportio *DBO* ad augmentum *DBO* super *DOE* eadem est *AB* circuli ad *K* superficiem per eversam proportionem. Ergo a pari que est *DBO* ad augmentum suum super *DOE* eadem est *AB* circuli ad *M* circulum. Sed que est *AB* circuli basis ad *M* circulum basim eadem est *DBO* piramidis ad *MST* piramidem, cum sint eiusdem altitudinis per XI duodecimi. Ergo proportio *DBO* ad augmentum suum supra *DOE* que *DBO* ad *MST* piramidem. Ergo augmentum *DBO* piramidis super *DOE* est equale *MST* piramidi, quod proposuimus. Principali ergo proposito insistamus.

Amplius, in *AOE*, *TOE* tam *O* quam *T* angulus est rectus, et *E* est communis, quod ut in anteproxima propostitione probari potest. Ergo *OAE*, *OTE* trianguli sunt similes. Ergo proportio *OE* ad *OT* est que *EA* ad *OA*. Et que est *AE* ad *AO* eadem est curve superficiei *AOE* ad circulum quem describit *AO*. Ergo que est proportio *OE* ad *OT* eadem est curve superficiei piramidis *AOE*, quam superficiem describit *AE*, ad circulum quem describit *AO*. Ergo solidum quod fit ex ductu *OE* in circulum quem describit *AO* est equale solido

100 DE: OD *C*
102 M: AN *A* / equalis: est equalis *A*
103 DBO: ABO *A*
104, 106, 107, 108 AB: IB *A*
104 basem *A* / XI *A* IX *BC*
107 DBO: proportio DBO *A*
109 basem *AC* / est *om. C*
111 supra: super *C* / DBO: DBC *A*
114–15 in...communis *A* AOC angulu[s] est rectus *BC*
115 potest *A* potest et E est communi[s] *BC*
117 EA: AE *A* / OA: AO *A* / est *om. AC*
118–20 Ergo...AO *om. A*
118 est *om. B*

height DB [see Fig. 71]. And [let there be] a smaller cone ODE of the same altitude, whose base is circle GE, and whose slant height is DE. The difference [in area] between circles AB and GE we let be designated as K. Then let there be a circle M equal to K, and erected upon circle M a cone MS, whose altitude is equal to OD.

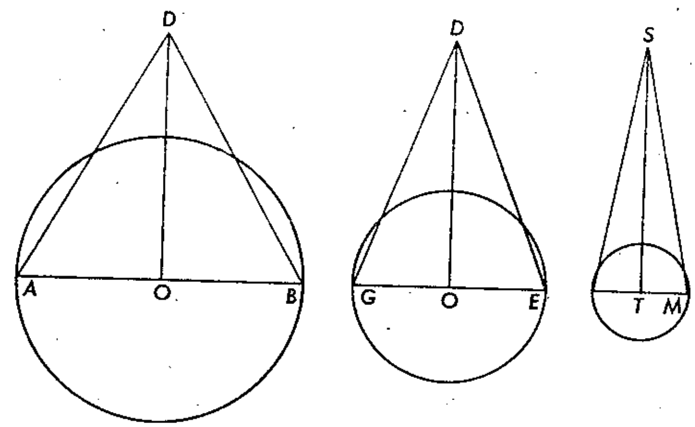

Fig. 71

Now proceed with the proof: $\frac{\text{cone } DBO}{\text{cone } DOE} = \frac{\text{base circle } AB}{\text{base circle } GE}$ by XII.11 [of the *Elements*]. Therefore, $\frac{\text{cone } DBO}{\text{cone } DBO - \text{cone } DOE} = \frac{\text{circle } AB}{\text{surface } K}$, by the conversion of ratios. Therefore, by equality,

$$\frac{\text{cone } DBO}{\text{cone } DBO - \text{cone } DOE} = \frac{\text{circle } AB}{\text{circle } M}.$$

But base circle AB/base circle M = cone DBO/ cone MST, since they are of the same altitude; this is by XII.11 [of the *Elements*]. Therefore,

$$\frac{\text{cone } DBO}{\text{cone } DBO - \text{cone } DOE} = \frac{\text{cone } DBO}{\text{cone } MST}.$$

Therefore, cone DBO − cone DOE = cone MST, which we proposed. Therefore, let us press on to the principal proposition [see Fig. 70, above].

Further, in [triangles] AOE and TOE, O as well as T is a right angle, and $\angle E$ is common, which can be proved as in the proposition before the preceding one [i.e., as in Proposition V]. Therefore, triangles OAE and OTE are similar. Therefore, $OE/OT = EA/OA$. And

$$\frac{AE}{AO} = \frac{\text{lat surf cone } AOE}{\text{circle described by } AO}.$$

quod fit ex ductu *OT* in curvam superficiem piramidis *AOE*, quam superficiem describit *AE*. Sed quod fit ex *OE* in circulum quem describit *AO* est columpna tripla ad piramidem *AOE*. Et id quod fit ex ductu *OT* in curvam superficiem piramidis *AOE*, quam superficiem describit linea *AE*, est equale columpne triple ad piramidem, cuius altitudo *OT*, basis vero circulus equalis curve superficiei piramidis *AOE*, quam superficiem describit linea *AE*. Ergo piramis *AOE* est equalis piramidi cuius altitudo *OT*, basis vero circulus equalis curve superficiei piramidis *AOE*; subtriplum subtriplo sicut triplum triplo.

Deinde simili ratione sicut superius respiciantur *OTE*, *NVE* trianguli similes, cum tam *V* quam *T* sit rectus et *E* communis. Ergo proportio *OE* ad *OT* que *EN* ad *NV*. Sed que *EN* ad *NV* eadem est curve superficiei piramidis *NVE*, quam superficiem describit linea *NE*, ad circulum cuius semidiameter est *NV*, per corollarium prime huius. Ergo que est proportio *OE* ad *OT* eadem est curve superficiei piramidis *NVE*, quam superficiem describit *NE*, ad circulum quem describit *NV*. Ergo quod fit ex *OE* in circulum quem describit *NV* est equale illi quod fit ex ductu *OT* in curvam superficiem quam describit *NE*. Sed ex *OE* in circulum quem describit *NV* fit columpna tripla duplicis piramidis *NVE*, *NVO*, per precedentem descriptionem et per IX duodecimi. Et quod fit ex *OT* in curvam superficiem quam describit *NE* est columpna tripla piramidis, cuius altitudo *OT*, basis vero circulus equalis curve superficiei quam describit *NE*, per eandem descriptionem et per IX duodecimi. Ergo duplex piramis *NVO*, *NVE* est equalis piramidi cuius altitudo *OT*, basis vero circulus equalis curve superficiei quam describit *NE*. Sed piramis *AOE* preostensa est esse equalis piramidi cuius altitudo *OT*, basis vero circulus equalis curve superficiei quam describit *AE*. Ergo piramis cuius altitudo *OT*, basis vero circulus equalis curve superficiei quam describit *AN* circumvoluta, *OV* stanti, est equalis corpori quod est differentia piramidis *AOE* ad duplicem piramidem *NVE*, *NVO*, scilicet corpus quod describit triangulus *NOA*, dependens a linea *AE*, circumvolutus, *OE*

123, 125², 134, 137 superficiem *om.* *A*
132 cum: quorum *A* / sit: angulus est *A* / E: E est *A*
134 NVE *B* NVB *A* NVH (?) *C*
135 cuius semidiameter *A* *tr.* *BC* / corellarium *B*
141 duplicis piramidis *tr.* *A*
142 IX: XI *A* / fit *om.* *A*
143 *post* NE *add.* *B* per eandem descriptionem et per IX XII^mi / piramidis duplcis piramidis *A* / OT: est OT *A*
144 equalis *om.* *A*
145 IX: XI *A*
147–50 NE....describit *om.* *A*
152 NVO *om.* *A*

Therefore. $\frac{OE}{OT} = \frac{\text{lat surf cone } AOE \text{ described by } AE}{\text{circle described by } AO}$. Therefore, ($OE$ · circle described by AO) = (OT · lat surf cone AOE described by AE). But the product of OE and the circle described by AO is a cylinder which is three times cone AOE; and the product of OT and the lateral surface of cone AOE described by line AE is equal to a cylinder which is three times the cone whose altitude is OT and whose base circle is equal to the lateral surface of cone AOE described by line AE. Therefore, cone AOE is equal to a cone whose altitude is OT and whose base circle is equal to the lateral surface of cone AOE, since 1/3 is to 1/3 as 3 is to 3.

Then by an argument similar to that above, triangles OTE and NVE are regarded as similar since V as well as T is a right angle and $\angle E$ is common. Therefore, $OE/OT = EN/NV$. But

$$\frac{EN}{NV} = \frac{\text{lat surf cone } NVE \text{ described by line } NE}{\text{circle of radius } NV},$$

by the corollary of the first [proposition] of this [work]. Therefore,

$$\frac{OE}{OT} = \frac{\text{lat surf cone } NVE \text{ described by line } NE}{\text{circle described by } NV}. \text{ Therefore,}$$

(OE · circle described by NV) = (OT · lat surf described by NE).

But the product of OE and the circle described by NV is a cylinder which is three times the double cone NVE, NVO, by the preceding description and by [Proposition] XII. 9 [of the *Elements*]. And the product of OT and the lateral surface described by NE is a cylinder which is three times a cone whose altitude is OT and whose base circle is equal to the lateral surface described by NE, by that same description and by XII.9 [of the *Elements*]. Therefore, the double cone NVO, NVE is equal to a cone whose altitude is OT and whose base circle is equal to the lateral surface described by NE. But cone AOE was earlier shown to be equal to the cone whose altitude is OT and whose base circle is equal to the lateral surface described by AE. Therefore, the cone whose altitude is OT and whose base circle is equal to the lateral surface described by AN in rotation, with OV remaining fixed, is equal to the body constituted by the difference [in volume] between cone AOE and the double cone NVE, NVO, i.e., the body which triangle NOA describes when AE is revolved around OE as an axis, by the rule set out earlier. Let stanti, per premissam regulam. Illa piramis que est equalis huic differentie / [115r] sit G. Ergo si memineris priorum G, S, M piramides sunt equales medietati corporis poligonii, et earum tres bases sunt equales curve superficiei medietatis poligonii, et earum altitudo, OT sive OI sive OZ, que sunt equales medietati diametri spere que inscribitur poligonio.

Simili demonstrationis progressu probetur alia medietas corporis poligonii, scilicet AQB, esse equalis tribus piramidibus quarum altitudo sit equalis semidiametro spere inscripte solido, bases vero equales curve superficiei conice medietatis poligonii. Sit deinde piramis R fundata super basim circulum LK equalem sex basibus sex piramidum que probate sunt esse equales corpori poligonio. Sit etiam R equalis altitudinis illis sex piramidibus. Erit ergo piramis R illis sex piramidibus equalis, quia que est proportio piramidis ad piramidem eadem est basis ad basim, cum sint eiusdem altitudinis, per XI duodecimi. Erit ergo piramis R mediantibus sex piramidibus equalis corpori solido poligonio, eiusque basis circulus LK equalis conicis superficiebus corporis, et altitudo R equalis OT linee que est equalis semdiametro spere inscripte solido. Relinquitur ergo ratum quod longe diuque venati sumus.

Si tamen poligonium equilaterum et equiangulorum circulo inscribatur, cuius medietas sit laterum numero imparium, et stanti diametro circumvoluatur et faciat solidum poligonium, dubium esse potest utrum illud solidum sit equale piramidi cuius altitudo sit equalis semidiametro spere inscripte solido et basis sit circulus equalis conice superficiei solidi. Quod quia auctor non proposuit, et nos illud investigare omittimus et diligenti relinquimus posteritati*.

154 stante *C*
155 S, M *tr. A*
156–57 et...poligonii *om. A*
158 que *A* quod *BC*
160 demonstrationis...probetur: rationis progressu et demonstratione consimili probatur *A*
161 esse *om. A*
164 basem *A*
165 poligonii *A*
166–67 Erit...equalis *om. A*
167 eadem: ea *A*
168 basem *AC*
171 corporis *om. A*
172–73 longe diuque: diu longeque *A*
175 sit laterum *tr. A* / stante *C*
179–80 auctor...posteritati: non propositum ab auctore nos investigare omittimus et posteritatis diligentie relinquemus *A*

* See Section 5 below for the proof by the scribe of Manuscript *E* of the theorem for the case where the half-polygon has an odd number of sides.

the cone equal to this difference be *G*. Therefore, if you recall the earlier statements, cones *G*, *M*, and *S* are equal [in sum] to one half of the polygonal body, and their three bases are equal [in sum] to the lateral surface of half that polygonal [body], while their altitude, *OT* or *OI* or *OZ*, is equal to the radius of the sphere inscribed in the polygonal [body].

By a similar line of demonstration, let it be proved that the other half of the polygonal body, i.e., *AQB*, is equal to three cones each of whose altitudes is equal to the radius of the sphere inscribed in the solid and whose bases are equal [in sum] to the conical curved surface of the half polygonal [body]. Then let there be a cone *R** erected on base circle *LK* equal to the six bases of the six cones which have been proved to be equal [in sum] to the polygonal body. Also let *R* be of altitude equal to that of [each of] the six cones. Therefore, cone *R* will be equal to these six cones, for the ratio of cone to cone is the same as that of base to base when the cones are of the same altitude, by XII.11 [of the *Elements*]. Therefore, cone *R* by the mediacy of the six cones will be equal to the solid polygonal body, and its (*R*'s) base circle *LK* is equal to the conical surfaces of the body, and *R*'s altitude is equal to line *OT*, which is [itself] equal to the radius of the sphere inscribed in the solid. Therefore, what we sought a long time ago remains as established.

If, however, half of the regular polygon inscribed in the circle has an odd number of sides, it can be doubted whether the solid formed by the polygon's rotation about a fixed diameter is equal to a cone whose altitude is equal to the radius of the sphere inscribed in the solid and whose base is a circle equal to the conical surface of the solid. But since the author has not proposed it, we shall refrain from investigating it, leaving it to a diligent posterity.

* R is not actually constructed on the figure.

VIII. OMNIS COLUMPNA CUIUS ALTITUDO DIAMETRO SPERE ET BASIS MAXIMO CIRCULO IN SPERA FUERINT EQUALES SEXQUIALTERA EST SPERE, SICUT ET TOTA SUPERFICIES COLUMPNE SUPERFICIEI SPERE SEXQUIALTERA EST.

Ne quis veritati demonstratoris oblatret et demonstrationis tenorem interrumpat, priusquam propositum aggrediamur, quoddam elementum ad propositum perutile proponatur, et probetur quod tale est. Si fuerit proportio primi ad secundum que tertii ad quartum, fueritque id quod fit ex ductu quinti in primum maius eo quod fit ex ductu sexti in secundum, erit quoque id quod fit ex ductu quinti in tertium maius eo quod fit ex ductu sexti in quartum.

Exemplum. Sit *A* ad *B* ut *C* ad *D* et fiat ex ductu *E* in *A* quantitas *G* et ex *F* in *B* quantitas *H*; sitque *G* maior quam *H* [Fig. 72]. Et item ex *E* in *C* fiat quantitas *M* et ex *F* in *D* quantitas *S*. Dico quod *M* est maior *S*.

Dispositio. Ducatur *E* in *B* et fiat *R*, et *E* in *D* et fiat *N*. Ratio. Age. Ex *E* in *A* fit *G*; ex *E* in *B* fit *R*; ergo cum productorum et producentium eadem sit proportio, erit *G* ad *R* ut *A* ad *B*. Item ex *E* in *B* fit *R*; ex *F* in *B* fit *H*, ergo que est proportio *E* ad *F* eadem est *R* ad *H*. Rursum, ex *E* in *C* fit *M*; ex *E* in *D* fit *N*, ergo proportio *C* ad *D* est que *M* ad *N*. Item ex *E* in *D* fit *N*; ex *F* in *D* fit *S*, ergo que est proportio *E* ad *F* eadem est *N* ad *S*. Sic ergo collectis premissis, erit proportio *G* ad *R* que *M* ad *N* et *R* ad *H* que *N* ad *S*. Ergo per equam proportionem que est proportio *G* ad *H* eadem est *M* ad *S*. Sed *G* est maior *H*. Ergo *M* est maior *S*. Sic quod cum iam constet, principali attingamus proposito.

Esto exemplum *QM* columpna cuius altitudo linea *QM* est equalis

1 VIII: 8ª *D om. AC* II *B* 2ª *J* secunda conclusio *mg. B manu recentiori* | *mg. B manu recentiori*: glossum ex magna causa quod (quia *?*) includitur notabilis conclusio...
2 in spera *A om. BC*
6–7 tenorem interrumpat *A tr. BC*
7 quoddam *om. A*
8 perutile: utile *A*
13 fiat *B* fit *AC*
14 item: tunc *A*
15 fiat *AC om. B* | et *om. A*
17 E²: C *A*
19 producentium et productorum *A* | sit *om. A*
22 est *om. A*
23 eadem: ea *A*
24 R²: H *C* | N²: M *A*
26 S *om. C* | Sic *om. AC*
27 attingamur (*?*) *C*
28 est *om. A*

VIII. EVERY CYLINDER WHOSE ALTITUDE IS EQUAL TO THE DIAMETER OF A SPHERE AND WHOSE BASE IS EQUAL TO THE GREATEST CIRCLE IN THE SPHERE IS THREE HALVES THE SPHERE, JUST AS THE TOTAL SURFACE OF THE CYLINDER IS THREE HALVES THE SURFACE OF THE SPHERE.

Lest anyone carp at the soundness of the demonstrator and interrupt the course of the demonstration, let us—before we undertake that which we have proposed—posit and prove a certain principle useful for [proving] the proposition. [The principle is this:] If the ratio of the first [term] to the second is as that of the third to the fourth, and if the product of the fifth and the first is greater than the product of the sixth and the second, so also will the product of the fifth and the third be greater than the product of the sixth and the fourth.

Exemplification: Let $A/B = C/D$. Let $E \cdot A = G$, and $F \cdot B = H$, and let $G > H$ [see Fig. 72]. And also let $E \cdot C = M$, and $F \cdot D = S$. I say that $M > S$.

