CHAPTER 11

Mathematics and Narrative: An Aristotelian Perspective

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The idea that mathematics deals with timeless truths is forcefully stated in a famous exchange between Socrates and Glaucon in Plato's *Republic*, which makes the further point that the language of geometry, with its talk of manipulating figures, is absurd, since it conflicts with the idea of the timelessness of its objects. Let me quote the passage in full:

> SOCRATES: This at least will not be disputed by those who have even the slightest acquaintance with geometry, that the branch of knowledge is in direct contradiction with the language used by its adepts.
> GLAUCON: How so?
> SOCRATES: Their language is most ludicrous, though it cannot help that, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of "squaring" and "extending" and "adding" and the like, whereas the real object of the entire study is pure knowledge.
> GLAUCON: That is absolutely true.
> SOCRATES: And must we not agree on a further point?
> GLAUCON: What?
> SOCRATES: That it is the knowledge of what always is, and not of what at some particular time comes into being and passes away.
> GLAUCON: That is readily agreed. For geometry is the knowledge of what always is.[1]

So it is ridiculous to talk of squaring a circle, for instance, when nothing is *done* to the circle. Plato wishes to recommend mathematics for the education of those who are to rule in his ideal city, and that is because it provides training in abstract thought. That is its essential characteristic, so the philosopher-kings should not be misled by a vocabulary that might seem to suggest they were doing things with figures and numbers. Indeed, Greek mathematics made considerable use of terms that elsewhere were applied to physical, concrete operations, starting with the word for "construction," *kataskeue*, itself.

Later Platonists, convinced, no doubt, that they were being loyal to the master's own thought on the subject, developed the criticism further, suggesting that any talk of moving mathematical figures, or otherwise applying physical or mechanical concepts to them, should be banned. Plutarch is a case in point. Faced with the impressive evidence of Archimedes' interest in mechanics and of his skill as an engineer and practical whiz kid, Plutarch tried to insist that Archimedes was not really concerned with practical applications at all. In his *Life of Marcellus*, chapter 14, Plutarch stated that Archimedes did not devote himself to such applications as if that work were worth serious effort. Rather, as he went on to say (chap. 17), "Archimedes possessed such a lofty spirit, so profound a soul, and such a wealth of theoretical insight, that although his inventions had won for him a name and fame for no merely human, but rather some superhuman, intelligence, he was not willing to leave behind him any treatise on that subject. Regarding the work of an engineer [*ta mechanika*] and of every art that serves the needs of life as ignoble and vulgar, he devoted his ambition only to those studies the beauty and subtlety of which are not affected by the claims of necessity."[2]

That is, as is now widely recognized (Netz 1999, 303; Cuomo 2001, chap. 6), a highly tendentious set of comments on Archimedes. But Plutarch purports to quote Plato's attack on earlier mathematicians who had used mechanical methods in geometry. In one passage[3] Plutarch reports that the two mathematicians in Plato's sights were Archytas and Eudoxus, who had used such methods in their solutions to the problem of the duplication of the cube. Indeed, Eutocius gives Archytas's solution to that in some detail and further alludes to Eudoxus's work on it.[4] The solution depends on finding two mean proportionals, and to do this Archytas proposed a complex three-dimensional construction determining a certain point as the intersection of three surfaces of

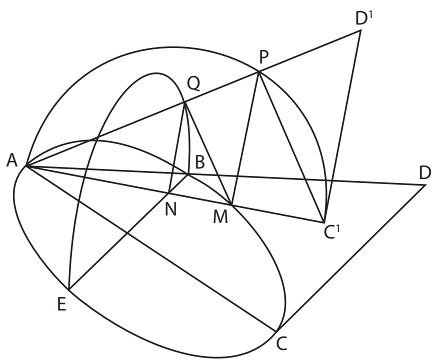

Figure 11.1. Thomas L. Heath's demonstration of Archytas's solution to the problem of the duplication of the cube. See note 5 for details. (From Heath, *A History of Greek Mathematics*, 1:247.)

revolution, namely (1) a right cone, (2) a cylinder, and (3) a torus with inner diameter zero. Heath comments (1921, 1:246–47) that the intersection of the two latter surfaces yields a certain curve, and "the point required is found as the point in which the cone meets this curve" (figure 11.1)—whereupon Archytas demonstrates how that point enables the two mean proportionals to be found.[5]

