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Propositions XII-XV of the Banu Musa (9th century Arabic text attributed to the three brothers, translated into Latin by Gerard of Cremona in the 12th century): comparison with the equivalent (related) propositions of Archimedes in *Sphere and Cylinder Book I.*

Proposition XII:

The statement of the Banu Musa Proposition XII amounts to a very precise verbal explanation of what is depicted in the following diagram (Fig. 46, although without reference to any letters in the diagram which followed as an "example"), and the assertion that that the sum of the verbally-described (green) lines in the left picture is equal to the (green) line DE in the duplicated picture on the right.

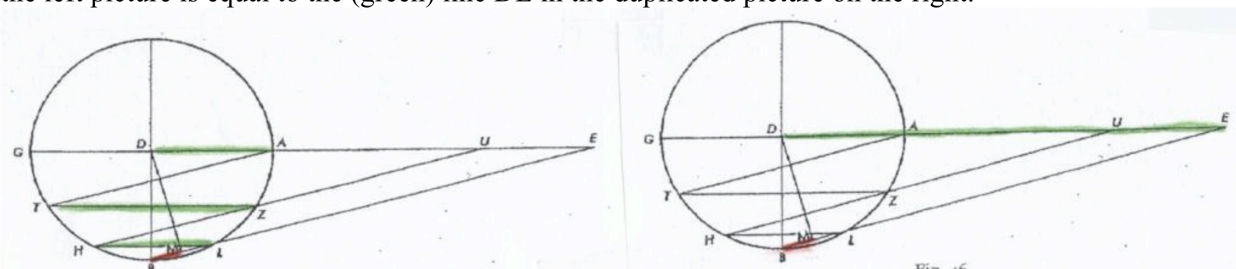


Fig. 46 Fig. 46

As the left picture suggests, the Banu Musa Proposition XII covers the same geometric consideration of parallel chords of a circle (and has the same ultimate end – to establish the surface area and volume of a sphere) as did **<u>Archimedes' Props. 21 and 22</u>** in *SCI*, although the proofs are significantly different and the Banu Musa advanced additional ends as well in the proof. Proposition XII in fact serves as a **<u>lemma</u>** to what follows in the proof – namely, an equivalent to **<u>Archimedes' Propositions 24 and 29</u>**:

> the Banu Musa establish the plane figure (rectangle) whose area equals the surface of a solid inscribed in (circumscribed about) a hemisphere in terms of the diagram of Prop. XII.

A key difference is that the Banu Musa treat a **hemisphere** rather than a full sphere, so their summation $(ZT + LH + AD)$ is **one-half** of Archimedes' $(\mathrm{EK} + \mathrm{Z\Lambda} + ... + \mathrm{\Theta M})$ in Prop.21, and since they <u>also</u> use **one-half** the side (i.e., MB) of the inscribed regular polygon rather than whole side (Archimedes' AE), they obtain a rectangle whose area is equal to **one-fourth** the surface area of the full sphere (surface area of a quadrant) which is then doubled to get the area of the full hemisphere.

<u>In Archimedes Props. 21 and 24</u>, recall that we obtain:

(long leg of triangle inscribed in hemisphere)^2 < area of rectangle < (diameter)^2

i,e,
$$(\mathrm{E\Gamma})^2 < \mathrm{A\Gamma} \cdot \mathrm{E\Gamma} < (\mathrm{A\Gamma})^2$$
[where $\mathrm{AE} \cdot (\mathrm{EK} + \mathrm{Z\Lambda} + ... + \mathrm{\Theta M}) = \mathrm{A\Gamma} \cdot \mathrm{E\Gamma}$].

<u>In the Banu Musa's XII</u>, we will obtain (note: apothem of a regular polygon is radius of circle inscribed in it):

Apothem^2 < 1/4 (area of Archimedes' rectangle) < Radius^2

This leads to the result **<u>in the Banu Musa's XIII</u>** re a solid inscribed in a hemisphere of radius R:
$$(2\pi)\cdot radius^2_{\text{inscr. sphere}} < \text{(surface of solid circ. - insc. between hemispheres)} < (2\pi)\cdot Radius^2_{\text{circumscr. sphere}}$$

This must be (is) half the result Archimedes obtained for a solid inscribed in a sphere of radius R:
$$(\pi)\cdot diameter^2_{\text{inscr. sphere}} < \text{(surface of solid circ. - inscr. between spheres)} < (\pi)\cdot Diameter^2_{\text{circumscr. sphere}}$$

Among other differences in the Banu Musa proofs:

Instead of arguing via the similar internal triangles of the inscribed polygon as in Archimedes, the Banu Musa use parallel lines and the equalities in a parallelogram to create a large external triangle EDB (which is similar to the contained triangles DMB and EMD created by the mean proportional DM).