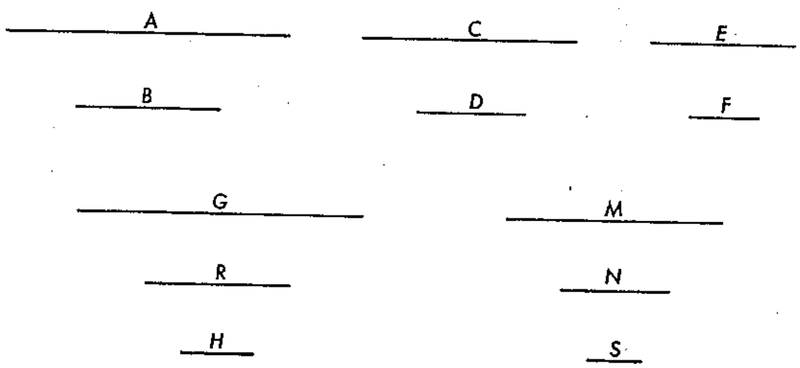

Fig. 72

Disposition: Let $E \cdot B = R$, and $E \cdot D = N$. Now proceed with the proof: With $E \cdot A = G$, and $E \cdot B = R$, therefore, since the ratio of the products and that of the quantities producing the products are the same, $G/R = A/B$. Also, with $E \cdot B = R$, and $F \cdot B = H$, therefore $E/F = R/H$; again with $E \cdot C = M$, and $E \cdot D = N$, therefore $C/D = M/N$. Also, with $E \cdot D = N$, and $F \cdot D = S$, therefore $E/F = N/S$. Therefore, with all of these statements set forth together, $G/R = M/N$, and $R/H = N/S$. Therefore, by the equality of ratios, $G/H = M/S$. But $G > H$; therefore $M > S$. Since this is now evident, let us take up the principal proposition.

For example, let there be a cylinder QM whose altitude, line QM, is

AE diametro spere *F* et basis eius circulus *RS* maximo circulo *F* spere, scilicet circulo *ADEC*, sit equalis [Fig. 73]. Dico quod columpna *QM* est sexquialtera spere *F*. Sin autem, sit *QM* sexquialtera maioris vel minoris, et primo *K* spere minoris.

Dispositio. Inscribatur *ACED* circulo poligonium equalium laterum et numero parium, circulum *H* minime contingentium. Deinde a *C* in *E* ducatur linea recta *EG*, et ab *O* centro utriusque circuli ducatur *OI* perpendicularis ad *AG*. Stanti ergo *AE*, circumvolvatur poligonium et fiat corpus conicarum superficierum. Et circumvolvatur semicirculus *ADE* fiatque spera *F*, et *HYT* semicirculus fiatque spera *K*. Medietas autem corporis conicarum superficierum dicatur *P*.

Ratio. Age. Quod fit ex ductu *AO* in circumferentiam *ADEC* est 115v duplum circuli *ADEC*, per / primam Archimenidis. Et quod fit ex *EG* in eandem circumferentiam est duplum *P* per quintam huius. Ergo que est proportio *AO* ad *EG* eadem est circuli *ADEC* ad superficiem *P*. Sed *AO* est maior *OI*, cum *I* sit rectus, et *AE* maior *EG*, cum *G* cadens in semicirculum sit rectus. Ergo maius est id quod fit ex ductu *AE* in *AO* quam ex *OI* in *EG*. Sit ergo *OA* primum, *EG* secundum, circulus *ADEC* tertium, *P* quartum, *AE* quintum, *OI* sextum. Ergo per id quod paulo ante principale propositum probavimus maius est id quod fit ex *AE* quinto in *ADEC* circulum tertium, scilicet columpna *QM*, que dicta est esse sexquialtera spere *K*, eo quod fit ex ductu *OI* sexti in *P* superficiem quartum. Sed quod fit ex ductu *OI* in superficiem *P* est triplum semisolidi, per proximam. Ibi enim probatum est quod tres piramides quarum altitudo est *OI*, bases vero sunt equales conice superficiei semipoligonii, sunt equales semipoligonio, et ad illas tres triplum est quod fit ex ductu *OI* in *P* superficiem conicam semipoligonii. Ergo quod fit ex *OI* in superficiem *P* sexquialterum est totius solidi. Ergo collectis prioribus columpna *QM*, que est sexquialtera ad speram *K*, est maior quam sexquialtera totius solidi conicarum superficierum. Ergo *K* subsexquialterum est maius quam solidum, pars suo toto, quod est impossibile. Relinquitur ergo

30 ADEC: ADOE *A* / quod: quoniam *A*
32 primo *tr. A post* minoris^2
33 ACED: ACDE *A*
39 corporis *tr. A post* superficierum
42 *ante* huius *del. B* IIII^{ti}; *C habet* IIII^{m}
43 AO: AD *A*
44 AE: AE est *A*
45 id *om. A*
48–49 principale...probavimus *AC* pr- missimus *B*
49 id *om. A* / circulum *om. A*
54 sunt^1 *om. A*
55 tres *A* tres est *BC* / triplum est *tr. A* quod: cum quod *B* / OI: M *A*
58 maior: maius *A* / sexquialterum *A*

equal to diameter AE of sphere F and whose base circle RS is equal to the greatest circle of sphere F, i.e., to circle $ADEC$ [see Fig. 73]. I say that cylinder QM is 3/2 sphere F. For if not, let QM be 3/2 [a sphere] greater or less [than F]. In the first place [let it be 3/2] a lesser sphere K.

Disposition: Let there be inscribed in circle $ACED$ a regular polygon of an even number of sides, which sides do not touch circle H at all. Then let straight line EG be drawn from G to E, and let line OI be drawn from center O of each circle perpendicular to AG. With AE as an axis, let the polygon be rotated, forming a body with conical surfaces. And let the semicircle ADE be rotated, forming sphere F; and let semicircle HYT be rotated, forming sphere K. Moreover, let half of the body with conical surfaces be designated as P.

Proceed with the proof: ($AO \cdot$ circum $ADEC$) = 2 circle $ADEC$, by the first [proposition of the *Measurement of the Circle*] of Archimedes. And ($EG \cdot$ circum $ADEC$) = (2 · surface of P), by the fifth [proposition] of this [work]. Therefore, AO/EG = circle $ADEC$/surface of P. But $AO > OI$, since $\angle I$ is a right angle; and $AE > EG$, since $\angle G$, falling in a semicircle, is a right angle. Therefore, $(AE \cdot AO) > (OI \cdot EG)$. Therefore, let OA be the first term, EG the second, circle $ADEC$ the third, [the surface of] P the fourth, AE the fifth, and OI the sixth. Hence, by that principle which we proved just before [undertaking] the principal proposition, the product of the fifth term AE and the third term circle $ADEC$, which product is equal to cylinder QM, itself said to be equal to 3/2 sphere K, is greater than the product of the sixth term OI and the fourth term surface of P. But the product of OI and the surface of P is three times the semisolid, by the previous [proposition]. For there it was proved that three cones each of whose altitudes is OI and whose bases are [in sum] equal to the conical surfaces of half the polygonal body are equal to the [volume of the] half of the polygonal body; and ($OI \cdot$ surface of P) = (3 · the three cones). Therefore, ($OI \cdot$ surface of P) = (3/2 the whole solid). Therefore, with all the prior statements brought together, [it is evident that] cylinder QM, which is 3/2 sphere K, is greater than 3/2 the whole solid with conical surfaces. Therefore, [sphere] K, which is 2/3 [cylinder QM], is greater than the solid, [that is,] the part is greater than quod columpna QM non est sexquialtera spere minoris quam F.

Dico etiam quod neque maioris. Sed ne tempus nugis teramus prior maneat figura. Et esto linea QM equalis linee HT, diametro spere K, et circulus RS equalis HYT maximo circulo spere K. Dico itaque quod columpna QM est sexquialtera spere K, falsigraphus immo spere F maioris. Inconcussis omnibus maneat dispostitio que et prius, addito eo quod AO secetur in Z, ita quod proportio AO ad HO sit tanquam HO ad ZO.

Ratio. Age. Que est proportio EG ad AO eadem est superficiei P ad circulum $ADEC$, sicut prius probavimus. Et que est AO ad ZO, id est, AO semidiametri ad HO semidiametrum, duplicata, eadem est $ADEC$ circuli ad HYT circulum, per primam duodecimi. Ergo per equam proportionem, que est proportio linee EG ad lineam ZO eadem est superficiei P ad HYT circulum. Sed quod fit ex OI in EG suum duplum maius est eo quod fit ex TH in OZ. Ergo per regulam in initio huius propositionis probatam, quod fit ex ductu OI in superficiem P est maius eo quod fit ex ductu TH diametri in HYT circulum. Sed quod fit ex ductu OI in superficiem P est sexquialterum totius solidi conicarum superficierum, per proximam. Et quod fit ex HT in circulum HYT est columpna QM, quam dixit sexquialteram spere F esse falsigraphus. Ergo maius est sexquialterum solidi conicarum superficierum quam columpna QM sexquialtera spere F. Ergo sub-sexquialterum subsexquialtero maius, solidum spera F, pars suo toto, quod est impossibile. Relinquitur ergo columpnam non esse sexquialteram maiori spere vel minori quam illi cuius diameter axi columpne et maximus circulus basi columpne equatur. Sic ergo proposito non sumus defraudati.

62 teramus: teneamus *A*
63 equalis linee *A* linea equalis *BC*
64 RS: KS *C* / circulo *tr. A ante* HYT
65 columpna QM *A* QM conica *BC* / est sexquialtera *tr. A*
75 fit *om. A*
75–76 in[2]...propositionis: superius *A*
77–78 est....P *om. A*
80 circulum HYT *tr. A*
80–81 sexquialteram...falisgraphus: falsigraphus esse sexquialteram spere F *A*
83 maius: maius est *A* / spera: spere *A*
84 columpnam: columpnam QM *C*
85 maiori spere *tr. A*
86–87 Sic...defraudate: Et sic constat propositum *A*

the whole, which is impossible. It remains, therefore, that cylinder *QM* is not 3/2 of a sphere less than *F*.

I say also that neither is it [three halves] of one greater [than *F*]. But in order not to waste time in idle speech, let us keep the prior figure [Fig. 73]. Let line *QM* be equal to line *HT*, the diameter of sphere *K*, and circle *RS* to *HYT*, the greatest circle of sphere *K*. And so I say that cylinder *QM* is 3/2 sphere *K*, while the pseudographer [says] rather that cylinder *QM* is 3/2 the larger sphere *F*. With all things unchanged, the disposition [of quantites and magnitudes] remains as before, it having been added that *AO* is cut at *Z* so that $AO/HO = HO/ZO$.

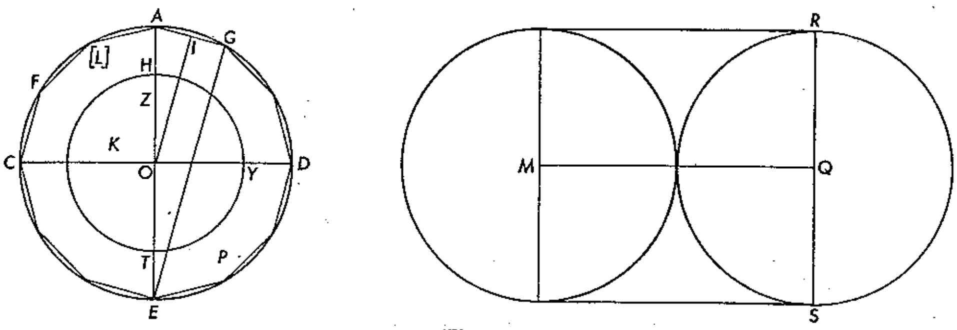

Fig. 73

Note: The figure of the cylinder *MQ* is rotaded so that altitude *MQ* stands as a vertical.

Proceed with the proof: $EG/AO =$ surface of *P*/circle *ADEC*, as we proved earlier. And $AO/ZO = (AO/OH)^2 =$ circle *ADEC*/circle *HYT*, by [Proposition] XII.1 [of the *Elements*]. Hence by the equality of ratios $EG/ZO =$ surface of *P*/circle *HYT*. But with $EG = 2\ OI$, $(OI \cdot EG) > (TH \cdot OZ)$. Therefore, by the rule proved in the beginning of this proposition, ($OI \cdot$ surface of *P*) > (diameter $TH \cdot$ circle *HYT*). However, ($OI \cdot$ surface of *P*) = (3/2 the whole solid with conical surfaces), by the preceding [proposition]; and ($HT \cdot$ circle *HYT*) = cylinder *QM*, which the pseudographer has said to be 3/2 sphere *F*. Therefore, 3/2 the solid with conical surfaces is greater than cylinder *QM*, i.e., greater than 3/2 sphere *F*. Therefore, 2/3 (3/2 the solid) > 2/3 (3/2 sphere *F*), and so the solid is greater than sphere *F*, i.e., the part is greater than the whole, which is impossible. It remains, therefore, that a cylinder is not 3/2 a sphere larger or smaller than the sphere whose diameter is equal to the axis of the cylinder and whose greatest circle is equal to the base of the cylinder. And so, therefore, we have not been deceived in our proposition.

IX. OMNIS SPERA EST EQUALIS ROTUNDE PIRAMIDI CUIUS BASIS EQUATUR SUPERFICIEI SPERE ET ALTITUDO SEMIDIAMETRO SPERE.

Esto exemplum *A* piramis rotunda, cuius basis circulus *NM* sit equalis superficiei spere *C* et *AB* altitudo *A* sit equalis *DC* semidiametro *C* spere [Fig. 74]. Dico quod spera *C* est equalis piramidi *A*.

Dispositio. Sumatur enim piramis *E*, cuius altitudo *EI* sit equalis *AB* et eius basis equalis maximo circulo spere *C*. Sumatur etiam columpna cuius basis sit equalis maximo circulo spere *C* et *GH* eius axis sit duplus ad *DC*.

Ratio. Age. *GH* columpna est duplo altior piramide *E* et sita est in equali basi cum *E*. Ergo columpna *GH* est sextupla ad *E* piramidem. Si enim essent eiusdem altitudinis columpna esset tripla ad *E* per IX duodecimi. Sed columpna *GH* sita est in basi *FG* circulo equali maximo circulo spere *C* et axis *GH* est equalis diametro spere *C* ex dispositione. Ergo columpna *GH* est sexquialtera ad speram *C*, per proximam. Sed columpna *GH* erat sextupla ad *E*. Ergo spera *C* est quadrupla ad *E*. Item ex ductu diametri *C* spere in circumferentiam 116r maximi circuli *C* spere, i.e., basis *E* pi/ramidis, sit curva superficies spere *C*, per sextam huius, scilicet *MN* circulus. Et quod fit ex ductu diametri spere *C* in circumferentiam maximi circuli spere *C* est quadruplum ad circulum maximum spere *C*, per primam Archimenidis de mensura circuli. Ergo a duplici pari circulus *MN* est quadruplus basi piramidis *E*. Sed *E* et *A* piramides sunt eiusdem altitudinis. Ergo *A*

1 IX *om.* *AC* 9ª *D* III *B* 3ª *J* / est equalis rotunde *A* rotunde est equalis *BC*
5 DC *om.* *A*
7 enim *om.* *A*
14 IX: XI *A* / FG: FH *A*
18, 19 C spere *tr.* *A*
21 C¹ *om.* *A*
22 circulum maximum *tr.* *A*
22–23 per...circuli *A* *om.* *BC*

IX. EVERY SPHERE IS EQUAL TO A CONE WHOSE BASE IS EQUAL TO THE SURFACE OF THE SPHERE AND WHOSE ALTITUDE IS EQUAL TO THE RADIUS OF THE SPHERE.

For example, let there be a cone *A*, whose base circle *NM* is equal to the surface of sphere *C* and whose altitude *AB* is equal to radius *DC* of sphere *C* [see Fig. 74]. I say that sphere *C* is equal to cone *A*.

Disposition: Let there be posited cone *E*, whose altitude *EI* is equal to *AB* and whose base is equal to the greatest circle of sphere *C*. Also let a cylinder be posited whose base is equal to the greatest circle of sphere *C* and whose axis *GH* is 2 *DC*.

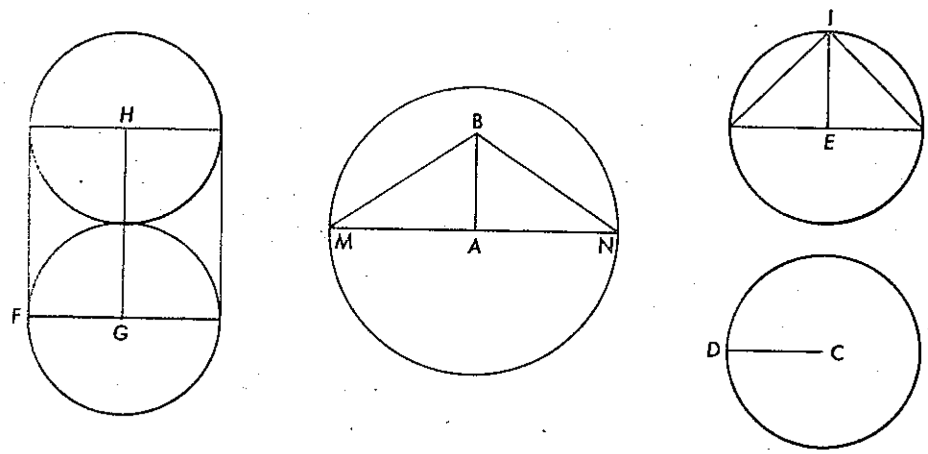

Fig. 74

Proceed with the proof: Cylinder *GH* is twice the height of cone *E* and is situated on a base equal to that of *E*. Hence cylinder *GH* = 6 cone *E*, for if the altitudes were the same, the cylinder would be three times *E*, by XII.9 [of the *Elements*]. But cylinder *GH* is situated on base circle *FG* equal to the greatest circle of sphere *C*, and axis *GH* is equal to the diameter of sphere *C* by construction. Hence cylinder *GH* = 3/2 sphere *C*, by the preceding [proposition]. But cylinder *GH* was [equal to] 6 *E*. Hence sphere *C* = 4 *E*. Also, the product of the diameter of sphere *C* and the circumference of the greatest circle of sphere *C* or of the base of cone *E* is equal to the surface of sphere *C*, by the sixth [proposition] of this [work], i.e., [this product is equal] to circle *MN*. And the product of the diameter of sphere *C* and the circumference of the greatest circle of sphere *C* is four times the greatest circle of sphere *C*, by the first [proposition] of Archimedes' *Measurement of the Circle*. Therefore, by equality twice, circle *MN* = 4 base of cone *E*. But cones *E* and *A* are of the piramis est quadrupla *E* piramidi per XII. Sed *C* spera fuerat quadrupla ad *E*. Ergo *A* piramis est equalis *C* spere. Ad quod astruendum aspiravimus.

X. CUIUSLIBET SPERE PROPORTIO AD CUBUM SUI DIAMETRI EST TANQUAM PROPORTIO UNDECIM AD VIGINTI UNUM.

Hoc respondet secunde propostitioni Archimenidis de quadratura curculi, et ex ea fidem sumit. Sed cum illa processerit per fere, et hec similiter, unde nec hec nec illa est vera; sed vulgariter per integra procedit.

Esto exemplum *D* spera et cubus fundatus super *EH*, quadratum linee *OM* equalis *AB* diametro *D* spere [Fig. 75]. Dico quod que proportio cubi *EH* ad speram *D* eadem est XXI ad XI. Fiat enim columpna *OM*, cuius tam axis quam diameter basis equatur *AB*.