These are mechanical procedures only in the sense that Archytas speaks of surfaces and solids in revolution, not that he employs any other physical concepts, such as Archimedes was to do in his "mechanical method," where it is assumed that mathematical areas can be thought of as having centers of gravity and as being balanced against one another around a fulcrum. Nevertheless, Plutarch has it that Plato "was angry" with Archytas and Eudoxus and "inveighed against them as the destroyers and corrupters of what is good in geometry, which thus ran away from incorporeal and intelligible things and made use of bodies that required much laborious manual work."[6]

So we have already in Greek antiquity powerful and influential proponents of the view according to which (1) mathematics deals with what is timeless and (2) the category boundary between it and physics must be strictly observed. Mathematics should make no use of physical devices or even allow itself appeal to mechanical concepts, as is the case when mathematical figures are imagined as being manipulated, for example by being put into motion. Equally, in that view, to suggest any link between mathematics and narrative would be tantamount to confusing categories. One category deals with quantity, the other with time. One sets out timeless truths, the other recounts chronological sequences of actions. A story, such as the plot of a tragedy, as Aristotle famously insisted (*Poetics* 7.1450b26–27), deals with a whole, and a whole is defined as having a beginning, a middle, and an end. The facts that it sets out are sequential, even though the order in the narrative may not follow the chronological order of the events narrated.

Yet that is not the end of the matter, and again I can cite sources from classical Greek antiquity to make the point. Two types of evidence can be adduced that may be thought substantially to modify the Platonic picture I have so far presented. On the one hand, there are comments from the perspective of a very different philosophy of mathematics from Plato's, and on the other, there is more to be said about the actual practices of Greek mathematicians.

Aristotle's philosophy of mathematics differs radically from Plato's in a number of fundamental respects. First, he did not postulate separate intelligible mathematical objects, such as Plato treated as intermediate between intelligible forms and perceptible particulars.[7] For Aristotle, mathematics studied the mathematical properties of physical objects, in abstraction from the physical properties those objects possessed. Of course, Aristotle agreed with Plato that while physical hoops or rings come to be and pass away, circularity does not.[8] But while the mathematician can study the circle in the abstract, circularity does not exist as a separate intelligible entity.

However, with respect to our concerns here, the more interesting divergence between Aristotle and Plato relates to comments that Aristotle made about the activity of mathematicians and the actualization of certain potentialities in their work. In the *Metaphysics* (1051a21–31), Aristotle makes the following remark:

> *diagrammata* too in mathematics are discovered by an actualization [*energeia*], for it is by a process of dividing up that they [the mathematicians] discover them. If the division had already been performed, they [the *diagrammata*] would have been manifest: as it is, they are present only potentially. Why

> does the triangle imply two right angles [i.e., that the angles sum to two right angles]? Because the angles at one point are equal to two right angles. If, therefore, the straight line parallel to the side had been drawn upwards, the reason why would at once [*euthus*] have been clear. Why is the angle in a semi-circle a right angle, universally? Because if there are three equal straight lines, and the base consists of two of them, while the *orthe* drawn from the middle point is the third, the truth is at once clear to anyone who knows the aforesaid theorem [i.e., that the angles of a triangle sum to two right angles]. Hence it is manifest that relations subsisting potentially are discovered by being brought into actuality [*energeia*]: the reason is that the exercise of thought is an actuality [*energeia*].

Several features of this intriguing passage are problematic. There are no less than three issues of translation of varying degrees of importance, namely, those of the terms *orthe*, *diagramma*, and *energeia* itself, and there are corresponding doubts over (1) the constructions Aristotle has in mind, (2) what he claims becomes clear, once the division is performed, and (3) the intellectual procedure he thinks is involved. Let me begin with the last term, the interpretation of *energeia*, which we shall return to in order to see the implications of the passage as a whole. This term can cover both the notion of "activity" (what the mathematician does) and that of "actualization" or "bringing into actuality," specifically of certain properties that come to light thanks to the divisions that the mathematician effects. Translators are accordingly divided in their preferences, and indeed there is some slippage in Aristotle's usage. The first sense may be uppermost at the end of the text I quoted, where Aristotle speaks of the mathematician's exercise of thought, but in the immediately preceding phrase we are dealing with the actualization of certain potentialities, namely, the relations that obtain in the mathematical objects studied. Moreover, it should be noted that the main issue in this book of the *Metaphysics* is the distinction between potentiality and actuality in all its complexity, and this has guided my own preference for "actuality" or "bringing into actualization."