We will see how these (and further) differences in the Banu Musa proofs of Props. XII-XV also led to a use of "a form of the formula for the area of the circle equivalent to $A = \pi r^2$ in addition to the more common Archimedean form $A = \frac{1}{2}\, c \cdot r$" (Clagett, p. 224).

Note that while Clagett's translation of the Latin translation of the Banu Musa uses the symbol π, Clagett states [p. 224 and 323] that the Banu Musa used the expression

"the quantity which when multiplied by the diameter produces the circumference."

The Banu Musa's statement and proof of XII:

> "When there is a circle whose diameter is drawn and there is drawn from its center a line perpendicular to the diameter and terminating at the circumference so that one of the two halves of the circle is bisected, and then when one of the two quadrants is divided into any number of equal parts and the chord of the segment, one of whose extremities is the point of intersection of the line erected on the diameter and the circumference, is produced while the diameter is produced in the direction of their intersection until the two lines intersect, and there are drawn in the circle from the points at which the quadrant arc of the circle is divided chords parallel to the diameter, [if all of this is done,] then the straight line between the point where the two extended lines meet and the center of the circle is equal to the sum of the radius plus the chords drawn in the circle parallel to the diameter."
>
> [the above is in all caps, as are all of the proposition statements; but it serves as a "lemma" to a much larger result]
>
> "For example, let there be a circle ABG whose diameter is line C and whose center is point D [see Fig. 46]. And form the center let line DB be drawn perpendicular to AG, thus bisecting arc ABG. And I shall divide the quadrant AB into as many equal parts as I wish, and I shall assume these parts to be AZ, ZL, LB. And I shall draw chord BL and make it continue. And I shall also extend line AG, the diameter, rectilinearly until they [e.g., BL and AG] meet at point E. And I shall draw from the two points Z and L the two chords ZT and LH parallel to diameter AG. I say, therefore, that line $DE = radius + ZT + LH$."

Proof (not always verbatim):

I shall draw line TA and I shall draw line HZ, continuing the latter rectilinearly until it meets line EG at U. [Likewise, as indicated above, BL meets line EG at E.]
I shall proceed in a similar way if the quadrant AB is divided into more parts than these [i.e., generalizes beyond the figure].
Hence lines TZ and HL are parallel, since they are so drawn.
And lines TA, HU, and BE are parallel, since $TH = AZ$ and $HB = ZL$.
Therefore the quadrilateral $TAUZ$ is a parallelogram. Therefore $TZ = AU$.
And quadrilateral $HUEL$ is a parallelogram. Therefore $HL = UE$.

Therefore, the whole line $ED = TZ + HL +$ radius. [i.e., $ED = TZ + HL + DB$]

End of proof of the Proposition XII (as stated above)
– i.e., end of proof of the lemma required for the proof of what followed.

The proof continued with what amounts to Archimedes's Proposition 24.

Hence in the figure we draw a line, e.g., line DM, from the center thus bisecting one of the chords of the quadrant, LB being the line bisected at point M.

Then it will already be known, from what we have recounted concerning this figure [i.e., from what we have established in the above lemma as depicted in the "example" drawn in Fig. 46],
that the multiplication of (1) ½ chord BL by (2) the sum of the two chords parallel to the diameter plus the radius is less than the square of the radius and is greater than DM^2,

$$\text{[i.e. } (DM)^2 < \frac{1}{2} \cdot BL \cdot (TZ + HL + \text{ radius}) < (DB)^2 \text{]}$$

because of the fact that the three triangles DMB, EDB, and EMD are <u>similar</u>.