Ratio. Age. Proportio *GF* quadrati ad *O* circulum est sicut XIIII ad XI, per secundam Archimenidis de quadratura circuli. Sed ex ductu *OM* altitudinis in *GF* quadratum fit cubus, et ex *OM* in *O* circulum sit columpna. Ergo cum productorum et producentium eadem sit proportio, erit proportio cubi ad columpnam tanquam *GF* basis ad *O* circulum. Ergo a pari proportio cubi ad columpnam sicut XIIII ad XI. Sed proportio columpne ad speram est sciut XXI ad XIIII, cum columpna sit sexquialtera ad speram, per anteproximam. Sit ergo cubus primum, columpna secundum, XIIII tertium, XI quartum, spera quintum, XXI sextum. Ergo per XXIIII quinti Elementorum in proportione equalitatis erunt quantus cubus ad speram tantus numerus XXI ad numerum XI.

Sicque tiphis noster portum tenet in quem iam dudum vela suc-

25 C spera *tr. A*
1 X *om. AC* 10ª *D* IIII *B* 4ª *J*
2 proportio *om. A*
4 Hec *A*
5 ex...sumit *om. A* / summit *C* / processerit *A* precesserit *BC*
8 EH: GH *A*
9 D spere *tr. A* / que *om. A*
10 cubi EH: est cubi CH *A* / eadem est: que *A*
11 basis *om. A*
12 quadrati *A* quadranguli *BC*
12–13 XIIII ad XI: 11 ad 14 *A*
20 secundum: quasi secundum *C*
21 XXIIII quinti *B* 23^{am} 12^{mi} *A* XXIIII primi *C*
23 numerum *om. A*
24–28 Sicque... Archimenidis *om. H*
24 typhis *F* / tiphis noster: timba nostra *E* classis nostra, i.e. navis *mg. D* classis nostra, i.e. navis D_1 / quem: quam *F* / iam *om. JF*
24–25 succinxerat *AEFDD*$_1$ succingserat *B* succingserat *C* succingcerat *J*

same altitude. Therefore, cone $A = 4$ cone E, by [the eleventh proposition of Book] XII [of the *Elements*]. But sphere C was 4 E. Therefore, cone $A =$ sphere C. And this is the porism to which we aspired.

X. THE RATIO OF ANY SPHERE TO THE CUBE OF ITS DIAMETER IS AS THE RATIO OF XI TO XXI.

This corresponds to the second proposition of Archimedes' *On the Quadrature of the circle*, and it assumes its credence from that [proposition]. But since that [proposition] proceeded only by approximation, as also does this one, neither this nor that one is true, but proceeds only in a common way by means of integers.

For example, let there be a sphere D and a cube based on EH which is the square of line OM, OM being equal to diameter AB of sphere D [see Fig. 75]. I say that cube EH/sphere $D = 21/11$. For let there be constructed

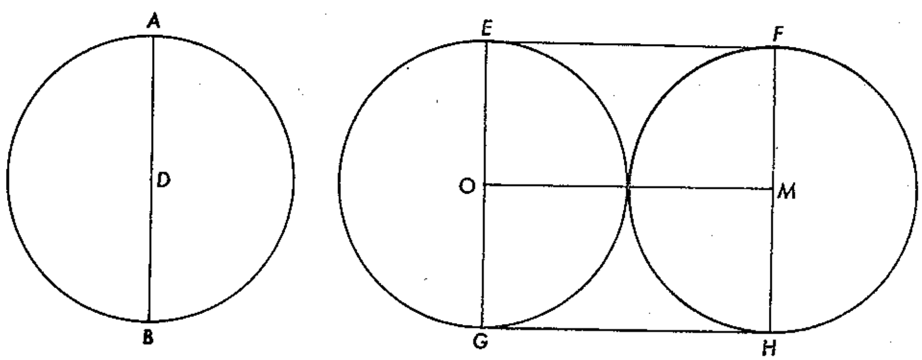

Fig. 75

Note: The figure of the cylinder OM is rotated so that altitude OM rotates as a vertical.

a cylinder OM, whose axis as well as its base diameter is equal to AB.

Proceed with the proof: Square GF/circle $O = 14/11$, by the second [proposition] of Archimedes' *Quadrature of the Circle*. But (altitude $OM \cdot$ square GF) = cube EH, and ($OM \cdot$ circle O) = cylinder OM. Therefore, since the ratios of the products and the quantities producing the products are the same, cube [EH]/cylinder [OM] = square GF/circle O. Therefore, by equality, cube [EH]/cylinder [OM] $= 14/11$. However, cylinder [OM]/sphere [D] $= 21/14$, since cylinder [OM] $= 3/2$ sphere [D], by the next to the last proposition [i.e., Proposition VIII]. Hence let the cube be the first term, the cylinder the second, 14 the third, 11 the fourth, the sphere the fifth, and 21 the sixth. Therefore, by [Proposition] V.24 of the *Elements*, cube/sphere $= 21/11$.

cinxerat. Iamque cum bibulis hereat harenis anchora Archimenidis remigii, Johannes navigationis grates ageret summo creatori.

Explicit commentum Johannis de Tinemue in demonstrationes Archimenidis.

25 bibulis *om.* D_1 *cum lacuna* / hereat: habeat D_1 / *post* anchora *add.* *BJ* simul / Archimenidis *AEJF* Archimenides *BCID*

25–28 anchora.... Archimenidis *BCID* anchora remigii Archimenidis. Explicit commentum Gervasii de Essexta (Assassia *E*) *AE*

26 Johannes: et Johannes *J* / grates ageret: grate agit *I* grates agit DD_1 / summo *tr.* *J ante* grates

27–28 Explicit... Archimenidis *om.* *JF*

27 Tinemue *expandi ex* Tin' *B* Tin̄ *C* Thin *D* Thin⁹ D_1 Tīn *I* ((Note: My expansion is based on the form given in the lower right margin of the first folio, 111r, of *B*.))

28 *post* Archimenidis *add. mg. B manu recentiori* super duos libros. hic finis *mg. superius B habet manu recentiori* Archimenides. Secundus liber. finis

And so our Tiphys reaches the port toward which he set sail but a short time ago. And now when the anchor of Archimedes the rower sits fast in the thirsty sands, let Johannes extend the thanks of navigation to the Supreme Creator.

Here ends the commentary of Johannes de Tinemue on the demonstrations of Archimedes.

COMMENTARY

Proposition I

1–5 "Cuiuslibet... basis." The form of this enunciation was obviously suggested by the first proposition of the *De mensura circuli*. This proposition must be distinguished from, but compared to, those given by Archimedes in the *De sphaera et cylindro* (Proposition 14 of Book I) and the Banū Mūsā in the *Verba filiorum* (Proposition IX). For the citation of these and the comparable propositions of Pappus and Leonardo Pisano, see Commentary, *Verba filiorum*, Proposition IX, page 63. The proof is for a right circular cone.

15–17 "Rationis... admitti." This supposition is comparable to the first postulate of the Cambridge Version of the *De mensura circuli* (see Chapter Three, Section 1, above). These lines, together with lines 18–22, form a rather intrusive comment, which perhaps suggests that Johannes was commenting on a primitive form of this work rather than on the *De sphaera et cylindro*.

23–102 "Triangulus.... proposui." We can represent the spirit of the proof in an abbreviated form (see Fig. 64), using modern symbols as follows.*

Proof:

(1) To prove: $S = F$, where S is the lateral surface of a right circular cone of slant height l and base circumference c, and F is the area of right triangle such that $F = \frac{1}{2}(l \cdot c)$.

(2) Either $F = S$ or $F \neq S$. If $F \neq S$, then $F > S$, or $F < S$.

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$l = SZ =$ slant height EB	$t =$ side of polygon $ACNP$
$c = ST =$ circum of circle $ACNP$	$h =$ altitude of face triangle EAB
$c_1 =$ circum of circle HK	

(3) Suppose first that $F < S$, and by the basic postulate $F = S_1$, where S_1 is the lateral surface of some cone whose altitude is the same as that of the cone of surface S, but whose base is less than that of cone S. Let c_1 be the circumference of the base of S_1. Hence, $c_1 < c$.

(4) Now inscribe in c a regular polygon, one of whose sides is t and whose perimeter is p, such that $c > p > c_1$ and p does not touch c_1 at any place. (Cf. Proposition III of the *Verba filiorum*.) Then construct a pyramid with a polygon of perimeter p as the base and with an identical altitude as that of cone of surface S. Let the surface of this pyramid be S_2. Let the altitude of one of its triangular faces be h.

(5) Then $S_2 = \frac{1}{2}(h \cdot p)$, since the area of any one triangle is $\frac{1}{2}(h \cdot t)$ and the sum of all t's is p.

(6) Hence $S_2 < F$, since $F = \frac{1}{2}(l \cdot c)$ and $l > h$ and $c > p$.

(7) Therefore $S_2 < S_1$, from (3) and (6).

(8) But S_2 was constructed as including S_1; hence, in actuality $S_2 > S_1$.

(9) Since (7) is the contradiction of the fact of (8), the supposition from which (7) was derived, namely, that $F < S$, is false.

(10) Then suppose, if it is possible, that $F > S$ and thus that $F = S_1$ where S_1 is the surface of a cone of the same altitude as cone S but whose circumference c_1 of its base is greater than c.

(11) Now circumscribe about c a regular polygon of side t and perimeter p such that $c_1 > p > c$ and p does not touch c_1. Erect on p a pyramid of surface S_2 with the same altitude as the cone of surface S, and, as before, let h be the altitude of one of its triangular faces.

(12) Then $S_2 = \frac{1}{2}(h \cdot p)$. (Cf. step 5.)

(13) Hence $S_2 > F$, since $h = l$ and $p > c$.

(14) Therefore $S_2 > S_1$, since $F = S_1$ by hypothesis.

(15) But S_2 was constructed, in fact, within S_1, and so $S_2 < S_1$.

(16) Since (14) is contradictory to the fact of (15), the supposition from which (14) was derived, namely, $F > S$, is false.

(17) Hence, since $F \not> S$ and $F \not< S$, then $F = S$, or $S = \frac{1}{2}(l \cdot c)$. Q.E.D.

49 "per [41] theorema primi." See Commentary, Cambridge Version of the *De mensura circuli*, Chapter Three, Section 1, line 33.

52 "primam secundi." See Commentary, Corpus Christi Version, Chapter Five, Section 5, lines 66–67.

105–117 "Ex.... basis." This corollary is identical to Proposition 15 of Book I of Archimedes' *De sphaera et cylindro* (ed. of Heiberg, p. 69). Incidentally, the actual wording of Johannes' quotation of Proposition I differs from that of the Gerard of Cremona translation. I would suspect that, if this were an original Latin composition rather than a translation, Gerard's translation would have been quoted rather exactly, as is usually the case with later Latin geometrical treatises. In short, the somewhat different form of the quotation from Proposition I of the *De mensura circuli* may be another indication that we are faced here with a translation rather than an original composition.

Proposition II

1–22 "Cuiuslibet.... propositum." Cf. the comparable but different theorem of Archimedes' *De sphaera et cylindro*, Book I, Proposition 13 (ed. of Heiberg, pp. 53–62). The proof is similar to that for the first proposition and so is merely sketched by the author. Again I use modern symbols to present the tenor of the proof.

The given rectangle of adjacent sides equal to the axis of the cylinder and the circumference of its base must be either equal to the lateral surface S of the cylinder or not equal to it. If it is not equal to S, then it must be equal to the surface S_1 of some other cylinder with the same axis but with a smaller or larger base. Suppose first that the quadrangle is equal to the lateral surface S_1 of a cylinder of smaller base. Then inscribe within the base of the cylinder of surface S a regular polygon whose perimeter is greater than the circumference of the base of the cylinder of S_1 and whose sides in no place touch that circumference. Then construct a prismatic figure within S the base of which figure is the regular polygon previously inscribed. The lateral surface of this prism can be shown to be at the same time smaller and larger than the surface S_1, an obvious contradiction negating the assumption that the surface S_1 of a cylinder with a base smaller than that of the cylinder of surface S is the desired surface to which the rectangle must be equated. In the second half of the proof the surface of a similar prism circumscribed around S is shown to be simultaneously larger and smaller than the surface S_1 of a cylinder with the same axis as that of the cylinder of surface S but whose base is larger. From this contradiction it is apparent that the desired surface S_1 cannot be of a cylinder of larger base. Since then S_1 cannot be the surface of a cylinder of either larger or smaller base, it must be the surface of a cylinder of the same base as that of which S is the surface. In short, S_1 must be identical with S, and so the given rectangle is equal to S. Q.E.D.

23–40 "Ex.... theoremati." Both corollaries are proved by the application of simple proportions.

Proposition III

1–35 "Quorumlibet.... sufficienter." For a discussion of the comparable theorems given by Pappus and the authors of the *Verba filiorum*, see the Commentary to Proposition V of the *Verba filiorum* in Chapter Four above. This proposition is patently an auxiliary theorem to be used later; it does not concern itself with a "curved surface," as do the other theorems of this treatise. Johannes' proof (see Fig. 65) of this theorem follows:*

(1) Let d_1, d_2 be the diameters of circles whose circumferences are c_1, c_2.

(2) Then either $d_1/d_2 = c_1/c_2$ or $d_1/d_2 \neq c_1/c_2$. And if $d_1/d_2 \neq c_1/c_2$, then $d_1/d_2 = c_1/c_3$ where c_3 is either greater or less than c_2.

(3) Suppose first that $d_1/d_2 = c_1/c_3$ where c_3 is less than c_2.

(4) Now describe a regular polygon within c_2 such that its perimeter p_2 is less than c_2 but greater than c_3 (Cf. Proposition I, step 4). Also describe a similar polygon within c_1 whose perimeter is p_1.

(5) Then $c_1/p_1 = c_3/p_2$ because (a) $d_1/d_2 = c_1/c_3$ by hypothesis, and (b) $d_1/d_2 = p_1/p_2$ by Euclid, XII.1. (See Commentary, line 14, below.)

(6) But $c_1 > p_1$ and thus $c_3 > p_2$. (Cf. *De Sphaera*, Book I, *pos* postulate 5.)

(7) However $p_2 > c_3$ by construction.

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$d_1 = CA$
$d_2 = GE$
c_1 = circum of circle *CBA*
c_2 = circum of circle *FEG*
c_3 = circum of circle *IK* (1st half of proof)
c_3 = circum of circle *MS* (2nd half of proof)
c_4 = circum of circle *H*
p_1 = perim of polygon *BAC*
p_2 = perim of polygon *FEG*

(8) Since (7) is in fact true and contradicts (6), the inference of (6) must be false and thus the supposition from which it was drawn, namely, that $c_3 < c_2$, must be false.
(9) Now suppose that $d_1/d_2 = c_1/c_3$ where $c_3 > c_2$.
(10) Then let $d_2/d_1 = c_2/c_4$ where $c_4 \neq c_1$.
(11) Then $c_3/c_2 = c_1/c_4$ because $c_3/c_1 = c_2/c_4$, this being true by eliminating d_1 and d_2 from the equations of (9) and (10).
(12) Hence $c_1 > c_4$ since from (9) $c_3 > c_2$.
(13) Therefore $d_2/d_1 = c_2/c_4$ where $c_4 < c_1$.
(14) But (13) can be shown to be false by steps similar to those of (3) to (8). Hence if (13) is false, the supposition from which it was derived is false, namely, that $c_3 > c_2$.
(15) Since $c_3 \not> c_2$ and $c_3 \not< c_2$, then $c_3 = c_2$ and $d_1/d_2 = c_1/c_2$. Q.E.D.

The author concludes the proof with the statement that the proposition can be proved in another way by the application of the third proposition of the *De mensura circuli* of Archimedes.

14 "primam duodecimi." Although the author of the *Liber de curvis* no doubt precedes the Adelard II translation in point of time, in order to keep uniformity of citation (and since the proposition as given in Adelard II is basically the same as in the Greek text) I shall quote the Adelard II Version (Brit. Museum, Add. 34018, 63r): "Omnium duarum superficierum multiangularum inter duos circulos descriptarum est proportio alterius ad alteram tanquam proportio quadratorum que ex diametris circulorum eas circumscribentium proveniunt." Now if $\text{polygon}_1/\text{polygon}_2 = d_1^2/d_2^2$ [by Proposition XII.1] and if $\text{polygon}_1/\text{polygon}_2 = p_1^2/p_2^2$, then it is obvious that $d_1/d_2 = p_1/p_2$, as stated in step (5) of the argument given in the previous comment.

Proposition IV

1–37 "Quarumlibet.... proposuimus." Again (see Fig. 66) let us restate Johannes' argument.* In the fourth proposition Johannes will prove that $S_2 - S_1 = \frac{1}{2}(l_2 - l_1) \cdot (c_2 + c_1)$ where S_2 and S_1 are the lateral surfaces of similar cones and l_2 and l_1 are their slant heights and

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$l_1 = IM = SZ$	$c_1 = ZY =$ circum of circle MNA
$l_2 = CE = ST$	$c_2 = TP =$ circum of circle EQB

c_2 and c_1 are the circumferences of their bases. Cf. Archimedes, *De sphaera et cylindro*, Book I, Proposition 16, and *Verba filiorum*, Proposition XI; both are cited in the commentary to Proposition XI of the *Verba filiorum* in Chapter IV above. The proof follows:

(1) Let there be a right traingle $F_2 = \frac{1}{2}(l_2 \cdot c_2)$, and thus, by Proposition I of the *De curvis*, we can say that $S_2 = F_2$.

(2) From l_2 subtract the length l_1 and erect a line x parallel to c_2. There will be formed then a triangle $F_1 = l_1 \cdot x$ similer to triangle F_2.

(3) Then $l_2/l_1 = c_2/x$ since F_1 is similar to F_2.

(4) Hence $x = c_1$ since (a) $l_2/l_1 = d_2/d_1$ by definition of similar cones, and (b) $d_2/d_1 = c_2/c_1$ by Proposition III of the *De curvis*, and hence (c) $l_2/l_1 = c_2/c_1$.

(5) Hence $F_1 = S_1$ by Proposition I of the *De curvis*.

(6) Thus $S_2 - S_1 = F_2 - F_1$.

(7) But $F_2 - F_1$ is a quadrangular area whose base is $l_2 - l_1$ and whose erect sides are c_2, c_1. This area can be divided into two triangles with bases respectively c_2 and c_1 and with the same altitude $l_2 - l_1$.

(8) $F_2 - F_1 = \frac{1}{2}(l_2 - l_1) \cdot (c_2 + c_1) = S_2 - S_1$, by the addition of the triangles formed in (7). Q.E.D.

Proposition V

1–117 "Si....propositum." Cf. *De sphaera et cylindro*, Book I, Proposition 24 (ed. of Heiberg, 95–97). The fifth proposition is crucial for Johannes' ultimate determination of the surface area of the sphere. This proposition finds the surface area of a solid composed of conoid segments generated by the revolution of a regular polygon inscribed in a circle around one of the diameters of the circle. Although the author does not stipulate in the statement of the proposition the number of sides of the original polygon whose revolution describes the solid in question, like Archimedes he assumes in his proof that the number of sides of this polygon is divisible by four.