Next there is the term *orthe*, which relates to the question of the construction Aristotle has in mind in the proof of the theorem that

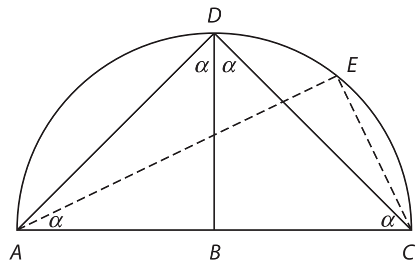

Figure 11.2. One demonstration that the angle in a semi-circle is a right angle. See note 9 for details. (From Heath, *Mathematics in Aristotle*, 73.)

the angle in a semicircle is right. Several commentators take the term to mean "perpendicular" (for which the more usual term is *kathetos*) and immediately before, at 1051a24, 25, and 27, the same term is used of a right *angle*. However, Burnyeat et al. (1984, 148–49) suggest that *orthe* used of the line in the construction may just mean "straight" (for which the usual term is *eutheia*). In the first interpretation the proof would not be universal but would relate only to the right-angled isosceles (figure 11.2).[9] In that view, as Heath (1949, 73–74) noted, the proof would have to be completed by reference to the theorem that shows that the angles in equal segments of a circle are equal (Euclid, *Elements* 3.21). For the theorem proving the angle in a semicircle is right, Euclid (*Elements* 3.31) uses a more complex diagram, with one of the sides of the triangle produced, in order to show not just that the angle in the semicircle is right (in virtue of the fact that the angles at the base of each of the two isosceles triangles formed by a diameter of a circle and any radius drawn in one of the semicircles are equal) but also that the angle in a greater segment is larger than a right angle, while one in a smaller segment is less than a right angle.

However, according to the second line of interpretation, taking *orthe* to mean "straight line" rather than "perpendicular," the proof is indeed from the start universal, and the construction Aristotle has in mind corresponds to that in Euclid, though without the complication of one side of the triangle being produced (figure 11.3).[10]

There is a similar question mark over the diagram Aristotle has in mind for the first theorem, that the angles of a triangle sum to two rights, where again we know of alternative proofs. Eudemus reports one

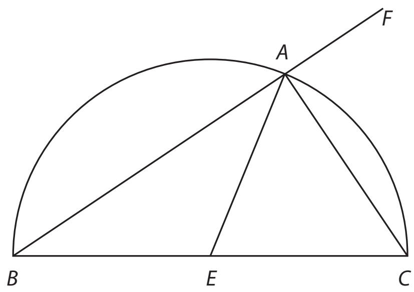

Figure 11.3. The Euclidean demonstration that the angle in a semi-circle is a right angle. See note 10 for details. (From Heath, *Mathematics in Aristotle*, 72.)

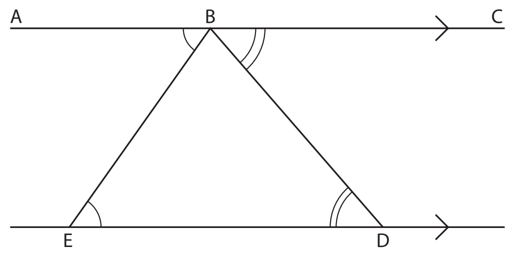

Figure 11.4. Eudemus' construction to show that the angles of a triangle sum to two right angles.

in which the key construction is drawing a line parallel to the base of the triangle (figure 11.4).[11] The three angles at the apex of the triangle can then be shown to be equal to the three internal angles (by the properties of parallels), and their sum is two right angles since the line is straight. In Euclid (*Elements* 1.32) and in Aristotle himself, the construction involves drawing a parallel to one side of the triangle and similarly invoking the propositions concerning parallels to get the result. Since Aristotle speaks of the straight line parallel to the side (not the base) being drawn "upward" (*anekto*, *Metaphysics* 1051a25), it seems likely that he has the Euclidean diagram in mind (figures 11.5. 11.6).[12]

But the most important issue relates to the term *diagrammata*, which I also left untranslated. That term can be used of diagrams in our sense, but more often it means "geometric propositions," including their "proofs."[13] W. D. Ross Ross (1924) took the word here, in *Metaphysics*

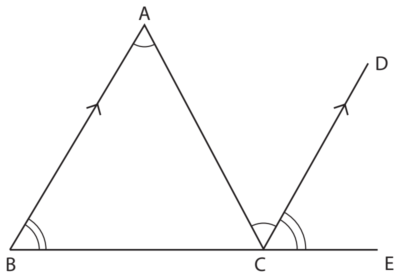

Figure 11.5. Euclidean construction to show that the angles of a triangle sum to two right angles.