> Note: in the large right triangle EDB, the line DM is a perpendicular [mean proportional] drawn from the vertex of the right angle at D to the hypotenuse BE, yielding the three similar triangles:

	DMB	EMD	EDB
short leg:	MB	MD	DB
long leg:	MD	ME	DE
hypotenuse:	DB	DE	BE

Therefore [since triangles DMB and EDB are similar], $\frac{MB}{BD} = \frac{DB}{BE}$. Hence, $(DB)^2 = MB \cdot BE$,
DB being the radius. Now line $BE > (ZT + LH + BD)$, since $(ZT + LH + BD) = DE$ and $BE > DE$
[that is, since the hypotenuse BE is greater than the long leg DE in triangle EDB].
Hence, line $MB \cdot (ZT + LH + BD) < [MB \cdot BE =](BD)^2$. [right hand side of inequality proved]

And similarly [since triangles DMB and EMD are similar, $\frac{BM}{MD} = \frac{MD}{ME}$, so that] $BM \cdot ME = (MD)^2$.
But line $ME < (ZT + LH + BD)$ since
$(ZT + LH + BD) = DE$ and $DE > EM$ [again, since hypotenuse greater than long leg]
Therefore, $MB \cdot (ZT + LH + BD) > [MB \cdot ME =](DM)^2$. [left hand side of inequality proved]

> "Therefore it has now been demonstrated that in every circle where the diameter is drawn and one of the two halves of the circle is bisected and one of the two quadrants [thus formed] is then divided into any number of equal parts and from the 'dividing' points of the parts are drawn chords in the circle parallel to the diameter, then the multiplication of one half of the chord of one of the segments of the quadrant by the sum of the radius plus all of the chords drawn in the circle parallel to the diameter is less than the square of the radius and greater than the square of the line going out form the center which meets and bisects the chord of one of the parts of the quadrant. And this is what we wished to show."

[This result can be expressed using the labels in their proof as follows]:

$$(DM)^2 < \frac{1}{2} \cdot BL \cdot (TZ + HL + \text{ radius}) < (DB)^2$$

End of proof of XII.

<u>Note:</u>
Regarding the relationship between the Banu Musa's and Archimedes' findings,
observe that the Banu Musa halved both the side and the sum used by Archimedes, obtaining
$\frac{1}{2} AE \cdot \frac{1}{2}(Archimedes' sum) = BanuMusd s(BD \cdot DM)$,
where BD is the radius and DM is the apothem,
so that the Banu Musa middle term is 1/4 of Archimedes' middle term.

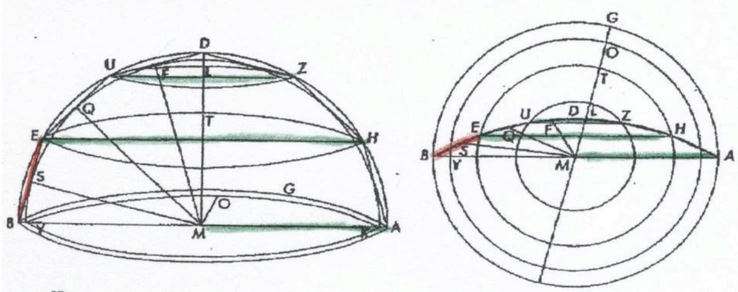
Proposition XIII:

As was the case with XII, the Banu Musa Proposition XIII amounts to a very precise verbal explanation of what is subsequently depicted in a diagram (Figure 47 from Clagett, again without reference to any letters in the intended diagram), which again was followed by a statement of the claims of the proposition – namely, the **surface area of the body** which is at once **inscribed in one hemisphere** and **circumscribed about a smaller hemisphere** is

> "less than double the area of the base of the hemisphere containing the body and
> more than double the area of the base of the hemisphere which the body contains."

Here the geometric figures described are truncated cones bounded by circles whose diameters are the green lines in the figure, and a non-truncated cone whose "vertex" is the "pole" of the hemisphere. (The concentric circles in the right figure are the bases of the cones; contrast with Figs. 49 and 50.)

For example, let there be a hemisphere *ABGD* [see Fig. 47], and *ABG* a great circle of it, whose surface is the base of the hemisphere *ABGD*.



[Reconstructed] Fig. 47 [As given in Arabic texts and in MSS *H* and *P*]

Note: *AMB* is drawn off center in MS *P*, and lines *MF*, *MQ*, and *MS* are incorrectly drawn in MS *H*.