To understand what Johannes is proving and the nature of his proof,* let us take an equilateral polygon of $4n$ sides, each side of magnitude t (see Fig. 67). Let this polygon be inscribed in a circle

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$c = ADIH$	$h_1 = YI$	$r_1 = B$[illegible]
$d = AI$	$h_2 = TY$	$r_2 =$ [illegible]
$f = EH$	$h_3 = ST$	$r = DE$

of circumference c and diameter d. Then let the circle with the inscribed polygon be rotated about a diameter of the circle. By this rotation the circle describes a sphere and the polygon describes a figure with conical surfaces (cf. *De sphaera*, Book I, Proposition 24).

The first part of Johannes' proposition states that the surface of this solid S will be given as follows: $S = t\,(c_1 + c_2 + \ldots + c_{n-1})$ where $c_1, c_2, \ldots, c_{n-1}$ are the circumferences of the circles described by the vertices of the angles of the polygon in the course of its rotation; t is the side of the polygon.

The second part of the proposition affirms further that the same surface will also be given by this relationship: $S = c \cdot f$, where f is one of the sides of the right triangles whose hypotenuse is the diameter of the circle, d, and whose other side is a side t of the polygon; c is the circumference of the circle.

The first part of the proposition is easily proved by adding up the conical surfaces $s_1, s_2, \ldots, s_n$ described by n sides of the polygon as it rotates. These surfaces are, of course, equivalent to the total surface S. The proof follows:

(1) $s_1 = \frac{1}{2}\, t \cdot c_1$ by *De curvis*, Proposition I.

(2) $s_2 = \frac{1}{2}\, t\,(c_1 + c_2)$ by *De curvis*, Proposition IV.

$s_3 = \frac{1}{2}\, t\,(c_2 + c_3)$ by *De curvis*, Proposition IV.

....

$s_{n-1} = \frac{1}{2}\, t\,(c_{n-2} + c_{n-1})$ by *De curvis*, Proposition IV.

$s_n = \frac{1}{2}\, t \cdot c_{n-1}$ by *De curvis*, Proposition I.

(3) Hence $S = t\,(c_1 + c_2 + \ldots + c_{n-1})$ by addition. Q.E.D.

The second part of the proposition, namely, $S = c \cdot f$, is proved as follows (referring to Fig. 67):

(1) $h_1/r_1 = h_2/r_1 = h_3/r_2 = \ldots\ h_{2n-3}/r_{n-1} = h_{2n-2}/r_{n-1} = t/f$ by the proportional sides of similar triangles.

(2) Now $h_1 + h_2 + h_3 + h_4 + \ldots + h_{2n-3} + h_{2n-2} = d$ and $r_1 + r_1 + r_2 + r_2 + \ldots + r_{n-1} + r_{n-1} = d_1 + d_2 + \ldots + d_{n-1}$.

(3) Thus $d/(d_1 + d_2 + \ldots + d_{n-1}) = t/f$ from (1) and (2) together.

(4) But $d/(d_1 + d_2 + \ldots + d_{n-1}) = c/(c_1 + c_2 + \ldots + c_{n-1})$ by *De curvis*, Proposition III.

(5) Hence $t/f = c/(c_1 + c_2 + \ldots + c_{n-1})$.

(6) But $S = t\,(c_1 + c_2 + \ldots + c_{n-1})$ from the first part of this proposition.

(7) Hence $S = c \cdot f$. Q.E.D.

An additional proof for a solid generated by the revolution of a semipolygon of odd-numbered sides follows the same form of the principal proof. It needs no further analysis.

78–79 "penultimam quinti." Although, as I have said, it is improbable that this work is an original Latin composition and hence the author had no access to the Adelard II version of the *Elements*, we can cite the Adelard wording of Proposition V.24 (Ms. cit., f. 21r) for the sake of uniformity of citation throughout this volume and since the Adelard reading of this proposition is equivalent to the reading of the Greek text: "Si fuerit proportio primi ad secundam tanquam tertii ad quartum, proportio vero quinti ad secundum tanquam sexti ad quartum, erit proportio primi et quinti pariter acceptorum ad secundum tanquam sexti et tertii pariter acceptorum ad quartum."

Proposition VI

1–41 "cuislibet....propositum." The comparative propositions of Archimedes and the Banū Mūsā are noted below in the Commentary to lines 42–56. Johannes is now prepared to treat the surface area of a sphere which he finds in this proposition to be equal to a rectangle, one of whose two adjacent sides is equal to the diameter of the sphere and the other to the circumference of the sphere. Restating Johannes' proposition symbolically, it reads $S = d \cdot c$ where S is the surface of the sphere, d and c the diameter and circumference of a great circle of the sphere. Proof* (see Fig. 69):

(1) Either $d \cdot c = S$, or $d \cdot c \neq S$.

(2) If $d \cdot c = S$, then $d \cdot c = S_1$ where $S_1 < S$ or $S_1 > S$ (by the basic postulate).

(3) Let us assume first that $d \cdot c = S_1$ where $S_1 < S$. Let us further suppose that S_1 is the surface of a sphere concentric with S. We designate the circumferences of great circles of these spheres c_1 and c. Of course $c > c_1$.

(4) Now inscribe within c an equilateral polygon of side t and of perimeter p such that $c > p > c_1$ and p does not touch c_1. Then let c revolve about d to produce S and similarly c_1 to produce

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$p = AF + FD + \ldots + ZA$	$c = ADBC$
$t = AF$	$c_1 = SNH$
$d = AB$	$q = FB$

S_1 and p to produce the surface S_2 composed of a series of conical surfaces.

(5) Then $d \cdot c > q \cdot c$ since $d > q$, d being opposite a right angle.

(6) Hence $S_1 > S_2$ since (a) $d \cdot c = S_1$ from (3) and (b) $q \cdot c = S_2$, by *De curvis*, Proposition V.

(7) But, in fact, $S_2 > S_1$ by construction.

(8) Since (6) is contrary to the fact of (7), then (6) is false, and the assumption from which it follows is false, namely, that $S_1 < S$.

(9) By a similar series of steps Johannes derives a contradiction from $S_1 > S$.

(10) Hence, if $S_1 \ngtr S$ and $S_1 \nless S$, $S_1 = S$, and thus $d \cdot c = S$. Q.E.D.

42–56 "Ex.... probandum." Three corollaries to this proposition are given and proved from it easily. The first states that the surface of the sphere is equal to four times the area of a great circle of the sphere. (For the statement of this by Archimedes and the Banū Mūsā, see the Commentary to Proposition XIV of the *Verba filiorum* in Chapter IV above.) From the first proposition of the *De mensura circuli*, Johannes asserts that the area of a circle is equal to one quarter the product of the diameter and the circumference. From his own Proposition VI he gives the surface of a sphere as the product of the diameter and circumference; hence the surface of the sphere is four times the area of a great circle.

The second corollary states that the surface of a sphere is equal to the lateral surface of a cylinder whose axis and base diameter are equal to the diameter of the sphere. This corollary is immediately evident from Proposition II of the *De curvis* taken together with Proposition VI. Similarly, if we add the area of the two end circles of such a cylinder to find its total surface, we shall have $S = \frac{3}{2}\, d \cdot c$, as the last corollary asserts.

Description and Proposition VII

2–4 "Omne... contineri." Although this so-called "description" is phrased in a very general way, it is used in Proposition VII only to justify the conclusion that the volume of a right cylinder is equal to the product of the base and altitude.

5–173 "Omne.... sumus." Having in the first six propositions determined the areas of certain "curved" surfaces including finally that of a sphere, Johannes is now ready to go on to the problem of the volumes of the solids bounded by the curved surface areas already discussed, with the primary objective of determining the volume of a sphere. Before going to the question of the volume of a sphere, Johannes introduces this crucial Proposition VII, which seeks the volume of the solid whose area has been found in Proposition V, i.e., the solid generated by the the rotation of a regular polygon inscribed in a circle around the diameter of the circle. This is virtually the same theorem as that given by Archimedes, *De sphaera et cylindro*, Book I, Proposition 26 (ed. of Heiberg, pp. 101–105). Archimedes' proof, however, is different from the one presented here by Johannes. Johannes again implies in this proposition, although he does not state it, that the number of the sides of the polygon is divisible by four.

Proposition VII states, and Johannes ventures to prove, that $V = P = \frac{1}{3}S \cdot r_i$ where V is the volume of the solid formed by the rotation of a regular polygon of $4n$ sides around the diameter of the circle in which it is inscribed, P is the volume of an equivalent cone, S is the surface of the solid of volume V, and r_i is the radius of a sphere inscribed in the solid of volume V. Proof* (see Fig. 70):

(1) Construction: Take a quarter of the circle circumscribing the polygon of $4n$ sides. Inscribed in that quarter there will be n sides (Johannes takes a figure which has 3 sides; I have generalized the proof for a figure with n sides), which we designate $t_1, t_2, \ldots, t_n$. If we connect the midpoints of these sides with the center of the circle, we shall have equal lines $r_{i1}, r_{i2}, \ldots, r_{in}$ each of them equal to the radius of a circle inscribed within the polygon. Then if we connect the center with the extremities of the sides of the polygon, we shall have equal lines $r_{c1}, r_{c2}, \ldots, r_{cn}$, each equal to the radius of the circle in which the polygon is inscribed. Now t_1 intersects r_{c1}; but extend t_2 by the line e_1 and extend r_{c1} by f_1. Similarly, extend t_3 until it joins the extension of r_{c1}. Call these extensions e_2 and f_2 respectively. Make similar extensions for the remaining sides. If we rotate this quarter of a circle with its appended lines, the quarter itself forms a hemisphere, and the quarter of the regular polygon forms a solid of volume $V/2$. Each side of the polygon

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$r_{c1} = OC$	$e_1 = FD$	$r_{i2} = OZ$	$t_3 = AN$
$r_{c2} = OF$	$e_2 = NE$	$r_{i3} = OT$	$f_1 = DC$
$r_{c3} = ON$	$r_1 = FP$	$t_1 = FC$	$f_2 = DE$
$r_{c4} = OA$	$r_{i1} = OI$	$t_2 = NF$	

describes conical surfaces $s_1, s_2, \ldots, s_n$. The circles forming the bases of these conical surfaces are of areas $a_1; a_1, a_2; \ldots a_{n-1}, a_n$; the radius of a is r_1, that of a_2 is r_2, etc. Furthermore, $V/2$ is composed of a series of special volumes, $v_1 + v_2 + \ldots + v_n$; v_1 is the volume of the double cone formed by the rotation around r_{c1} of the triangle of sides t_1, r_{c1}, and r_{c2}; v_2 is the volume generated by the revolution of the triangle of sides t_2, r_{c2}, r_{c3}. And similarly for $v_3 \ldots v_n$.

The following auxiliary volumes should be noted:

(a) v_{e1} formed by the rotation of triangle with sides e_1, $r_{c1} + f_1$, and r_{c2}.

(b) v_{e1t2} formed by the rotation of the triangle with sides $e_1 + t_2$, $r_{c1} + f_1$, and r_{c3}.

(c) v_{e2} formed by the rotation of the triangle of sides e_2, $r_{c1} + f_1 + f_2$, and r_{c3}.

(d) v_{e2t3} formed by the rotation of the triangle of sides $t_3 + e_2$, $r_{c1} + f_1 + f_2$, r_{c4}.

(2) Now $v_1 = \frac{1}{3}r_{i1} \cdot s_1$ because (a) $t_1/r_1 = r_{c1}/r_{i1}$ by similar triangles; and (b) $t_1/r_1 = s_1/a_1$ by *De curvis*, Corollary to Proposition I; and thus (c) $r_{c1} \cdot a_1 = r_{i1} \cdot s_1$; but (d) $r_{c1} \cdot a_1 = 3\ v_1$ by Euclid, Proposition XII.9 (Gr. XII.10).

(3) And $v_2 = \frac{1}{3}r_{i2} \cdot s_2$ because (a) $v_2 = v_{e1t2} - v_{e1}$ and (b) $v_{e1t2} = \frac{1}{3}r_{i2} \cdot (s_e + s_2)$ by steps like (2) (note: s_e is the surface generated by e_1 in its revolution); and (c) $v_{e1} = \frac{1}{3}r_{i2} \cdot s_e$.

(4) Similarly $v_3 = \frac{1}{3}r_{i3} \cdot s_3$.

....

$v = \frac{1}{3}r_{in} \cdot s_n$.

(5) Hence $V = \frac{1}{3}r_i \cdot S$ because (a) $r_{i1} = r_{i2} = r_{i3} = \ldots = r_{in} = r_i$; (b) $V/2 = v_1 + v_2 + \ldots + v_n$; and (c) $S/2 = s_1 + s_2 + \ldots + s_n$. Q.E.D.

22, 37, 44–45, 63, 83, 142, 145 "IX duodecimi." This is Proposition XII.10 in the Greek text. Notice that it is XII.9 in both the Adelard and Arabic texts. For the possible bearing of this fact upon the question of whether the *Liber de curvis superficiebus* is an original Latin composition or translation, see note 8 of the Introduction to this section. The text of the proposition runs in the Adelard II Version as follows (*ms. cit.*, 65v): "Omnis columna rotunda pyramidi sue tripla esse probatur."

50 "primam secundi." See the Commentary, the Corpus Christi Version, Chapter Three, Section 5, lines 66–67.

95–113 "Regula.... insistamus." Here Johannes digresses to prove by reference to Euclid that the difference between two unequal cones of the same altitude is equal to a cone of the same altitude whose base is equal to the difference between the bases of the original cones.

104, 110, 168 "XI duodecimi." Proposition XII.11 in the Adelard II Version of the *Elements* runs (*ms. cit.*, 66r): "Omnes duas rotundas pyramides sive columnas eque altas suis basibus proportionales esse necesse est."

174–80 "Si.... posteritati." The significance of this passage for the question of whether this tract is an original composition or transaction is discussed in note 8 of the Introduction to this section.

Proposition VIII

1–5 "Omnis.... est." The eighth proposition of the *De curvis* is the familiar corollary of Archimedes' *De sphaera et cylindro*, Book I, Proposition 34 (ed. of Heiberg, pp. 131–33), asserting that a cylinder with an altitude equal to the diameter of a sphere has a volume 3/2 that of the sphere. It was also known to the Latin readers of the anonymous *De ysoperimetris*. (See Appendix III, paragraph 8.) It should be noted that with Johannes this is the fundamental theorem regarding the volume of a sphere, while with Archimedes this proposition is derived from an earlier more fundamental proposition. Hence there is no direct correspondence between the proofs of Johannes and Archimedes.

6–27 "Ne.... proposito." Johannes initiates the proof with an *elementum* on proportions to the effect that if there are six terms, a through f, such that $a/b = c/d$ and $e \cdot a > f \cdot b$, then $e \cdot c > f \cdot d$.

28–87 "Esto.... defraudati." The proof (see Fig. 73) can be summarized as follows*: $C = \frac{3}{2} V$, where V is the volume of a sphere whose diameter is d and C is the volume of a cylinder whose altitude and base diameter both are equal to d.

(1) Either $C = \frac{3}{2} V$ or $C \neq \frac{3}{2} V$.

(2) If $C \neq \frac{3}{2} V$, then there is some sphere V_1 such that $C = \frac{3}{2} V_1$. And either $V_1 < V$ or $V_1 > V$.

(3) Assume first that $V_1 < V$, with V and V_1 being concentric.

(4) Construction: Inscribe within c, the circumference of a great

* The following table relates the symbols used in this commentary to the quantities given in the figure and text:

$c = ADEC$	$h = OI$	$r = AO$
$c_1 = HYT$	$q = GE$	$d = AE$
$p = AG + \ldots + [L]A$		

circle of V, a regular polygon of perimeter p such that $c > p > c_1$ where c_1 is the circumference of a great circle of sphere V_1 and where p does not touch c_1. Construct h and q as in Proposition VI. Let V_2 be the volume of the solid formed by the rotation of the polygon around the diameter d. Let S be the surface of that solid. P is the surface of half that solid. The radius of sphere V is r. The area of great circle of radius r we designate as A.

(5) Then $r/q = A/P$ because (a) $r \cdot c = 2\,A$ by *De mensura circuli*, Proposition I; and (b) $q \cdot c = 2\,P$ by *De curvis*, Proposition IV.

(6) And $r \cdot d > h \cdot q$ since $r > h$ and $d > q$.

(7) Hence $d \cdot A > h \cdot P$ since in (5) and (6) we have six terms fulfilling the conditions of the *elementum* advanced at the beginning of the proposition.

(8) Thus $C > \frac{3}{2}\,V_2$ because (a) $d \cdot A = C$ and (b) $h \cdot P = \frac{3}{2}\,V_2$.

(9) But $C > \frac{3}{2}\,V_1$ since $V_2 > V_1$ by construction.

(10) Thus the fact of (9) contradicts the assumption of $C = \frac{3}{2}\,V_1$ where $V_1 < V$; and hence that assumption must be false.

(11) Assuming the other possibility, that $C = \frac{3}{2}\,V_1$ where $V_1 > V$, Johannes shows by similar steps that a contradiction follows.

(12) Hence, if $V_1 \nless V$ and $V_1 \ngtr V$, $V_1 = V$ and $C = \frac{3}{2}\,V$. Q.E.D.

The proof for the surfaces was given in Proposition VI.

72 "primam duodecimi." See the Commentary to Proposition III, line 14.

Proposition IX

1–27 "omnis.... aspiravimus." The ninth proposition equates the volume of a sphere to a cone whose base is equal to the surface of the sphere and whose altitude is equal to the radius of the sphere, that is, $V = \frac{1}{3}\,d \cdot c \cdot r$. This formulation is equivalent to the more familiar modern form using π, namely, $V = \frac{4}{3}\pi\, r_3$. Johannes' proof is simple. The relationship between a cylinder and cone is known and also that between a cylinder and a sphere. Hence the relationship between a cone and a sphere immediately follows. Cf. the statements of this proposition by Archimedes, the Banū Mūsā, Leonardo Pisano, and the author of the *De ysoperimetris* in the Commentary to Proposition XV of the *Verba filiorum* (Chapter Four above).

14 "IX duodecimi." See the Commentary to Proposition VII, line 22.

25 "XII." See the Commentary to Proposition VII, line 104.

Proposition X

1–23 "Cuislibet.... XI." The tenth and the last proposition of the *De curvis* has no parallel proposition in the *De sphaera et cylindro*. But it represents, as Johannes says, a proposition similar to that of the second proposition of the *De mensura circuli*. Proposition II of the *De mensura circuli* showed that, if we assume a value of π as $3\frac{1}{7}$, the ratio of the area of a circle to the square on its diameter is as 11 to 14. Making a similar approximation for π, this last proposition of the *De curvis* asserts that the ratio of the volume of a sphere to cube of its diameter is as 11 to 21. Johannes carefully points out at the beginning of the proof that both he and Archimedes are proceeding by approximation, and actually neither Archimedes' nor his own proposition is true (*vera*). Inverting his ratios, we can represent his proof as follows:

To prove: $V/d^3 = 11/21$

(1) $C/d^3 = 11/14$, C being the volume of a cylinder of axis d and base diameter d, because (a) $C/d^3 = A/d^2$, A being the area of a circle of diameter d; (b) $A/d^2 = 11/14$, by *De mensura circuli*, Proposition I.