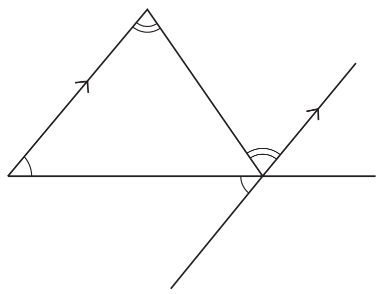

Figure 11.6. Alternative view of the Euclidean construction in *Elements* 1.32.

1051a22, in the former sense,[14] but that is surely wrong. When Aristotle says that "if the divisions had been performed, they [the *diagrammata*] would have been manifest," it is the proofs that he has in mind, not the constructions, since the "divisions" *are* in a sense the constructions, and it would be banal to the point of tautology to say that they become clear when the divisions have been made. Rather, Aristotle's point is that the proofs of the propositions concerned are discovered by the discovery of the constructions. Once the constructions have been made, it is "clear" (*delon*) that the result follows, although we should remark that for all its obviousness, Euclid still gives a proof in each case. Merely inspecting a diagram is, of course, not enough for Euclidean demonstration.

The nub of the matter relates to the claim that Aristotle here makes in relation to the *energeia*, "actualization/actuality," that geometry implies. When Aristotle says that proofs are discovered by an actualization, what does that involve? The term *energeia* is standardly contrasted with *kinesis*, movement or change. Aristotle is certainly not committed to some idea that a mathematical proof involves either movement or change. Recall that his view is that the mathematician discusses the mathematical properties of physical objects in abstraction from the physical properties that they possess. What he has in mind at *Metaphysics* 1051a21ff is the kind of actuality that is brought about by an exercise of thought—as indeed the end of the text I quoted makes explicit. Whereas a *kinesis* is a process that takes time, an *energeia* can be complete at every moment. We do not need time to see; any act of vision is complete as the act of vision it is throughout the time the vision takes place. Nevertheless, in the case of mathematical reasoning we can still talk (as Aristotle does) of discovering certain truths by means of the divisions or constructions that the mathematician uses. Given that those constructions relate to abstract entities, those entities are not altered by the construction being performed: yet what the construction does is to actualize what is there in potentiality, but only in potentiality until the construction is carried out.

We have, then, a twofold contrast to pay attention to. On the one hand, any *energeia* is complete at any moment of time. We must bear in mind the contrast between seeing or thinking, which are complete at any time, and an activity such as walking to Athens, which is complete only when you have reached Athens. On the other hand, scanning a complex visual field, or—more to the point—going through a complex piece of logical or mathematical reasoning, does take time. When the task is specified not just as "seeing" or "thinking" but looking over a complex visual object or going through a sequence of arguments, the seeing or thinking as such are complete at any moment, but that is not true of the task as so specified.

In this, Aristotelian view, then, we have to do justice to two facets of the work of the mathematician. On the one hand, she reveals what is there in potentiality all along. Yet on the other that revelation depends on an actualization, the discovery of what will reveal the truth in question. The proof takes time to set out, to be actualized, in other words, though that does not militate against the timelessness of the truth that it reveals. In many cases the proof would not be possible without the construction. But though the construction is set out in a series of steps, and the proof is not complete until the steps are complete, those steps should be viewed as the actualization of a potentiality, not as a process that involves change.

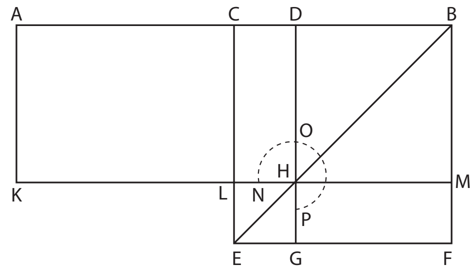

Figure 11.7. Proposition 2.5. The diagram Euclid used to prove *Elements* 2.5. (From Heath, A *History of Greek Mathematics*, 1:382.)

So, while Plato and the Platonic tradition wanted ideally to purge mathematical reasoning of any vocabulary that might suggest activity or doing of any sort, Aristotle was an influential spokesman for a contrary view according to which the actualization of a potentiality was of the essence of geometric proof. When we turn from philosophical commentators to mathematical practitioners, there is plenty of evidence that tends to bear out Aristotle's point of view and in particular that points to a realization of the feature I have just remarked on, namely, the complexity of some mathematical demonstrations.