The Banu Musa Proposition XIII covers the same geometric content established in **Archimedes' Propositions 25 and 30**. But again, the Banu Musa version is applied to a hemisphere rather than the whole sphere, so that the bounds established in the inequality involve not Archimedes' $4\pi R^2$ but $2\pi R^2$ where (for both Archimedes and the Banu Musa) the R alternatively is the radius of an inner sphere (lower bound re a circumscribed object) and an outer sphere (upper bound re an inscribed object).

Also different:
The Banu Musa **retain the rectangle (product)** as the area which the surface equals, **rather than** following Archimedes Props. 25 and 30 which make the surface of the solid equal to "a circle the square on whose radius is equal to the rectangle."

Likewise, the Banu Musa establish the product "(slant height)*(circumference)" in Proposition XI for the surface area of a cone, rather than following Archimedes' use of a "circle" to express that area.

Moreover, the Banu Musa feature the quantity "circumference/diameter" [precisely, "the quantity which when multiplied by the diameter produces the circumference"] – an expression for the quantity π -- and accordingly substitute for "circumference" the expression "diameter*(circumference/diameter)" to obtain the product "(slant height)*(diameter)*(π)" as a new expression for the lateral surface of a cone.

The resulting inequalities are:

$$\text{Apothem}^2 < \text{product obtained} < \text{Radius}^2$$

$$2\pi r^2 < \text{surface of “circumscribed object = inscribed object”} < 2\pi R^2$$

The Banu Musa's statement and proof of XIII:

> "When there is a body which falls within a hemisphere – and which [consequently] the hemisphere contains – and the body is composed of any number of segments of cones* such that the upper [plane] surface of any segment is the base of the segment [immediately] above it and the base surfaces of all the segments are parallel, and such that the base of the bottom segment is the base of the hemisphere, while the top segment is itself a cone with its vertex a pole of the hemisphere, and such that the slant heights of the segments are equal, and when there is inscribed within the body a hemisphere which the body contains and whose base is placed within the surface of the base of the body – [when all of this is true,] then the surface area of the body is less than double the area of the base of the hemisphere containing the body and more than double the area of the base of the hemisphere which the body contains."

Proof:

Draw in hemisphere $ABGD$ a semicircle of one of the great circles of the sphere passing through polar point D. This semicircle is the arc ADB.
Draw line AB as a diameter of the sphere and bisect it at point M, which is the center of the sphere.
Draw two lines HE and UZ, which are both parallel to each other and parallel to line AB, since lines AB, HE, ZU are the common sections of circle ADB and the parallel surfaces of circles ABG, ETH, and ULZ.
And it is evident that lines AB, HE, UZ are chords of circles ABG, ETH, and ULZ which are the bases of the segments of which body $ABGD$ is composed, since the pole of the axis of all these circles is point D, through which the semicircle ADB passes.
And draw the slant heights BE, EU, UD, which are posited to be all equal.

Hence a diameter of semicircle ADB has already been drawn and it is AB.
And the semicircle [i.e., arc ADB] has been bisected at D.
And arc DB has been divided into equal parts, namely, the arcs BE, EU, UD.
And from the two points E and U the two chords UZ and EH have been drawn parallel to the diameter.
Therefore,
the multiplication of one half of any one of the chords BE, EU, UD by the sum $\left(UZ + HE + \frac{1}{2}AB\right)$ is less than $\left(\frac{1}{2}AB\right)^2$, as demonstrated in Proposition XII.

Furthermore, body $ABGD$ is composed of **segments of cones** such that the bases of all the segments are parallel, the upper segment is a cone, and the straight lines drawn in all [the surfaces of] the segments from their bases to their upper [plane surfaces] rectilinearly [i.e., the straight lines constituting the slant heights] are equal.

Therefore,
the multiplication of (1) one of these slant heights by (2) the sum of one half the circumference of the base of the lowest segment plus all the circumferences of the bases of the segments above the lowest one is equal to the surface area of the body, as demonstrated in Proposition XI.