(2) $C/V = 3/2$, by the *De curvis*, Proposition VIII.

(3) Hence $V/d^3 = 11/21$. Q.E.D.

21 "XXIIII quinti." Proposition V.24 runs in the Adelard II Version (*ms. cit.*, 21r): "Si fuerit proportio primi ad secundum tanquam tertii ad quartum, proportio vero quinti ad secundum tanquam sexti ad quartum, erit proportio primi et quinti pariter acceptorum ad secundum tanquam sexti et tertii pariter acceptorum ad quartum."

3. Two Propositions Interposed in the Text of the *Liber de curvis superficiebus*

In the fourteenth-century manuscript *D* (Florence, Bibl. Naz., Conv. Soppr. J.V.30) of the *Liber de curvis superficiebus* there occur on the margin of folio 1r and 1v two additional (or interposed) propositions concerned, respectively, with the difference between the lateral surfaces of two cones and with the difference between the lateral surfaces of two cylinders. I have not found these additional propositions in any other medieval manuscript but they are present in manuscript D_1 (Paris, Bibl. Nat., Fonds latin 11247), which is a careless copy made from D in the sixteenth century. In D_1 the first additional proposition is placed in the body of the text after Proposition I, with a marginal note stating that "it was in the margin" (*erat in margine*), which means, of course, that the additional proposition was in the margin of manuscript D from which D_1 was copied. Similarly, the second additional proposition was placed by the scribe of D_1 in the text of the *Liber de curvis superficiebus* after Proposition III. Again, the scribe of D_1 noted that *hoc erat in margine*.

It seems evident that these two additional propositions had some currency from the early fourteenth century in spite of our not being able to find further manuscripts including them. For, as I indicated in Section 2 of this chapter (see notes 21, 22, and 24), from at least the year 1328, geometers tended to cite Proposition III of the *Liber de curvis superficiebus* as the fifth proposition. The most plausible way to account for this is to assume that there existed a commonly known text (the source of D) in which the two additional propositions had been added to the body of the text before the regular Proposition III.

The first additional proposition depends directly on Proposition I of the *Liber de curvis superficiebus*. It holds that the difference between the lateral surfaces of two unequal cones having equal slant heights is equal to the lateral surface of a third cone with the same slant height but with a base circumference (or diameter) equal to the difference between the base circumferences (or diameters) of the first two cones. By Proposition I of the *Liber de curvis superficiebus*, each of the surfaces of the two unequal cones is equated to a right triangle in which the right angle is contained by a line equal to the slant height and a line equal to the circumference of the base. The triangular difference between the two right triangles is then shown to be equal to a third right triangle which is equal to the lateral surface of the third cone. This proposition should be compared with Proposition IV of the *Liber de curvis superficiebus*.

The second additional proposition depends on Proposition II of the *Liber de curvis superficiebus*. It states that the difference between the lateral surfaces of two unequal cylinders having the same altitude is equal to the lateral surface of a third cylinder with the same altitude but having a base circumference (or diameter) equal to the difference between the base circumferences (or diameters) of the first two cylinders. By Propositions II of the *Liber de curvis superficiebus*, each of the surfaces of the two cylinders is equated to a rectangle with sides equal respectively to the altitude of the cylinder and the circumference of its base. The rectangular difference between these two rectangles then can be shown to be equal to the lateral surface of the third cylinder. It is at the end of this proposition that we find the term "interposed" (*interposite*) applied to the first proposition (and by implication to the second proposition as well).

I have already commented upon the characteristics of the two manuscripts D and D_1 in the Sigla of Section 2 of this chapter. I have followed D exclusively in my text, noting, however, the variant readings of D_1. The marginal references are to D. The figures are as in D except that I have rotated the quadrilateral $DEMN$ in Fig. 75 through 90 degrees.

[Interposed Propositions]

[Propositiones interposite]

1r /[I.] OMNIUM DUARUM ROTUNDARUM PYRAMIDUM INEQUALIUM, QUARUM YPOTHENUSE FUERINT EQUALES, SUARUM CURVARUM SUPERFICIERUM DIFFERENTIA EST EQUALIS CURVE SUPERFICIEI ROTUNDE PYRAMIDIS, CUIUS YPOTHENUSA EST EQUALIS YPOTHENUSE RELIQUARUM PYRAMIDUM, CIRCUMFERENTIA VERO BASIS EQUALIS DIFFERENTIE CIRCUMFERENTIARUM BASIUM RELIQUARUM, VEL DIAMETER BASIS EQUALIS DIFFERENTIE DIAMETRORUM BASIUM RELIQUARUM.

Sint due pyramides inequales rotunde, quarum ypothenuse sint equales. Maior quidem *DHE*, eius basis circulus *GE*, ypothenusa vero *DE* [Fig. 76]. Minor vero *DCK*, cuius basis circulus *FK*, ypothenusa *DK*. Et sit circumferentia circuli *AL* equalis differentie circumferentiarum duorum circulorum *GE* et *FK*. Et super circulum *AL* erigatur pyramis *DBL*, cuius ypothenusa *DL* ponatur equalis ypothenuse *DE*; fiatque, per hanc primam huius, triangulus orthogonius *DEN* equalis curve superficiei pyramidis *DHE*, ita quod *DE* latus trianguli sit equalis *DE* ypothenuse, et *DN* equale circumferentie circuli *GE*, et angulus *D* rectus. Et ex lateri *DN* resecetur *DL*, equalis circumferentie circuli *FK*. Et quia *DE*, *DK* sunt equales et angulus *D* rectus,

16 triangulis D_1 19 resetur D_1

[Interposed Propositions]

[I.] IN THE CASE OF TWO UNEQUAL CONES WHOSE SLANT HEIGHTS ARE EQUAL, THE DIFFERENCE BETWEEN THEIR LATERAL SURFACES IS EQUAL TO THE LATERAL SURFACE OF A [THIRD] CONE WHOSE SLANT HEIGHT IS EQUAL TO THE SLANT HEIGHT OF THE OTHER CONES AND WHOSE BASE CIRCUMFERENCE IS EQUAL TO THE DIFFERENCE BETWEEN THE BASE CIRCUMFERENCES OF THE OTHER CONES, OR WHOSE BASE DIAMETER IS EQUAL TO THE DIFFERENCE BETWEEN THE BASE DIAMETERS OF THE OTHER CONES.

Let there be two unequal cones whose slant heights are equal. The greater one we let be *DHE*, its base circle *GE*, and its slant height *DE* [see Fig. 76]. The smaller cone we let be *DCK*, its base circle *FK*, and

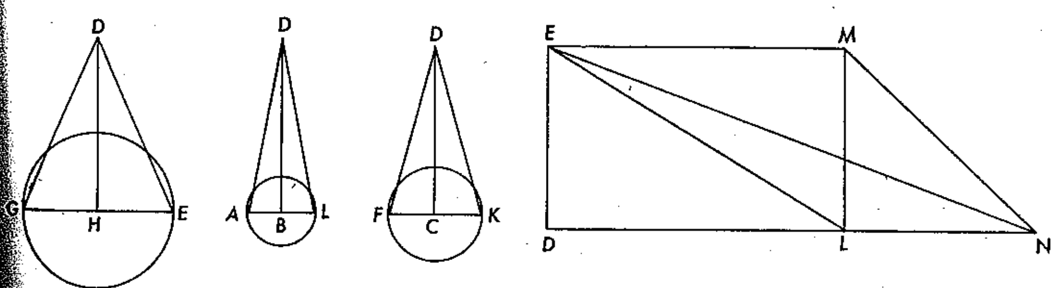

Fig. 76

its slant height *DK*. And let the circumference of circle *AL* be equal to the difference between the circumferences of the two circles *GE* and *FK*. And on circle *AL* let cone *DBL* be erected; its slant height *DL* is posited as equal to slant height *DE*. Then by the first [proposition] of this [work], let right △ *DEN* be constructed as equal to the lateral surface of cone *DHE*, so that side *DE* of the triangle is equal to slant height *DE* and [side] *DN* is equal to the circumference of circle *GE*, *D* being a right angle. And from side *DN* let *DL*, equal to the circumference of circle *FK*, be cut. And because [side] *DE* and [slant height] *DK* are equal and ergo, per primam huius, triangulus *DEL* equalis curve superficiei pyramidis *DCK*. Ergo differentia curve superficiei pyramidis *DHE* et pyramidis *DCK* erit triangulus *ELN*. Et differentia circumferentiarum circulorum *GE* et *FK* erit linea *LN*. Dico tunc curvam superficiem pyramidis *DBL* esse equalem triangulo *LEN*.

Quod sic probatur: A puncto *L* educatur *LM* perpendicularis super lineam *DN* et equalis ypothenuse *DL*, et trahatur linea *MN*. Eritque triangulus *MLN*, per hanc primam, equalis curve superficiei pyramidis *DBL*. Sic triangulus *MLN*, per 37 et 33 primi, et per 28 primi Euclidis, est equalis triangulo *ELN*. Quare et curva superficies *DBL* erit etiam equalis triangulo *ELN*, quod est propositum.

Patet etiam, per 3 huius, quod sicut se habet circumferentia *GE* ad circumferentiam *AL*, ita se habet diameter *GE* ad diametrum *AL*. Et iterum, per eandem, sicut se habet circumferentia *FK* ad circumferentiam *AL*, ita se habet diameter *FK* ad diametrum *AL*. Ergo, per penultimam quinti, sicut se habent circumferentie *GE*, *FK* simul ad circumferentiam *AL*, ita se habent diametri *GE*, *FK* simul ad diametrum *AL*. Ergo, per conversam proportionalitatem, sicut se habet circumferentia *AL* ad circumferentias *GE*, *FK* simul, ita se habet diameter *AL* ad diametros *GE*, *FK* simul. Sed circumferentia *AL* est equalis differentie circumferentiarum *GE*, *FK*. Ergo diameter *AL* est equalis differentie diametrorum *GE*, *FK*, quod fuit ultimo propositum.

1r / [II.] OMNIUM DUARUM ROTUNDARUM COLUMNARUM INEQUALIUM, EQUE ALTARUM, SUARUM SUPERFICIERUM CURVARUM DIFFERENTIA EST EQUALIS CURVE SUPERFICIEI ROTUNDE COLUMNE, EQUE ALTE, CUIUS BASIS CIRCUMFERENTIA DIFFERENTIE CIRCUMFERENTIARUM RELIQUARUM BASIUM EST EQUALIS, VEL CUIUS BASIS DIAMETER DIFFERENTIE DIAMETRORUM RELIQUARUM BASIUM EST EQUALIS.

Sint predicte columne *AB*, *CD*; et fac tetragonum *GL* [Fig. 77], per 2 huius, equalem curve superficiei columne *AB*, et tetragonum *GK*

21 superficiem (?) D_1
24 circuli D_1 / LN: in D_1 / tunc curvam: curva tunc D_1
28 MLN: MRA D_1
29 Sic: dt D_1 / MLN: MIN D_1 / primi et: et primam D_1
33, 35 diameter: diametrum D_1
36 quinti *om.* D_1 *et habet lacunam* / GE: E D_1
39, 40 simul: similiter D_1
40 diameter: diametri D_1
42 est equalis: et circulis D_1
4 eque *om.* D_1
7 diametri D_1

D is a right angle, therefore by the first [proposition] of this [work] $\triangle DEL$ is equal to the lateral surface of cone DCK. Therefore, $\triangle ELN$ will be [equal to] the difference between the lateral surfaces of cone DHE and cone DCK, and line LN will be [equal to] the difference between the circumferences of circle GE and circle FK. Then I say that the lateral surface of cone DBL is equal to $\triangle LEN$.

This is proved as follows: From point L let line LM be erected as a perpendicular to line DN and equal to slant height DL; and let line MN be drawn. And $\triangle MLN$, by this first [proposition], will be equal to the lateral surface of cone DBL. So $\triangle MNL = \triangle ELN$, by [Propositions] I.37, I.33, and I.28 of Euclid. Therefore, the lateral surface of DBL will also be equal to $\triangle ELN$, which has been proposed.

It is also evident, by the third [proposition] of this [work], that

$$\frac{\text{circum } GE}{\text{circum } AL} = \frac{\text{diam } GE}{\text{diam } AL},$$

and further that $\frac{\text{circum } FK}{\text{circum } AL} = \frac{\text{diam } FK}{\text{diam } AL}$. Therefore, by the penultimate [proposition] of [Book] V [of the *Elements*],

$$\frac{\text{circum } GE + \text{circum } FK}{\text{circum } AL} = \frac{\text{diam } GE + \text{diam } FK}{\text{diam } AL}.$$

Therefore, by inverse proportionality,

$$\frac{\text{circum } AL}{\text{circum } GE + \text{circum } FK} = \frac{\text{diam } AL}{\text{diam } GE + \text{diam } FK}.$$

But circum AL = circum GE − circum FK. Therefore, diam AL = diam GE − diam FK,* which was the last thing proposed.

[II.] IN THE CASE OF TWO UNEQUAL CYLINDERS WHOSE ALTITUDES ARE EQUAL, THE DIFFERENCE BETWEEN THEIR LATERAL SURFACES IS EQUAL TO THE LATERAL SURFACE OF A [THIRD] CYLINDER OF THE SAME ALTITUDE WHOSE BASE CIRCUMFERENCE IS EQUAL TO THE DIFFERENCE BETWEEN THE BASE CIRCUMFERENCES OF THE OTHER [CYLINDERS], OR WHOSE BASE DIAMETER IS EQUAL TO THE DIFFERENCE BETWEEN THE BASE DIAMETERS OF THE OTHER [CYLINDERS].

Let the aforesaid cylinders be AB and CD [see Fig. 77], and by the second [proposition] of this [work] make the rectangle GL equal to the lateral surface of cylinder AB and the rectangle GK equal to the lateral

* See the Commentary, lines 35–43.

equalem curve superficiei columne *CD*. Et proba tetragonum *HL* esse equalem curve superficiei columne *EF*, que ponitur eque alta columne *AB*, et circumferentia *EF* equalis differentie circumferentiarum *AB* et *CD*, qui quidem modus demonstrandi similis demonstrationi alterius interposite [propositionis].

11 HL: HI D_1 12 eque: equi D_1

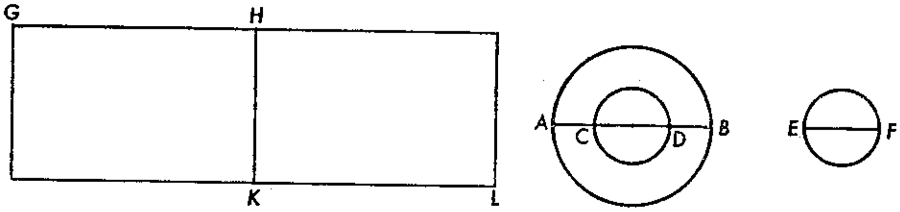

Fig. 77

surface of cylinder *CD*. And prove that rectangle *HL* is equal to the lateral surface of cylinder *EF*, which is posited as being equally high as cylinder *AB*, and circumference *EF* is equal to the difference between circumferences *AB* and *CD*, which method of demonstration is similar to the demonstration of the other interposed [proposition].

COMMENTARY

Proposition I

29 "per 37...28." Proposition I.37 in the Adelard II version of the *Elements* runs (Brit. Mus. Addit. 34018, 6r): "Equales sunt sibi cuncti trianguli qui super eandem basim atque inter duas lineas equidistantes fuerint constituti." Proposition I.33 (*ibid.*, 5v): "Si summitatibus duarum linearum equidistantium et equalis quantitatis alie due linee applicentur, ispe* etiam equales et equidistantes erunt." Proposition I.28 (*ibid.*, 5r): "Si linea una duabus [lineis rectis] supervenerit, fuertitque angulus eius intrinsecus** angulo intrinseco sibi opposito equalis, aut duo anguli intrinseci ex une parte duobus [angulis] rectis equales, ille due linee erunt equidistantes." Words in brackets in the Campanus edition of Basel, 1546.

35–43 "Ergo....propositum." Actually, what the author tacitly assumes at this point in the proof is that if Proposition V.24 of the *Elements*, using *componendo*, is accepted, then an equivalent proposition using *separando* is true, that is, if $(a + b)/c = (d + e)/f$, then $(a - b)/c = (d - e)/f$. Our author's argument, in effect, is that if $c = a - b$, then $f = d - e$, a, b, and c being the circumferences *GE*, *FK*, and *AL*, and d, e, and f being the diameters *GE*, *FK*, and *AL*. The transformation of

* ipse etiam *corr. m. rec. ex* utrobique ** *corr. ex* extrinsecus

V.24 assumed here is Simson's first corollary to Proposition V.24 (see T. L. Heath, *Euclid, The Elements* (vol. 2, p. 184).

36 "penultimam quinti," See above, this chapter, Commentary to Section 2, the *Liber de curvis superficiebus*, Proposition V, lines 78–79.

4. Three Propositions Added to the *Liber de curvis superficiebus*

In speaking of the popularity of the *Liber de curvis superficiebus* in Section 2 of this chapter, I noted that a reworking of the text appears in manuscript *M* (Florence, Bibl. Naz., Conv. Soppr. J.V.18, 92r–96v, 14c). After giving Proposition I of the Cambridge Version of the *De mensura circuli* (see Chapter Three, Section 1), the Latin author of this text then abridges the ten propositions of the *Liber de curvis superficiebus* and finally gives three further propositions on curved surfaces. Unfortunately, *M* is a very poor copy and needs much editorial emendation, both as to figures and text, in order to make mathematical sense. I can only hope that a better manuscript of this version will be found, but until such time I shall limit myself to editing and reconstructing the three additional propositions. The extensive revision needed for these three propositions will, I am sure, show the wisdom of my temporarily setting aside the preparation of the text of the abridgment of the ten propositions from the *Liber de curvis superficiebus*. The three additional propositions on curved surfaces I have numbered III (XI), IV (XII), and V (XIII). The numbers III, IV, and V indicate that these are three further propositions beyond the two propositions given in Section 3 of this chapter, while the numbers XI, XII, and XIII show that these propositions follow directly upon the ten propositions of the *Liber de curvis superficiebus* as given in manuscript *M*. I see no reason to doubt that these three additional propositions were, like the abridgement of the *Liber de curvis superficiebus* here given, done by a Latin author. For our author specifically distinguishes these additional propositions from those of the *Liber de curvis superficiebus* by referring the latter to Archimedes ("Arch."). Furthermore, if we assume that the author of these additional propositions was also responsible for the inclusion of Proposition I of the *De mensura circuli* in the Cambridge Version, then their Latin origin is assured, since that version was shown in Chapter Three above to be of Latin origin. Incidentally, the addition of Proposition I from the *De mensura circuli* as an introductory proposition makes considerable mathematical sense, since this proposition is frequently employed in the *Liber de curvis superficiebus* and the additional propositions.

Proposition III (XI), it should be noted, applies the so-called exhaustion method to a problem involving the surface of a segment of a sphere. It is evident that the author did not have access to Propositions 42 and 43 of Book I of the *De sphaera et cylindro* of Archimedes, for if he had had such access, he would not have had to apply the exhaustion procedure but only to apply simple proportions, as Archimedes did in proving Proposition 3 of Book II. It will be noted that Archimedes' Proposition II.3 is our author's corollary, except that Archimedes presented it as a problem:[1] "To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio." The whole proof of Proposition III (XI) was no doubt somewhat confused in its original form and the scribe has increased that confusion. But I am reasonably sure that my restored text represents the tenor of the original proof. Proposition IV (XII) is auxiliary to the last of the three propositions, while the last proposition itself tells us how to determine the volume of a truncated cone.

My text of the three propositions is freely corrected from *M* (although all of the original readings are, of course, included in the variant readings). There are certain peculiarities of the text that should be noted. The author often in a shorthand way refers to the "segment of the sphere," when he means "the surface of the segment of the sphere." I have added "surface" in my translation. Similarly, the author uses "poligonium" when he means the solid formed by the rotation of the polygon. The scribe of *M* uses both *ti* and *ci* before a vowel. I have always used *ti*, for example, *circumferentia*. The diagrams (Figs. 78 and 79) for Proposition III (XI) I have drawn from the text since they were missing in *M*. The only changes I have made in Fig. 80 are to put the circles in perspective, to draw a circle about *DEF*, and to relocate *K* slightly to indicate that it falls on the inner circle. The marginal references are to manuscript *M*, which has been described in Section 2 above.

[1] In the Moerbeke translation (MS Vat. Ottob. lat. 1850, 32r), Proposition II. 3, runs: "Tertium problema erat hoc: datam speram plano secare ita ut portionum superficies ad invicem habeant proportionem eandem date."

[Propositiones addite Libro de curvis superficiebus]

95v c. 1 / [III (XI).] ⟨SI⟩ CIRCULUS SPERAM PER INEQUALIA SECET, DIAMETRO SPERE PER CENTRUM EIUS TRANSEUNTE, ERIT PROPORTIO EADEM, SET ALTERNATIM SUMPTA, INTER PORTIONES DIAMETRI ET DIAMETRUM ET INTER SUPERFICIES PORTIONUM SPERE ET DIVIDENTEM CIRCULUM.

Fiat ergo spera *ABGD*, diameter *AG*, circulus *M*, cuius diameter est *BD* [Fig. 78]. Dico ergo quod que est proportio totalis diametri ad illam partem que est *EG* ea est curve superficiei *ABD* ad circulum secantem.

Si non, ergo ea est proportio minoris spere portionis vel maioris

4 portiones *correxi ex* proportiones / diametrum *corr. ex* diameterum 5 portionum *corr. ex* proportionum

[Propositions Added to the Book on Curved Surfaces]

[III (XI).] IF A CIRCLE CUTS A SPHERE INTO UNEQUAL [SEGMENTS] AND THE DIAMETER OF THE SPHERE PASSES THROUGH THE CENTER OF THE CIRCLE, THE INVERSE RATIO [OF EACH] OF THE SEGMENTS OF THE DIAMETER TO THE DIAMETER IS THE SAME AS THE RATIO [OF THAT ONE] OF THE SURFACES OF THE SEGMENTS OF THE SPHERE [WHOSE AXIS IS THE OTHER SEGMENT OF THE DIAMETER] TO THE [AREA OF THE] CUTTING CIRCLE.

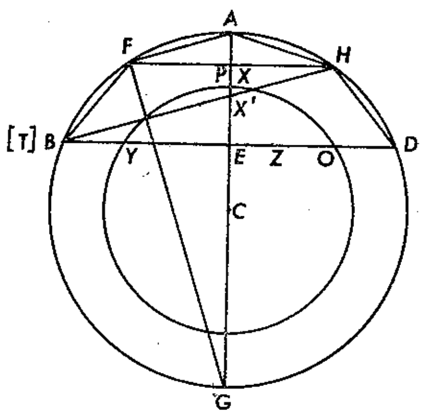

Fig. 78
Note: I have added the prime sign to X'.

Therefore, let sphere $ABDG$ be constructed [see Fig. 78] with diameter AG, and let the [cutting] circle be M with diameter BD [and center E]. I say, therefore, that the ratio of the whole diameter to its segment EG is the same as that of the curved surface [of sphere segment] ABD to the [area of the] cutting circle.

If not, the ratio is as that of [the surface of a] segment of a smaller or larger sphere to that circle. If it is of a [segment of a] smaller [sphere], let it be inscribed in it [i.e., in the given sphere]. Let the smaller sphere ad illum circulum. Si minoris, inscribatur ei et sit *YXO*. Deinde per c. 2 XII Euclidis inscribere poligonium maiori, scilicet quod non / tangat interiorem. Deinde duobus lateribus poligonii subtrade (!) basim *FH* et a *B* ad *H* duc lineam rectam et aliam ab *F* ad *G*. Age ergo ita V[a] Arch[imenidis] probatur, et id quod fit ex ductu *AF* in circumferentiam *FH* et circumferentiam *BE* est equale curve superficiei quam describunt *AF* et *BF*.

Item *AFG* et *AFP* sunt similes. Ergo que est proportio lateris *FG* ad *FA* ea est *FP* ad *AP*. A simili que est *FG* ad *FA* ea est *PH* ad *PX'* et ea est a simili *BE* ad *X'E*. Ergo per coniunctum que est unius antecedentis ad unum consequens ea est omnium antecedentium ad omnia consequentia. Ergo que est *FG* ad *FA* ea est *FPH* et *BE* ad totalem lineam *AE*. Set que est linee ad lineam ea est circumferentie ad circumferentiam, ut probabitur inferius. Ergo que est proportio *FG* ad *FA* ea est circumferentie *FH* et circumferentie *BE* ad circumferentiam *AE*. Ergo quod fit ex *FG* in circumferentiam *AE* est equale curve quam describunt *AF* et *BF*.

Item *AG* et *BD* secant se ad rectos in circulo. Ergo que est proportio *EG* ad *BE* ea est *BE* ad *AE* et ea est *AE* ad aliam; sit illa *ZE*. Et sit *TE* equalis *BE*. Age ergo quod fit ex ductu *BE* in *ZE* est equum quadrato *AE*. Ergo que est proportio *GE* ad *BE* ea est *BE* ad *EA* et ea est *EA* ad *EZ*. Ergo eversim que est *AE* ad *BE* ea est *ZE* ad *EA*. Sint ergo *GE* cum *EA* primum, *BEZ* secundum, *BE* tertium, *EA* quartum. Ergo id quod fit ex *GA* in *EA* est equum ei quod fit ex *BZ* in *BE*. Set *EB* est equale *TE*. Ergo quod fit ex *ZT* in *BE* est equum ei quod fit ex *GA* in *EA*. Et quod fit ex *GA* in circumferentiam *AE* est equum ei quod fit *ZT* in circumferentiam *BE*. Set

12 YXO *corr. ex* YZO
14 *ante* basim *add. M* poligonii *quod delevi*
15 ad[2] *corr. ex* a
17 BE *corr. ex* BL (?)
18 describunt *corr. ex* describitur
20 ea[2] *corr. ex* eam
21 BE ad X'E *corr. ex* BL ad XL
22 unius *corr. ex* minus (?)
23 BE *corr. ex* BL
24 AE *corr. ex* AL
26 *post* FA *scr. M* in circumferentiam ea est PH ad PZ FA *sed del. M* in circumferentiam *et delevi* ea est PH ad PZ FA / BE *corr. ex* BL
31 ZE[1] *corr. ex* AZE / ZE[2] *corr. ex* XE
32 *post* AE *iter. M lineas 29–32* (Ergo... AE) *per errorem* ergo que est proportio EG ad BE, ea est BE ad AE, et etiam AE ad aliam; sit ille XE (!) est equum quadrato AE. Ergo que est et sit TE equalis BE. Age ergo que (!) fit ex ductu BE in XE (!) est equum quadrato AE. *Sed delevi totum.*
33 *post* BE *add. M* ad; *delevi*
34 BEZ *corr. ex* BET
36 quod *corr. ex* que
37 Et *corr. ex* est equum ei / GA *corr. ex* BX in BE *quod est sublineatun in M*

segment be YXO. Then by the [14th proposition of the] twelfth [book] of Euclid*: [the problem is] to inscribe a polygonal body [i.e., a solid] in the greater [sphere] so that it does not touch the interior [sphere]. Then from two sides of the polygon draw base FH, then draw a straight line from B to H and another one from F to G. Proceed, therefore, as in the proof of the fifth proposition [of the *Book on Curved Surfaces*] of Archimedes. And so, $AF \cdot$(circum FH** + circum BE***) = lat surf described by AF and BF.

Also, triangles AFG and AFP are similar. Therefore $FG/FA = FP/AP$. Similarly, $FG/FA = PH/PX' = BE/X'E$. Therefore, by the composition of ratios, the ratio of one antecedent to one consequent is as the ratio of all the antecedents [together] to all the consequents [together]. Therefore, $\frac{FG}{FA} = \frac{(FP + PH + BE)}{(AP + PX' + X'E)} = \frac{FH + BE}{AE}$. But [since] line is to line as circumference to circumference, as will be proved below, therefore $\frac{FG}{FA} = \frac{(\text{circum } FH + \text{circum } BE)}{\text{circum } AE}$.

Therefore, ($FG \cdot$ circum AE) = lat surf described by AF and BF.

Also, AG and BD intersect each other at right angles in the circle. Therefore, $EG/BE = BE/AE = AE/ZE$ [ZE being so constructed to complete the proportion]. And let $TE = BE$. Therefore, $BE \cdot ZE = AE^2$. Therefore, [since] $GE/BE = BE/EA = EA/EZ$, by the inversion of ratios, $AE/BE = ZE/EA$. Therefore, [by composition of ratios,] $\frac{GE + EA}{BE + EZ} = \frac{BE}{EA}$. Therefore, $GA \cdot EA = BZ \cdot BE$. But $EB = TE$. Hence $ZT \cdot BE = GA \cdot EA$. And ($GA \cdot$ circum AE) = ($ZT \cdot$ circum BE). But circum BE = 1/2 circum BD, because diameter is to diameter as circum-

* Proposition XII. 14, Adelard text, is XII. 17 in the Greek text.

** Where FH is considered as a diameter.

*** That is, 1/2 circum BD, where BD is a diameter equal to twice BE considered as a diameter.

circumferentia *BE* est subdupla ad circumferentiam *BD*, quia que diametri ad diametrum et circumferentie ad circumferentiam. Ergo quod fit ex *AG* in circumferentiam *AE* equum est ei quod fit ex *ZT* in medietatem circumferentie *BD*. Ergo maius est quod fit ex *AG* in circumferentiam *AE* quam quod fit ex *FG* ⟨in circumferentiam *AE*⟩. Ergo quod fit ex *AG* in circumferentiam *AE* est maius quam curva *AF* et *FB*. Ergo quod fit ex *ZT* in medietatem circumferentie *BD* est maius illa eadem curva. Set id quod fit ex *ZT* in medietatem circumferentie *BD* se habet ad circulum *BD* ut se habet *ZT* ad *BE* lineam, per primam sexti Euclidis; et per primam de mensura circuli sic circulus diametri *BD* fit ex linea *BE* in dimidiam circumferentiam ipsius circuli, ut patet. At sicut *GE* ad *EA* ita est *BE* ad *EZ*. Ergo eversim sicut *AE* est ad *EG* ita *ZE* ad *EB*. Ergo coniunctim sicut est *AEG* ad *EG* ita est *ZET* ad *EB*. Ergo sic se habet id quod fit
96r c. 1 ex *TZ* / in dimidiam circumferentiam ipsius circuli *BD* ad circulum diametri *BD* ita se habet *AEG* ad *EG*. Set superficies portionis spere *YXO* ita se habet ad circulum *BD* sicud habet ⟨*AG* ad *EG* per⟩ ipothesim. Ergo eodem modo se habent ad illum circulum portio spere *XYO* et id quod fit ex *TZ* in dimidiam circumferentiam *BD*. Ergo id quod fit ex *TZ* in dimidiam circumferentiam *BD* est equale illi portioni spere. Set positum est quod illud est maius curva poligonii. Ergo illa curva spere est maior curva poligonii; set includit illum. Ergo inclusum est maius includente. Relinquit ergo quod non sit proportio portionis minoris spere ad circulum sicut diametri ad illam partem.

Ergo maioris; et sit idem centrum; et sit proportio portionis spere maioris ad circulum *BED* sicut *AG* ad *EG*; et sit illa portio *Y'OX* [Fig. 79]. Dividatur iterum arcus *ABD* in arcus quos in medio contingentes linee convenient infra *Y'OX*, ut sit una superficies inclusa lineis

39 circumferentia *corr. ex* circumferentiam / BE *corr. ex* B
41 ZT *corr. ex* FG ergo quod fit ex AG et
42 *post* BD *habet M lacunam et postea* AG maior (?) FG *quod delevi*
42 medietatem circumferentie *corr. ex* circumferentiam
47 habet² *corr. ex* habent / BE *corr. ex* BD
49 *post* fit *add. M* linea *quod delevi*
50 EZ *corr. ex* EX
53 ad circulum *bis M*
54 AEG ad EG *corr. ex* AET ad G
55 YXO *corr. ex* XEX
57 XYO *corr. ex* XYE (?)
57, 58 TZ *corr. ex* TX
59 positum *corr. ex* posterius
60 maior *corr. ex* maiori
62 *post* portionis *add. M* diametri *quod delevi*
64 *ante* Ergo *add. et. del. M* Set arcus BAD in arcus quos in medio contingentes linee convenient infra YOX ut sit una
65 portio *corr. ex* proportio

ference is to circumference. Hence($AG \cdot$ circum AE)=($ZT \cdot 1/2$ circum BD). Hence($AG \cdot$ circum AE) > ($FG \cdot$ circum AE). Hence($AG \cdot$ circum AE) > surf described by AF and FB. Therefore, ($ZT \cdot 1/2$ circum BD) > surf described by AF and FB. But $\frac{ZT \cdot 1/2 \text{ circum } BD}{\text{circle } BD} = \frac{ZT}{BE}$, by [Proposition] VI.1 of Euclid. And by the first [proposition] *On the Measurement of the Circle* [of Archimedes], the circle of diameter BD thus arises from the product of the line BE and one half the circumference of that circle, as is evident. But $GE/EA = BE/EZ$. Hence, by inversion, $AE/EG = ZE/EB$. Hence, by composition of ratios,

$$\frac{AE + EG}{EG} = \frac{ZE + ET}{EB}$$

[, EB being equal to ET]. Hence, ($TZ \cdot 1/2$ circum BD)/circle BD = ($AE + EG$)/EG. But the surface of sphere segment YXO is related to circle BD as AG is to EG, by hypothesis. Therefore, in the same way,

$$\frac{\text{Surf segment of sphere } YXO}{\text{circle BD}} = \frac{TZ \cdot 1/2 \text{ circum } BD}{\text{circle BD}}$$

Therefore, ($TZ \cdot 1/2$ circum BD) = (surf segment of sphere XYO). But it was posited that it was greater than the curved surface of the [rotated] polygon. Therefore, that curved surface of the sphere is greater than the curved surface of the [rotated] polygon. But the latter includes it and so the "included" is greater than the "including," [which is impossible]. Therefore, it remains that the ratio of the [surface of the] segment of a lesser sphere to the circle is not as the diameter to its segment.

Therefore, [let us try the segment of a] greater [sphere]. Let it have the same center [as the given sphere]. And let the ratio [of the surface] of a segment of greater sphere to circle BED be as AG to EG. And let the segment be $Y'OX$ [see Fig. 79]. Again let arc ABD be divided into arcs at whose middle points tangent lines within $Y'OX$ meet so that a single

FH, ⟨*HK*,⟩ *KL*, *LM*, *MF*. Eductis hinc modo (?) eius superficiei medietas circumducti describit solidum circa portionem spere; et contingit *HK* in puncto *Y*, et protrahantur linee *YC*, *FL*, *HL*. Quoniam ergo illud poligonium circumscribitur, erunt trianguli *YCK* et *KPH* similes, quia angulus *Y* est rectus propter angulum contingentie et angulus *P* rectus et angulus *K* communis. Ergo tertius tertio. Ergo que est proportio linee *CY* ad lineam *YK* eadem est proportio *HP* ad *KP*. A simili que est proportio *CY* ad *YK* eadem *PL* ad *PN*, propter triangulos similes; et eadem sunt ad ea. Ergo que est *CY* ad *YK* eadem est *HL* et *FE* ⟨ad *KE*⟩. Ergo quod fit ex ductu *YC* in circumferentiam *KE* est equum ei quod fit ex ductu *YK* in circumferentiam *HL* et circumferentiam *FE*. Ergo quod fit ex ductu dupli *CY* in eandem circumferentiam est equum ei quod fit ex ductu dupli *YK* in easdem circumferentias. Set duplum *CY* est *AG* diameter, scilicet duplum *YK* est *HK*. Ergo illud quod fit ex ductu *GA* in circumferentiam *KE* est equum ei quod fit ex ductu *KH* in circumferentiam *HL* et *FE*. Set ultimum est equale illis duabus superficiebus solidi, ut in quinta Arch[imenidis] probatur. Ergo quod fit ex ductu *AG* in circumferentiam *KE* equum illis duabus superficiebus. Set id quod fit ex ductu *AG* in circumferentiam *KE* est maius eo quod fit ex *GA* in circumferentiam *AE*. Set illud est equum ei quod fit ex ductu *TZ* in circumferentiam *BE*, ut ostensum est in priori parte huius propositionis, eadem facta dispositione de *T* et *Z*. Ergo quod fit ex ductu *AG* in circumferentiam *KE* est maius eo quod fit ex ductu *TZ* in circumferentiam *BE*. Set id quod fit ex *TZ* in circumferentiam *BE* est equum portioni superficiei sperice *Y'OX*. Set probatur illud omne equum predictis duabus superficiebus. Ergo ille due superficies sunt maiores illa portione. Ergo inclu/sum est maius includente. Et sic patet pro-
c. 2
positum.