Therefore [i.e., in terms of the labeled diagram],

$$BE \cdot \left(circumULZ + circumETH + \tfrac{1}{2}circumABG\right) = \text{surface area of body } ABGD.$$

Now,

$$BE \cdot \left(circumULZ + circumETH + \tfrac{1}{2}circumABG\right) = BE \cdot \left(UZ + EH + \tfrac{1}{2}AB\right) \cdot \pi,$$

since UZ, EH, and AB are the diameters of circles ULZ, ETH, and ABG.

Therefore, $BE \cdot \left(UZ + EH + \frac{1}{2}AB\right) \cdot \pi =$ surface area of body $ABGD$.

But $\frac{1}{2}BE \cdot \left(UZ + EH + \frac{1}{2}AB\right) \cdot \pi = \frac{1}{2} \cdot$ area of body $ABGD$,

and so $\frac{1}{2}BE \cdot \left(UZ + EH + \frac{1}{2}AB\right) \cdot \pi < \left(\frac{1}{2}AB\right)^2 \cdot \pi$.

But $\left(\frac{1}{2}AB\right)^2 \cdot \pi =$ area of circle ABG,
since line AB is its diameter.*

Therefore, circle ABG, the base of the body and the hemisphere which contains the body, is equal to more than one half the area of the body falling within the hemisphere.

[The Banu Musa then similarly determine the surface area of a body circumscribed about a hemisphere.]

Now, further, I shall describe in body $ABGD$ a hemisphere which the body contains and let the base of the hemisphere be inside the surface of the base of the body, i.e., inside the surface of circle ABG, and it (the base of the hemisphere) is the surface of circle OKY.
And I shall bisect lines BE, EU, UD at points S, Q, F, and I shall draw lines MS, MQ, MF.
And it is known that these lines are equal, since point M is the center of circle ABG and the chords BE, EU, UD are equal.

And I shall produce in circle OKY line OM, which will not be in the surface of circle ADB.
Therefore, the four points S, Q, F, and O are not in a single [plane] surface.
And to these points the equal lines MS, MQ, MF, MO have been drawn from point M.
Therefore, point M is the center of the sphere which body $ABGD$ contains and line MS is its radius.
And the circle KOY is the base of the hemisphere.
Therefore, $MS^2 \cdot \pi =$ area of circle KOY.

But $BE \cdot \left(UZ + EH + \frac{1}{2}AB\right) > MS^2$, as we have demonstrated earlier.
Therefore, since $MS^2 \cdot \pi =$ area of circle KOY, then

area of circle $KOY < \frac{1}{2}BE \cdot \left(UZ + EH + \frac{1}{2}AB\right) \cdot \pi$.

But area of circle $KOY = \frac{1}{2}$(surface area of body $ABGD$).

Therefore, the surface area of body $ABGD$ is greater than double the area of circle KOY, which is the base of the hemisphere which body $ABGD$ contains.

Therefore, it has now been demonstrated that the surface area of body $ABGD$ is
less than double the area of the base of the hemisphere which contains the body
and greater than double the area of the base of the sphere which body $ABGD$ contains.
[As on p.1:

$$(2\pi) \cdot radius^2_{\text{inscr. sphere}} < (\text{surface of solid circ.-insc. between hemispheres}) < (2\pi) \cdot Radius^2_{\text{circumscr. sphere}}$$]

And this is what we wished to show.
And this is its form [Fig. 47, as reproduced above].

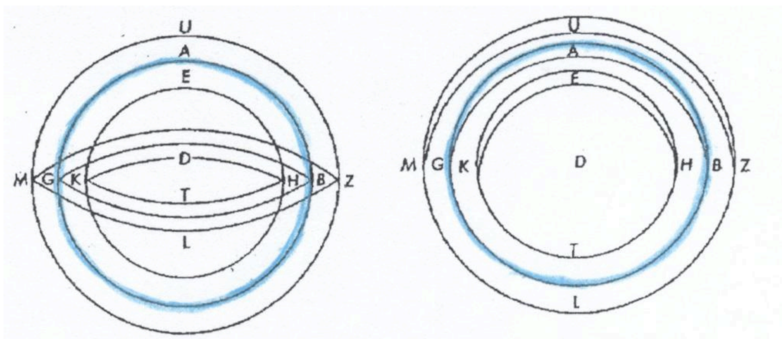
End of proof of XIII.
*(see page 11 regarding lines of "doubtful authenticity" not included in the above)

Proposition XIV:

The Banu Musa's Proposition XIV has the same geometric meaning as Archimedes' Proposition 33, although they calculate the surface area of a hemisphere rather than a sphere.