71 erunt trianguli *corr. ex* eȓt anguli
74 CY *corr. ex* CX
79 FE *corr. ex* BE
81 CY *corr. ex* EX
84 FE *corr. ex* FD / duabus *corr. ex* duobus
86 *post* equum *add. et del. M* ei quod fit ex ductu HK
95 portione *corr. ex* proportione

surface is included by lines FH, HK, KL, LM, and MF. Now with these lines so drawn the half of this surface when rotated describes a solid about the segment of the sphere; and it [the sphere] touches HK in point Y. And let lines YC, FL, and HL be drawn. Therefore, since that polygon is circumscribed, triangles YCK and KPH will be similar because $\angle Y$ is a right angle, being an angle of tangency, and $\angle P$ is a right angle, and $\angle K$ is common. Therefore, the third [angle is equal]

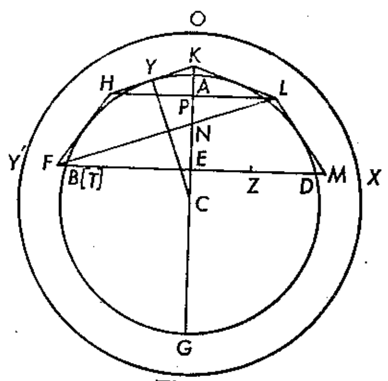

Fig. 79
Note: I have added the prime sign to Y'.

to the third [angle]. Hence $CY/YK = HP/KP$. Similarly, $CY/YK = PL/PN$, because of similar triangles. Therefore, by equality [and composition], $CY/YK = (HL + FE)/KE$ [and hence

$$\frac{CY}{YK} = \frac{\text{circum } HL + \text{circum } FE}{\text{circum } KE}\Big].$$

Therefore, $(YC \cdot \text{circum } KE) = YK \cdot (\text{circum } HL + \text{circum } FE)$. Therefore, $(2\ CY \cdot \text{circum } KE) = 2\ YK \cdot (\text{circum } HL + \text{circum } FE)$. But $2\ CY =$ diameter AG, and $2\ YK = HK$. Therefore, $(GA \cdot \text{circum } KE) = KH \cdot (\text{circum } HL + \text{circum } FE)$. But $KH \cdot (\text{circum } HL + \text{circum } FE)$ is equal to the two surfaces of the solid, as is proved in the fifth [proposition] of [the *Book on Curved Surfaces* of] Archimedes. Therefore, $(AG \cdot \text{circum } KE)$ is equal to those two surfaces. But $(AG \cdot \text{circum } KE) > (GA \cdot \text{circum } AE)$, [since circum $KE >$ circum AE]. But when T and Z are placed as before, $(GA \cdot \text{circum } AE) = (TZ \cdot \text{circum } BE)$, as was shown in the earlier part of this proposition. Therefore, $(AG \cdot \text{circum } KE) > (TZ \cdot \text{circum } BE)$. But $(TZ \cdot \text{circum } BE) =$ surf spherical segment $Y'OX$; but it is proved that $(AG \cdot \text{circum } EK) =$ surf described by KH and HF.

Similiter proportio alterius portionis ad eundem circulum non erit maior vel minor quam est proportio *AG* ad *AE*. Relinquitur ergo quod eadem proportio superficiei portionis sperice *BAD* ad superficiem alterius sicut proportio *AE* ad *EG*. Quoniam que est proportio *BAD* ad circulum *BED* eadem est *AG* ad *EG*, at similiter que est proportio circuli *BED* ad superficiem sperice portionis *BGD* ea est *AE* ad *AG*; ergo per eversam proportionalitatem que est portionis sperice ad portionem spericam, ea est *AE* ad *EG*.

[IV (XII).] SI ROTUNDE PIRAMIDIS AXIS IN TERTIAM PARTEM BASIS DUCATUR, PRODUCETUR SOLIDUM EQUALE PIRAMIDI.

Quia quod fit ex ductu axis in totalem basim piramidis est triplum piramidis, ut probatur in XII Euclidis, ergo quod fit ex ductu eiusdem ⟨in⟩ tertiam partem illius est equale piramidi.

[V (XIII).] IN OMNI CURTA PIRAMIDE SI A CENTRO BASIS AD YPOTHENUSAM DUCATUR PERPENDICULARIS, QUOD FIT EX EADEM PERPENDICULARI IN TERTIAM SUPERFICIEI PIRAMIDIS CUM QUOD EX AXE IN TERTIAM CIRCULI SUPERIORIS EQUATUR IPSI PIRAMIDI.

Sit ergo circulus *BYC*, cuius centrum *E*, diameter *BC*, et sit basis piramidis *BAC*, et sit *DLF* equidistans *BC* [Fig. 80]. Et circulus illius sit *MK* circa centrum *E*, axi tam sit eadem. Sumatur [*CEY*] sector circuli basis, tertia pars scilicet. Et deinde ad ypothenusam *AC* ducatur a centro perpendicularis *EP*. Est igitur *EK* equalis *LF* et sector *KEM* tertia pars circuli *KM*. Inde sic, id quod fit ex ductu *AE* in illum sectorem *CEY* est equale hec totali piramidi. Set quod fit ex *AL* in *KME* est equale hec piramidi parti *DLF*. Ergo quod fit ex ductu relique partis eius linee in eundem sectorem cum eo quod fit ex ductu

97, 99, 102, 103 portionis *corr. ex* proportionis
Prop. IV (*XII*)
4 triplum *corr. ex* duplum
Prop V (*XIII*)
1 curta *corr. ex* curva
3–4 superficiei *corr. ex* superficie
6 BC *corr. ex* BI
11 *post* ductu *add. et del. M* relique
13 DLF *corr. ex* DEF

Hence those two, surfaces [described by KH and HF] are together greater than [the surface of] that segment [$Y'OX$] Therefore, the "included" is greater than the "including", [which is impossible]. And so that which was proposed is evident.

Similarly, the ratio of the [surface of the] remaining segment to the [area of the] same circle will be neither greater nor less than the ratio of AG to AE. It remains, therefore, that

$$\frac{\text{surf spherical segment } BAD}{\text{surf spherical segment } BGD} = \frac{AE}{EG}, \text{ for}$$

$$\frac{\text{surf segment } BAD}{\text{circle } BED} = \frac{AG}{EG}, \text{ and similarly, } \frac{\text{circle } BED}{\text{surf segment } BGD} = \frac{AE}{AG}.$$

Hence, by inverse proportionality [and the elimination of the product of AG and circle BED], the [surface of the one] spherical segment is to the [surface of the other] spherical segment as AE is to EG.

[IV (XII).] IF THE AXIS OF A CONE IS MULTIPLIED BY ONE THIRD OF ITS BASE, A SOLID EQUAL TO THE CONE WILL BE PRODUCED.

Since the product of the axis and the whole base is three times the cone, as is proved in [Proposition 9* of Book] XII of Euclid's [*Elements*], therefore the product of the same [axis] and one third of its base is equal to the cone.

[V (XIII).] IN EVERY TRUNCATED CONE THE VOLUME IS EQUAL TO THE SUM OF (*a*.) THE PRODUCT OF (1) A PERPENDICULAR DRAWN FROM THE CENTER OF ITS [LOWER] BASE TO A SLANT HEIGHT [OF THE TRUNCATED CONE] AND (2) ONE THIRD OF THE [LATERAL] SURFACE OF THE [TRUNCATED] CONE AND (*b*.) THE PRODUCT OF THE AXIS AND ONE-THIRD PART OF THE UPPER [BASE] CIRCLE.

Let there be a circle BYC, whose center is E and whose diameter is BC [see Fig. 80]. And let it be the [lower] base of cone BAC, and let DLF be parallel to BC. And let there be a circle MK of center E within that [base circle which is equal to the circle of diameter DLF], and whose axis is the same. Let there be taken a sector [CEY] which is one third of the base circle. And then let $\perp EP$ be drawn from the center to the slant height AC. Hence $EK = LF$, and sector $KEM = 1/3$ circle KM. Then, ($AE \cdot$ sector CEY) = whole cone [BAC]. But ($AL \cdot$ sector KME) = partial cone DLF [A]. Hence the product of the difference in altitudes

* XII. 9 in the Adelard II text; XII. 10 in the Greek text.

totalis linee in reliquam partem, in *KCMY*, est equale relique parti eius totalis piramidis, scilicet curte piramidi. Ergo quod fit ex ductu *LE* in *EKM* cum eo quod fit ex *AE* in *KCMY* est equale curte piramidi.

Item anguli *L* et *E* sunt recti et *A* est communis; ergo trianguli *ALF* et *AEC* sunt similes. Ergo que proportio *LF* ad *FA* ea est *EC* ad *CA*. Set que ypothenuse ad semidiametrum basis ea est curve piramidis ad circulum basis, per primam Arch[imenidis]. Item *P* angulus est rectus et *A* communis. Ergo *APE* et *CEA* sunt similes. Ergo que est proportio *CE* ad *CA* ea est *EP* ad *AE*. Set que est *EC* ad *AC* eadem est curve totalis piramidis ad circulum basis. Set que totius ad totum ea est tertie ad tertiam. Ergo tertia pars superficiei piramidis *ADF* ad sectorem *EMK* sicut *AC* ad *EC*. Ergo..., i.e., superficiei pramidis *CF* ad superficiem residuam, i.e., *YCKM*, sicut *AC* ad *EC* et etiam sicut *AE* ad *EP*, propter similitudinem triangulorum. Ergo quod fit ⟨ex *EP*⟩ in tertiam superficiei *CF* est equum ei quod ex *AE* in superficiem *KCMY*. Set illud est equale curte piramidi cum eo quod *EL* in sectorem *MEK*, hoc est, per tertiam partem
96v c. 1 superioris circuli. Set quia ⟨*EP*⟩ est perpendicularis / ad *APC*, relinquitur ergo propositum.

15 *post* equale *add. et del. M* curte piramidi
16 totalis *corr. ex* c'te
17 KCMY *corr. ex* KC et MY
20 FA *corr. ex* LA
21 ypothenuse *corr. ex* altitudinis
23 APE *corr. ex* APA
25 AC *corr. ex* AP
27 ... *hic habet M lacunam*
29–30 *ante* triangulorum *del. M* c'te
30 superficiei *corr. M ex* superficiem
33 perpendicularis *corr. ex* perpendiculares / APC *corr. ex* IPO
34 *post* propositum *habet M* est *quod delevi*

[i.e., LE] and the same sector [i.e., KEM] plus the product of the whole altitude [i.e., AE] and the difference between the sectors, i.e., $KCMY$, is equal to the difference between the cones, i.e., to the truncated cone. Hence ($LE \cdot$ sector EKM) + ($AE \cdot$ sector $KCMY$) = the truncated cone.

Also, angles L and E are right angles and $\angle A$ is common. Therefore, triangles ALF and AEC are similar. Therefore, $LF/FA = EC/CA$. But the slant height is to the radius of the base as the lateral surface of a cone is to the [area of the] base circle, by the first [proposition of the *Book on Curved Surfaces*] of Archimedes. Also, $\angle P$ is a right angle and $\angle A$ is common. Therefore, [triangles] APE and CEA are similar.

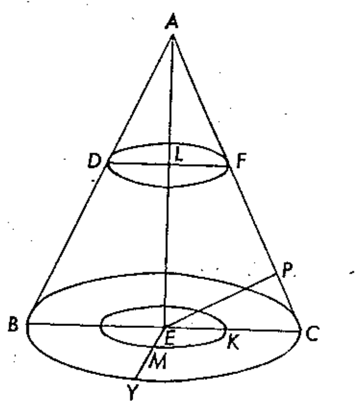

Fig. 80

Hence $CE/CA = EP/AE$. But EC is to AC as the lateral surface of the whole cone is to the base circle. But the ratio of whole to whole is as the ratio of their third parts. Therefore, $\frac{1/3 \text{ lat surf cone } ADF}{\text{sector } EMK} = \frac{AC}{EC}$.

Hence [the ratio of the difference between the surfaces of the two cones,] i.e., of the surface of the [truncated] cone of [slant height] CF, is to the difference between the surfaces [of the circular sectors], i.e., $YCKM$, as AC to EC, and also as AE to EP, because of the similarity of triangles. Therefore, ($EP \cdot 1/3$ surf of trunc cone CF) = ($AE \cdot$ surf $KCMY$). But the product of AE and surface $KCMY$ is equal to the truncated cone together with* the product of EL and sector MEK, i.e., the third part of the upper circle. But since EP is a perpendicular to APC, therefore the proposition remains.

* Should be "minus" rather than "together with"; see the Commentary, Proposition V, step (7).

COMMENTARY

Proposition III (XI)

1–104 "⟨Si⟩....EG." It will perhaps be helpful to recapitulate the confusing argument so that it might be more easily followed:

Suppose that Sps is the surface of a segment of a sphere generated by the rotation of circular segment BAD around its axis AEG. Suppose that Cd is the area of the circle which cuts the sphere into unequal parts. BD is the diameter of that circle. It is perpendicular to the diameter of the sphere, namely, AEG. Then Proposition III holds that $Sps/Cd = AG/EG$.

Proof (see Fig. 78):

(1) [Either $Sps/Cd = AG/EG$ or $Sps/Cd \neq AG/EG$. If $Sps/Cd \neq AG/EG$, then there is some surface Sx of a segment of a sphere concentric with a given sphere such that $Sx/Cd = AG/EG$ and] $Sx < Sps$, or $Sx > Sps$.

(2) Suppose that $Sx = Syox < Sps$.

(3) $AF \cdot$ (circum FH + circum BE) = Spoly ABD, where Spoly ABD is the surface formed by the rotation of the two sides of a regular polygon inscribed within segment BAD. This follows from the procedures used in Proposition V of the *De curvis superficiebus* (q.v.).

(4) $\triangle AFG$ is similar to $\triangle AFP$, since both have a right angle and a common angle. Hence, $FG/FA = FP/AP$. By similar procedures, $FG/FA = FP/AP = PH/PX' = BE/X'E$.

(5) $FG/FA = (FPH + BE)/AE =$ (circum FPH + circum BE)/circum AE, by addition and [from Proposition III, *De curvis superficiebus*].

(6) Hence, $FG \cdot$ circum AE = Spoly ABD, from (3) and (5) together.

(7) $EG/BE = BE/AE$, since BE is perpendicular to AG within a semicircle.

(8) Then $BE/AE = AE/EZ$ where EZ is constructed to make the proportion hold true. Thus $BE \cdot EZ = AE^2$. By conversion, $AE/BE = EZ/EA$, and $(GE + AE)/BEZ = BE/EA$, by

composition. Or $GA \cdot EA = BZ \cdot BE$. But $BE = ET$ (i.e., B and T appear to be identical). Hence, $ZT \cdot BE = GA \cdot EA$.

(9) Therefore, $ZT \cdot$ circum BE = $GA \cdot$ circum AE, since BE/EA = circum BE/circum EA [by *De curvis superficiebus*, Proposition III]. But circum $BE = \frac{1}{2}$ circum BD. Hence, $GA \cdot$ circum $AE = ZT \cdot \frac{1}{2}$ circum BD.

(10) $GA \cdot$ circum $AE >$ Spoly ABD, since Spoly $ABD = FG \cdot$ circum AE from step (6), and $GA > FG$, GA being opposite a right angle.

(11) Therefore, $ZT \cdot \frac{1}{2}$ circum $BD >$ Spoly ABD.

(12) But $(ZT \cdot \frac{1}{2}$ circum $BD)/(BE \cdot \frac{1}{2}$ circum $BD) = ZT/BE$.

(13) But $ZT/BE = AG/EG$, since from (8) $GE/EA = BE/ZE$, and conversely $AE/EG = ZE/EB$, and by addition $AEG/EG = ZET/EB$.

(14) Therefore, Syox $= ZT \cdot \frac{1}{2}$ circum BD, since by hypothesis Syox$/Cd = AG/EG$, and $Cd = BE \cdot \frac{1}{2}$ circum BD, by Proposition I of the *De mensura circuli* and by substituting known and equivalent quantities in the equation of (12).

(15) Therefore, Syox $>$ Spoly ABD, from (11) and (14) simultaneously.

(16) But in actuality Spoly ABD includes Syox and hence must be greater. This contradicts the deduction of (15) and hence the assumption on which it is based, namely, that Syox $<$ Sps.

Part II of Proof (see Fig. 79).

(17) Since Syox $\nless$ Sps, let us suppose that Syox $>$ Sps.

(18) As before, Syox has the same center as Sps, but is now circumscribed. Polygon $FHKLM$ is circumscribed about BAD, so that its sides are tangent to BAD but do not touch the outer circle (by *Elements* XII.14 = Greek XII.17). The rotation of $FHKLM$ forms Spoly FKM. Note: side HK is tangent to BAD at Y and the perpendicular YC is constructed. HPL and FNL are drawn as indicated. Z and T are marked, as in the first part of the proof.

(19) $\triangle CYK$ is similar to $\triangle HPK$, since $\angle P = \angle Y$, both being right angles, and $\angle K$ is common. CYK is also similar to KPL, PLN, and FNE.

(20) $CY/YK = HP/KP = PL/PN = FE/NE$, because of similar triangles.

(21) $CY/YK = (HL + FE)/KE$, by addition of proportions.

(22) $CY \cdot$ circum $KE = YK \cdot$ circum $(HL + FE)$, by *De curvis superficiebus*, Proposition III. Thus $2\ CY \cdot$ circum $KE = 2\ YK \cdot$ circum $(HL + FE)$.

(23) Therefore, $AG \cdot$ circum $KE = KH \cdot$ circum $(HL + FE)$, since $2\ CY = AG$, and $2\ YK = HK$.

(24) Hence, $AG \cdot$ circum $KE =$ Spoly FKM, since by *De curvis*, Proposition V, Spoly $FKM = KH \cdot$ circum $(HL + FE)$.

(25) $AG \cdot$ circum $KE > AG \cdot$ circum AE [since $KE > AE$], and $AG \cdot$ circum $AE = ZT \cdot$ circum BE, as shown in the first part.

(26) Hence, $AG \cdot$ circum $KE > ZT \cdot$ circum BE.

(27) But $ZT \cdot$ circum $BE =$ Syox, as in first part [by the assumption that Syox/circle $CD = AG/EG$].

(28) Therefore, $AG \cdot$ circum $KE >$ Syox, or Spoly $FKM >$ Syox. But this is impossible since Syox includes Spoly FKM. Hence Syox $\ngtr$ Sps. Since Syox $\nless$ Sps and Syox $\ngtr$ Sps, then Syox $=$ Sps, and the proposition follows.

In the same way it can be shown that when Sps is the surface of the segment BGD, then Sps/Cd $= AG/AE$. Thus, by eliminating Cd, surface seg BAD/surface seg $BGD = AE/EG$, as the corollary tells us. Cf. Archimedes, *De sphaera et cylindro*, Book II, Proposition 3.

13 "XII Euclidis." Proposition XII.14 (= Gr. XII.17) runs in the Adelard-Campanus version (ed. Basel, 1546): "Duabus sphaeris unum centrum habentibus propositis, intra maiorem earum solidum multarum basium superficiem minoris sphaerae minime tangentium figuraliter constituere...."

25 "ut probabitur inferius." It is not "proved below," but rather follows from Proposition III of the *Liber de curvis superficiebus*.

48 "primam sexti." See the Commentary, Chapter Three, Section 3, the Cambridge Version, line 70.

Proposition IV (XII)

5 "XII Euclidis." See above, the Commentary to Section 2, Proposition VII, line 22.

Proposition V (XIII)

1–34 "In....propositum." This proof is recapitulated as follows:

To prove that the volume of a truncated [right circular] cone is equal to the sum of (a) the product of the perpendicular from the center of the base to the slant height multiplied by one third of the surface of the truncated cone and (b) the product of the altitude multiplied by one third of the area of the superior circle, or, referring to Fig. 80, trunc cone $BDFC = [EP \cdot (\frac{1}{3}$ surf $BDFC) + LE \cdot (\frac{1}{3}$ circle $DF)]$.

(1) $CEY \cdot AE =$ cone ABC, and $KME \cdot AL =$ cone ADF, assuming $EK = LF$, $KEM = \frac{1}{3}$ circle $KM = \frac{1}{3}$ circle DF [and $CEY = \frac{1}{3}$ circle BYC, and applying Proposition IV above].

(2) Trunc cone $= (LE \cdot EKM) + (AE \cdot KMYC)$, since trunc cone $=$ cone $ABC -$ cone ADF, or $[(AL + LE) \cdot (KEM + MYKC)] - (AL \cdot EKM) =$ trunc cone.

(3) $\triangle ALF$ is similar to $\triangle AEC$, since L and E are right angles, and A is common. Hence $LF/AF = EC/AC$.

(4) $\triangle APE$ is similar to $\triangle CEA$, since P is a right angle and A is common. Hence $AE/EP = AC/EC = AF/LF$.

(5) Surf ADF/circle $KM = AF/LF$ and surf ABC/circle $BYC = AC/EC$, by Corollary of Proposition I of the *De curvis superficiebus*. Hence, $[\frac{1}{3}$ surf $ADF]/EKM = [\frac{1}{3}$ surf $ABC]/CEY$, EKM and CEY each being one third of their circles.

(6) Hence, by the subtraction of proportions, $[\frac{1}{3}$ surf trunc cone $BDFC]/YMKC = AE/EP$.

(7) But $AE \cdot YMKC =$ trunc cone $BDFC - (LE \cdot EKM)$, from (2).

(8) Hence, $[EP \cdot \frac{1}{3}$ surf $BDFC] + (LE \cdot EKM) =$ vol trunc cone $BDFC$. Q.E.D.

5. An Anonymous Comment on Proposition Seven of the *Liber de curvis superficiebus*

It will be recalled that after proving the seventh proposition of the *Liber de curvis superficiebus* for the case of a solid produced by the rotation of an inscribed regular polygon whose half has an even number of sides (that is, a polygon of $4n$ sides), Johannes de Tinemue remarks that there is some doubt as to whether the proposition holds for a solid produced by the rotation of a half polygon with an odd number of sides, but that since the author (i.e., Archimedes) had not taken the matter up, he, Johannes, would leave its treatment to a diligent posterity. At the end of the text of the *Liber de curvis superficiebus* in manuscript E (British Museum, Harleian

625, 139v, 14c), a Latin author has taken up Johannes' challenge and a proof is there presented for the case where the half polygon has an odd number of sides. Of course, the middle side in such a case does not describe a conical surface as to the other sides. It describes rather a cylindrical surface. One could say, accordingly, that such a case does not fall under the proposition as stated, for that proposition is stated in terms of a body with "conical surfaces," and such a body with exclusively conical surfaces would be formed only by a half polygon having an even number of sides. Hence both Johannes' remarks and the further remarks of the Latin commentator of manuscript *E* are to some extent inappropriate unless we extend the definition of the body under investigation.

In his additional comment the Latin author accepts from Proposition VII that the volumes of each of the solids formed by the rotation of triangles *OFC*, *OFN*, *OAP*, and *OAQ* around diameter *CQ* is equal to a cone whose altitude is the line drawn from the center of the circle to the middle point of each side of the polygon and whose base is the surface described by that side of the polygon. There remains for proof, then, only the case of the volume of the solid formed by the rotation of the middle triangle *ONA* about diameter *CQ*. The author easily shows that it too is equal to a cone whose altitude is equal to *OT* and whose base is the cylindrical surface described by side *NA*. Since the altitude is the same in each of the various cones to which the various parts of the total solid are equal, it is evident that the whole solid will be equal to a cone whose altitude is the same as that of each of the individual cones (that is, which is equal to the radius of a sphere inscribed in the solid) and whose base is the sum of all the bases of the individual cones (and that sum is comprised of all the surfaces described by the sides of the half polygon in revolution, or, in short, to the whole exterior surface of the solid). Hence the proposition applies equally well to a solid formed by a half polygon with an odd number of sides as it does to a solid formed by a half polygon with an even number of sides.

The Latin commentator concludes with a short proof that the proposition applies to the solid formed by the rotation of a polygon where a pair of opposite sides are bisected by the diameter about which the rotation is performed. The only thing that is necessary to prove in connection with this case is obvious, namely, that the solids formed by the rotation of the sides which the diameter bisects are equal to cones each of which has as its altitude a line drawn from the center of the circle to the middle point of the rotated side and has as its base a circle equal to that described by the side in rotation. All of the other partial solids are covered by the proof in Proposition VII. Hence Proposition VII applies to such a solid. The commentator then errs in stating that if we rotate a regular polygon of this last type about a diameter which connects two opposite angles of the polygon, the solid formed in this rotation will be equal to the solid formed when we rotate the same polygon about the diameter which bisects two opposite sides. For in fact, the surfaces of these solids will not be equal, and so even though the same formula of Proposition VII applies to both cases, their volumes cannot be equal. MS *E* has been described in Section 2 above.

[Commentum in septimam propositionem Libri de curvis superficiebus]

139v / Quia iste commentator Archimenidis non probat conclusionem septimam indistincte sed in illo casu solummodo quo corpus conicarum superficierum inscriptibile spere et circumscriptibile habeat superficies numero pares, cum tamen indistincte loquatur conclusio, ideo ne illius conclusionis probatio claudicet uno pede probabo conclusionem in illo casu quo superficies prenominati corporis sunt impares numero, ut sic posteritatis diligencia cui ipse commentator in fine conclusionis septime hoc probandum relinquit nubem obfuscantis dubietatis extenuet et lucem certitudinis imprimat illustrantis.

Sit igitur corpus conicarum superficierum *CNAQB* numero imparium [Fig. 81] inscriptibile et circumscriptibile spere. Tunc ex modo arguendi in septima conclusione patet quod piramis cuius altitudo est *OI* et basis equalis curve superficiei quam describit linea *FC* circumvoluta cum linea *FO*, stanti linea *CO*, est equalis corpori incluso infra *CFO* lineas circumvolutas, stanti eidem *CO*. Et per idem piramis cuius altitudo est *ZO* et basis equalis superficiei quam describit linea *FN*

Comment on Proposition VII of the Book on Curved Surfaces

Because this commentator on Archimedes does not prove the seventh conclusion generally but only in the case where the body with conical surfaces which is inscribable and circumscribable in a sphere has an even number of surfaces—and yet the conclusion is stated in a general fashion—therefore in order that the proof of the conclusion does not limp along on one foot I shall prove the conclusion for the case where the surfaces of the above designated solid are odd in number. Thus the "diligence of

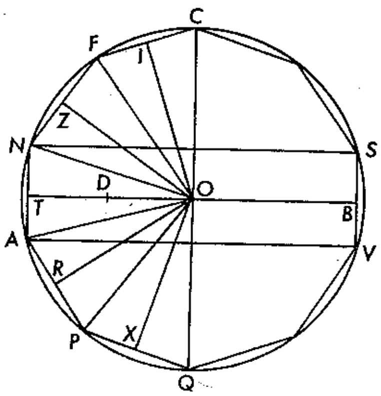

Fig. 81

posterity" to which the commentator in the end of the seventh conclusion leaves the proof may thin out the cloud of obscuring doubt and impress [on it] the light of illuminating certitude.

Therefore, let the body which has an odd number of conical surfaces and which is inscribable and circumscribable in a sphere be *CNAQB* [see Fig. 81]. Then by the manner of arguing followed in the seventh conclusion it is evident that the cone whose altitude is *OI* and whose base is equal to the lateral surface described by the revolution of *FC* when it is revolved along with *FO*, and *CO* is the axis, is equal to the body included within lines *CF* and *FO* when they are revolved about *CO* as an axis. And by the same argument the cone whose altitude is *ZO*, and whose in circumvolucione poligonii est equalis corpori incluso infra *FNO* lineas circumvolutas per circumvolucionem poligonii. Et eadem ratione arguitur de piramide cuius altitudo est *OR* et basis equalis superficiei descripte ab *AP* et etiam de piramide cuius altitudo est *OX* et basis equalis superficiei descripte a *PQ* circumvoluta. Igitur non restat aliud probare nisi quod piramis cuius altitudo est *OT* et basis equalis curve superficiei quam describit linea *AN* in circumvolucione poligonii sit equalis corpori incluso infra *ANO* lineas circumvolutas in circumvolucionem poligonii.

Hoc autem sic probatur: Linea *AN* est equedistans linee *COQ*, eo quod *CN* arcus est equalis arcui *AQ*. Igitur ex circumvolucione linee *AN* in circumvolucione poligonii fit columpna *AS* cuius axis est equalis linee *AN* et basis est circulus cuius diameter est linea *AV*, seu linea *NS* sui equalis vel linea *TB* utrique equalis. Cum igitur linea *TOB* dividat tam *AN* quam *SV* in duo equalia, patet quod columpna *TS* sit medietas columpne *AS*, posito quod *TB* sit diameter basis columpne *TS*.

Arguatur tunc sic: Sicut *NT* linea, que est altitudo columpne *TS*, se habet ad quartam partem diametri basis, que sit *DO*, illa curva superficies columpne *TS* quam describit *TN* circumvoluta se habet ad superficiem basis cuius diameter est linea *TB*, per secundam partem correlarii secunde huius. Sed ex ductu *TN* primi in basem columpne quartum resultat illa columpna *TS*. Igitur ex ductu *DO* secundi in curvam superficiem columpne tertii resultat eadem columpna, per 15 sexti vel 19^am^ septimi Euclidis. Tunc sic: ex ductu *OD* linee in curvam superficiem columpne *TS* fit columpna *TS*. Igitur per 15 quinti ex ductu *OT* semidiametri in eandem curvam superficiem columpne *TS* fit columpna dupla ad *TS*. Igitur piramis cuius altitudo est linea *TO* et basis equalis curve superficiei quam describit linea *TN* est tertia pars duplicis columpne ad *TS*. Illa igitur piramis est sexquitertia ad columpnam *TS*, sed corpus quod includit triangulus *TNO* in circumvolucione poligonii est sexquitertia ad columpnam *TS*, eo quod ille triangulus circumvolutus includit totam columpnam excepta piramide *NOS* eiusdem basis et eiusdem altitudinis que est subtripla ad columpnam per 9^am^ duodecimi Euclidis. Igitur piramis cuius altitudo est linea *TO* et basis eqalis curve superficiei columpne *TS*, quam describit linea *TN* circumvoluta, est equalis corpori incluso a trian-

20 OX *corr. ex.* AX

base is equal to the surface described by line *FN* when the polygon is rotated, is equal to the body included within △ *FNO* as it is rotated with the rotation of the polygon. And by the same reasoning it is argued concerning the cone whose altitude is *OR* and whose base is equal to the surface described by *AP* and also concerning the cone whose altitude is *OX* and whose base is equal to the surface described by *PQ* in revolution. Therefore, nothing remains to be proved except that the cone whose altitude is *OT* and whose base is equal to the lateral surface described by *AN* in the rotation of the polygon is equal to the body included within △ *ANO* as it revolves with the rotation of the polygon.

This is proved as follows: Line *AN* is parallel to line *COQ* because arc *CN* is equal to arc *AQ*. Therefore, by the revolution of line *AN* as the polygon rotates a cylinder *AS* is produced whose axis is equal to line *AN* and whose base is the circle whose diameter is line *AV* or line *NS* equal to it or line *TB* equal to each of them. Therefore, since line *TOB* bisects both *AN* and *SV*, it is evident that cylinder *TS* is equal to one half of cylinder *AS*, it having been posited that *TB* is the diameter of the base of cylinder *TS*.

Then let it be argued as follows: Line *NT*, which is the altitude of cylinder *TS*, is related to one fourth of the diameter of the base, which is *DO*, as the lateral surface of cylinder *TS*, described by *TN* in revolution, is related to the base circle whose diameter is line *TB*, by the second part of the corollary of the second [proposition] of this [work]. But the product of the first term, *TN*, and the fourth term, the base of the cylinder, is equal to cylinder *TS*. Therefore, the product of the second term, *DO*, and the third term, the lateral surface of the cylinder, is equal to the same cylinder, by VI.15 (VI.16, *Greek text*) or VII.19 of [the *Elements* of] Euclid. Then as follows: The product of line *OD* and the lateral surface of cylinder *TS* is equal to cylinder *TS*. Therefore, by V.15 [of the *Elements*], the product of radius *OT* and the same lateral surface of cylinder *TS* is equal to a cylinder double *TS*. Therefore, the cone whose altitude is line *TO* and whose base is equal to the lateral surface described by line *TN* is one third the cylinder which is double *TS*. Therefore, that cone is four thirds cylinder *TS*; but the body which △ *TNO* includes in its revolution is four thirds cylinder *TS*, because that triangle when it has revolved includes all of the cylinder except the cone *NOS* having the same base and altitude—and cone *NOS* is one third the cylinder by XII.9 (XII.10, *Greek text*) of Euclid. Thus the cone whose altitude is line *TO* and base is equal to the lateral surface of cylinder *TS*, described by line *TN* in revolution, is equal gulo *TNO* circumvoluto et eadem ratione piramis cuius altitudo est linea *TO* et basis equalis curve superficiei quam describit linea *TA* circumvoluta est equalis corpori incluso a triangulo *TAO* circumvoluto. Igitur piramis cuius altitudo est linea *TO* et basis equalis toti superficiei curve quam describit totalis linea *NA* circumvoluta est equalis corpori incluso a toto triangulo *ANO* circumvoluto. Sic igitur liquet protracta (?) probatio quam commentator in conclusione septima aggredi recusavit.

Item licet medietas poligonii que circumvoluta facit corpus conicarum superficierum circumscriptibile et inscriptibile spere non tangat extremitates diametri—immo diameter secet duo eius latera ut unum in duo equalia—adhuc nichilominus in illo casu verificabitur conclusio. Nam statim ad oculum liquet quod in illo casu linea *DE* circumvoluta describit superficiem planam circularem [Fig. 82] et per consequens *ODE* triangulus circumvolutus describit piramidem cuius altitudo est linea *OE* et basis circulus cuius diameter est linea *DF*. Cum igitur in aliis partibus poligonii stat probatio facta in commento septime conclusionis, igitur piramis cuius altitudo est linea *EO* et basis equalis omnibus superficiebus exterioribus corporis descripti per semipoligonium *GHE* est equalis illi corpori sic descripto. Et ex hoc liquet quod si poligonium equilaterum et equiangulum circulo inscribatur, quod sit *CHK* gratia exempli, sive medietas eius tangens extrema diametri, scilicet *HEK*, circumvolvatur, stanti diametro *HK*, sive alia to the body included by △ *TNO* when it has been revolved. And by the same argument the cone whose altitude is line *TO* and whose base is equal to the lateral surface described by line *TA* in revolution is equal to the body included by △ *TAO* when it has revolved. Therefore, the cone whose altitude is line *TO* and whose base is equal to the whole lateral surface described by the whole line *NA* in revolution is equal to the body included by the whole △ *ANO* when it has revolved. Therefore, the protracted proof, which the commentator refused to take up in the seventh conclusion, is clear.

Also, even though the half polygon which when rotated produces a body with conical surfaces which is circumscribable and inscribable in a sphere

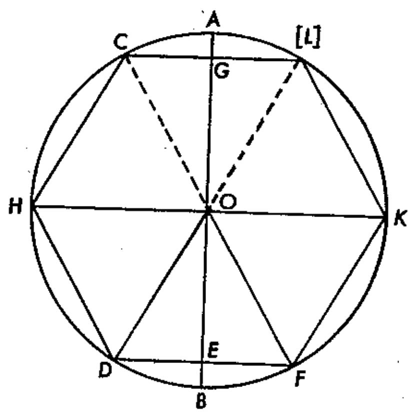

Fig. 82
Note: I have added the dotted lines.

does not touch the extremities of the diameter [about which the polygon rotates]—and in fact the diameter bisects two of its sides—still the conclusion will be verified in this instance. For it is immediately clear to the eye that in this case line *DE* when rotated describes a plane circular surface [see Fig. 82]. Consequently, △ *ODE* when rotated [about *AB*] describes a cone whose altitude is line *OE* and whose base circle has as its diameter line *DF*. Since the proof produced in the comment to the seventh consion is valid for the other parts of the polygon, therefore the cone whose altitude is line *EO* and whose base is equal to all of the exterior surfaces of the body described by the half polygon *GHE* is equal to the body so described. And from this it is evident that, in the case of the regular polygon *ÇHK* inscribed in the circle, if the half polygon which touches the extremes of the diameter, i.e., [half polygon] *HEK*, is rotated about *HK* as an axis, the body described by that half will be equal to the body which eius medietas, scilicet *GHE*, cuius media puncta oppositorum laterum tangunt diametrum *AB*, circumvolvatur, stanti diametro *AB*, corpus ab una medietate descriptum erit equale corpori quod ab alia medietate describitur. Hoc enim patet, eo quod utrumque corpus uni et eidem piramidi est equale; patet etiam quod eorum superficies sunt equales.

is described by another half in which the middle points of opposite sides touch the diameter, namely, [half polygon] *GHE*, when it rotates about diameter *AB*. This is evident because each body is equal to one and the same cone; it is also evident that their surfaces are equal.

COMMENTARY

7 "posteritatis diligencia." The reader will recall that manuscript *E* is of the second tradition of the manuscripts of the *Liber de curvis superficiebus* and hence it is this reading which is quoted rather than the reading "diligenti...posteritati" given in the first tradition (see Section 2, above, Proposition VII, line 180, and the variant reading for lines 179–80).

41 "15 sexti vel 19^am^ septimi." Proposition VI.15 [Greek VI.16] in the Adelard II Version of the *Elements* runs (Brit. Mus. Addit. 34018, 23v): "Si fuerint quatuor linee proportionales, quod sub prima et ultima rectangulum continetur equum erit ei quod sub duabus reliquis...." Proposition VII.19 (*ibid.*, 29r): "Si fuerint quatuor numeri proportionales, quod ex ductu primi in ultimum producetur equum erit ei quod ex ductu secundi in tertium...." Actually VII.19 does not properly apply, since in this comment the Latin commentator is concerned with geometric magnitudes rather than with numbers.

42–43 "15 quinti." See Commentary, Chapter Three, Section 5, the Corpus Christi Version, line 278.

51 "9^am^ duodecimi." See above, Commentary to Section 2, Proposition VII, line 22.

73–82 "Et...equales." Here the commentator makes a mistake—since the two solids do not equal the same cone and their surfaces are not equal. He was thrown off by the fact that Proposition VII can be used to determine the volume of each of the solids.