They also use the same logical structure of Archimedes' proof, but the details are different (beyond treating only the surface area of a hemisphere). As did Archimedes, the Banu Musa break up the proof into two mutually exclusive cases (each of which result in a contradiction), but rather than using the mean proportional construction (so central to Archimedes' proofs) to establish proportions relating the sides and the areas of, alternately, an inscribed and circumscribed solid, they compare smaller and larger spheres with the surface areas of the respective object and the sphere.

Thus (in contrast to Fig.47 of Prop. XIII) the concentric circles in this figure represent great circles of both a larger sphere and a smaller sphere with which the sphere (blue circle) is compared. But the object which is circumscribed about (inscribed in) sphere A **does not touch sphere U (sphere E)**.



[Reconstructed] Fig. 49 [As given in MS *P* and in Arabic texts]

Here is my **outline of the structure of the proof** (using the Dijksterhuis notation for surface area). Let

E_R represent surface area of the (blue) hemisphere $ABGD$ of radius R,

E_{R^-} represent surface area of the smaller hemisphere $EHTK$ (radius $R^- <$ apothem), and

E_{R+} represent the surface area of the larger hemisphere $UZLM$.

Case I: $2\pi R^2 < E_R$

Given: $2\pi R^2 < E_R$

Let : $E_{R^-} = 2\pi R^2$

By XIII: surface area of inscribed object $< 2\pi R^2$

By containment: $E_{R^-} <$ surface of the object

These give: $2\pi R^2 = E_{R^-} <$ surface of the object $< 2\pi R^2 < E_R$

Contradiction.

Case II: $E_R < 2\pi R^2$

Given: $E_R < 2\pi R^2$

Let: $2\pi R^2 = E_{R+}$

By XIII: $2\pi R^2 <$ surface area of circumscribed object

By containment: surface of object $< E_{R+}$

Thus we have: $E_R < 2\pi R^2 <$ surface of object $< E_{R+} = 2\pi R^2$

Contradiction.

Thus the surface of the hemisphere must equal $2\pi R^2$.

Banu Musa statement and proof of XIV:

"The surface area of every hemisphere is double the area of the greatest circle which falls in it."

Proof:
If double the area of circle *ABG* is not equal to the area of hemisphere *ABGD*, then it is less than the area of hemisphere *ABGD* or greater than it.

Case I:
Let double the area of circle *ABG* be less than the area of hemisphere *ABGD*, if that is possible.
And let double the area of circle *ABG* be equal to the area of a hemisphere smaller than hemisphere *ABGD*, namely, hemisphere *EHTK*.
When, therefore, there is described in hemisphere *ABGD* a body composed of segments of cones, the base of which body is the surface of circle *ABG* and its vertex is point *D*, and it is posited that the body does not touch hemisphere *EHTK*, then from what we have proved in Proposition XIII it will follow that the surface area of body *ABGD* is less than double the area of circle *ABG*.
But the surface area of body *ABGD* is greater than the surface area of hemisphere *EHTK*, since the one contains the other.
Therefore, the surface area of hemisphere *EHTK* is much less than double the area of circle *ABG*.
But it was posited as equal to it.
Contradiction.

Case II:
Let double the area of circle *ABG* be greater than the area of hemisphere *ABGD*, if that is possible.
And let it be equal to the area of a hemisphere greater than hemisphere *ABGD*, namely, hemisphere *UZLM*.
When, therefore, there is inscribed in hemisphere *UZLM* a body composed of segments of cones, the base of which body is circle *UZLM* and its vertex is point *D* and the body does not touch hemisphere *ABGD*, then it will follow from what we have proved before that the surface area of hemisphere *UZLM* is greater than the surface area of body *UZLM*.
But the surface area of hemisphere *UZLM* is greater than the surface area of [the contained] body *UZLM*.
Therefore, the surface area of [the contained] hemisphere *UZLM* is greater than double the area of circle *ABG*.
But it was posited as equal to it.
Contradiction.

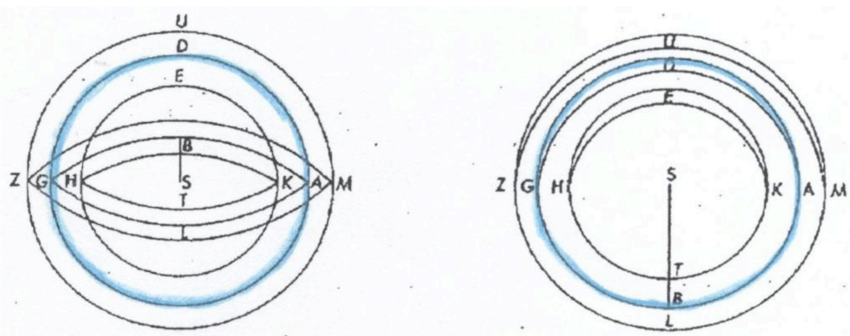
Therefore, it has now been demonstrated that the surface area of any sphere is quadruple the area of the greatest circle falling in it.

End of Proof of XIV.

Proposition XV:
This proposition is equivalent to Archimedes' Proposition 34 in that it determines the volume of a sphere (here a hemisphere). But whereas Archimedes Prop. 34 states that the volume of the sphere is four times the volume of a cone with base the great circle and height equal to the radius of the sphere, the Banu Musa proposition instead states that the volume of the sphere is one third of the surface area of the sphere multiplied by the radius.

As occurred with XIV, the proof structure here is the same as that of the corresponding Archimedes proposition, namely, breaking up the proof into two cases (each of which result in a contradiction) and following the same logic.

The circles in this figure are the same as those in XIV. That is, the concentric circles in this figure represent great circles of a larger sphere and a smaller sphere with which the sphere (blue circle) is compared **but which the constructed object does not touch**. (Line SB is the radius of the blue sphere.)



[Reconstructed] Fig. 50 [As given in MS *P* and in Arabic texts]

As with Prop. XIV, here is my **outline of the structure of the proof** of XV. Let

V_R represent the volume of the (blue) hemisphere *ABGD* of radius R,

V_{R^-} represent the volume of the smaller hemisphere *EHTK* (radius R^- < apothem), and

V_{R+} represent the volume of the larger hemisphere *UZLM*.

Case I: $R^*(1/3\ E_R) < V_R$

Given:	$R^*(1/3\ E_R) < V_R$
Let:	$R^*(1/3\ E_{R+}) = V_R$
By construction:	R^*(1/3 surface area of object circumscribed about blue sphere) < V_{R+}
By containment:	V_R < R^*(1/3 surface of the circumscribed object)
These give:	$R^*(1/3\ E_{R+}) = V_R <$ R^*(1/3 surface of the object)
Hence:	(1/3 E_{R+} [the sphere containing the object]) < (1/3 surface of the object)
	Contradiction.

Case II: $R^*(1/3\ E_R) > V_R$

Given:	$R^*(1/3\ E_R) > V_R$
Let:	$R^*(1/3\ E_{R-}) = V_R$
By construction:	V_{R-} < R^*(1/3 surface area of object inscribed in blue sphere)
By containment:	R^*(1/3 surface area of the inscribed object) < V_R
These give:	R^*(1/3 surface inscribed object) < $V_R = R^*(1/3\ E_{R-})$
Hence:	(1/3 surface inscribed object) < (1/3 E_{R-} [the sphere it completely contains])
	Contradiction.

Thus the volume of the hemisphere must equal $R^*(1/3\ E_R)$.

Banu Musa statement and proof of XV:

> "The multiplication of the radius of every sphere by one third of its surface area is the volume of the sphere."

Proof:
If this is not so, then let the multiplication of line SB [radius blue circle] by one third of the surface area of a sphere either larger than or smaller than sphere $ABGD$ be equal to the volume of sphere $ABGD$.

Case I:
Let the multiplication of line SB by one third of the surface area of a sphere larger than sphere $ABGD$ be equal to the volume of sphere $ABGD$.
Let this sphere be $ZULM$, concentric with sphere $ABGD$.
Hence, $SB \cdot \frac{1}{3}(\text{area of sphere } UZLM) = \text{volume of sphere } ABGD$.
Hence, when there is described about sphere $ABGD$ a body having surfaces which bound it, but which body does not touch sphere $UZLM$, it will follow from what we have proved before that the multiplication of line SB by one third of the surface area of the body which contains sphere $ABGD$ is greater than the area of sphere $ABGD$.
But the multiplication of line SB by one third of the area of sphere $UZLM$ is equal to the volume of sphere $ABGD$.
Therefore, one third of the area of sphere $UZLM$ is less than one third of the surface area of the body having surfaces, while the sphere $UZLM$ contains the body.
This, however, is a contradiction.

Case II:
Let the multiplication of line SB by one third of the surface area of a sphere less than sphere $ABGD$ be equal to the volume of sphere $ABGD$.
Let that lesser sphere be sphere $EHTK$, concentric with sphere $ABGD$.
Therefore, the multiplication of SB by one third of the surface area of sphere $EHTK$ is equal to the volume of sphere $ABGD$.
When, therefore, there is inscribed in sphere $ABGD$ a body having surfaces which bound it but which body does not touch sphere $EHTK$, it will follow from what we have proved that the multiplication of line SB by one third of the surface area of the body having surfaces which the sphere $ABGD$ contains is less than the area of sphere $ABGD$.
But the multiplication of line SB by one third of the surface area of sphere $EHTK$ is [equal to] the volume of sphere $ABGD$.
Therefore, one third of the surface area of sphere $EHTK$ is greater than one third the surface area of the body having surfaces, while the body contains sphere $EHTK$.
This is a contradiction.

Therefore, it has been shown that the multiplication of the radius of the sphere by one third of its surface area is equal to its volume.

End of proof of XV.

Lines of "doubtful authenticity" in proof of Prop. XIII

Clagett provides evidence (e.g., terminological differences in the Latin, p.231,n8) that material in lines 97-118 in the proof of Prop. XIII was neither part of the Banu Musa text nor marginal notes made by Gerhard "but rather a later explanatory addition" (p.233) which the scribe of copy H subsequently inserted into the proof at an inappropriate point (p.230-231,note 8). Clagett thus bracketed the section.

[That the **multiplication** of the square of ther radius **by π** is equal to the area of the circle is demonstrated as follows (see Fig. 48).

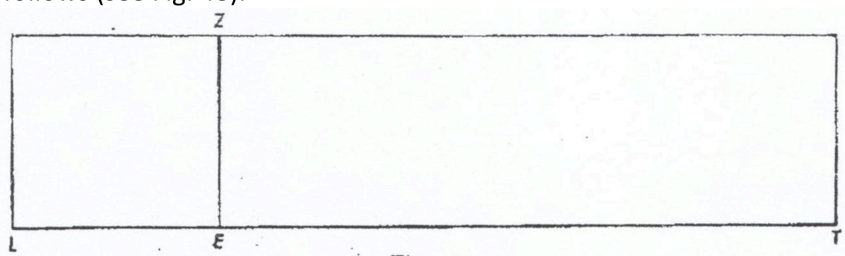


Fig. 48

Note: I have not reproduced line *RU* (which equals π).

Since I assume ET to be equal to one half the circumference and EZ equal to the radius
and [since] I shall multiply one into the other,
therefore surface ZT will be equal to the surface of the circle.

And I shall construct a square on ZE, which square is ZL.

I shall **posit** RU as the **quantity** which when multiplied by the diameter produces the circumference (i.e., as **π**).
And since the multiplicaton of the diameter by RU will produce the circumference,
therefore, when the circumference is divided by the diameter, RU is produced.
Hence, RU = circumference/diameter.

But the ratio of **the whole to the whole** is as that of **the half to the half**,
and ET is equal to half the circumference while EL is half the diameter.
Hence, $TE / EL = RU$.

But $(TE / EL) = (areaTZ / areaZL)$.
Therefore, $(areaTZ / areaZL) = RU$. [corrected printer's error, which read: $(areaTZ / areaTL) = RU$]
Hence, $(areaTZ / RU) = areaZL$.
Hence, $(areaZL \bullet RU) = areaZT$.

But ZL is the square of the radius, $RU = \pi$, and ZT is the area of the circle.

Hence, the multiplication of the square of the radius by π is equal to the area of the circle. And that is what we wished. This is its form (Fig. 48).]