We owe to Proclus (in the fifth century CE) the formal analysis of geometric reasoning in six steps.[15] (1) First there is the *protasis*, or enunciation of the proposition to be proved. (2) Next there is the *ekthesis*, or setting out. (3) Third is the *diorismos*, the definition of the goal, sometimes one that specifies the conditions of possibility for achieving it. (4) There then follows, fourth, the *kataskeue*, the construction of the diagram. (5) Next comes the *apodeixis*, or proof proper, and finally (6) the *sumperasma*, or conclusion. Of course, this pattern is not always rigidly adhered to, and how far it was followed self-consciously by earlier mathematicians is a moot point. But it fits a good deal of Euclid pretty well. Thus Netz, in his discussion in *The Shaping of Deduction in Greek Mathematics* (1999) opens with an illustration of the six steps taken in Euclid, *Elements* 2.5 (figure 11.7).[16] This establishes the complex proposition that as the enunciation has it, "if a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, with the square on the line between the cuts, is equal to the square of the half." That statement gives us step 1, the enunciation. The *ekthesis* then states: "for let some line, *AB*, be cut into equal segments at the point *C* and at unequal segments at the point *D*." The *diorismos* follows: "I say that the rectangle contained by *AD* and *DB*, together with the square on *CD*, is equal to the square on *CB*." Step 4, the construction, describes the squares in play, and adds three lines parallel to given ones. In step 5 the proof shows the equality of certain areas, drawing on propositions proved earlier in the *Elements*, and that includes the equality of certain areas taken together, leading to the equality sought. Finally, the conclusion (step 6) repeats the enunciation as a statement of what was to be proved, but this time includes the conjunction *ara*, "therefore," to indicate that it has indeed been shown.

Two points stand out as fundamental. First, the steps are sequential and the theorem as a whole involves what Aristotle would call an *energeia*, the actualization of a certain potentiality, and indeed the activity of the mathematical reasoner to bring that about. Second, the proof depends crucially on the construction. Without the appropriate construction the demonstration could not be given. The *ekthesis* in step 2 presents the reader with the situation to be explored, that of a line cut into equal and unequal segments. But it takes the construction in step 4 to give the diagram that will enable the proof to proceed. Before that construction is effected, the reader will be at a loss as to how the proposition can be shown. Merely inspecting the diagram will not be enough to give the proof, since certain equalities must be established, directly or by appealing to earlier results. But the proof depends on reasoning carried out on the figures as constructed, not on them as originally given in the enunciation.

Euclid, *Elements* 2.5, is an example of medium complexity. But if we consider both simpler and more complex cases, a similar pattern is common. Take the proof that the base angles of an isosceles triangle are equal (1.5). Already in the *ekthesis* we are told that the sides of the triangle are to be produced, while in the construction an arbitrary point is taken on each of those lines as produced and joined to the opposite point at the base of the triangle. That gives the diagram to be used in the proof (the diagram is reproduced in figure 11.8), but it is still the case that the proof has to be given, that is, that certain triangles (*ABG* and *ACF*, and again *BCG* and *BCF*) have to be shown to be similar, and so that certain angles are equal. Even though the proof is obvious to anyone who has

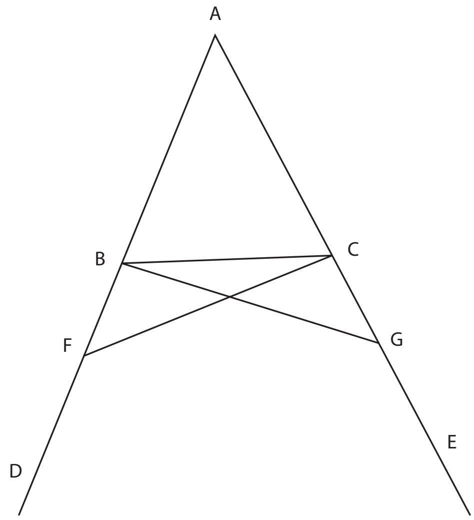

Figure 11.8. The diagram used to prove *Elements* 1.5.

been taken through it before, it is not complete until the sequence of its constituent steps has been completed. It takes a sequence of such steps, indeed, to reveal the truth of the theorem to be proved.

Similarly, in a more complex case, such as the proof of the Pythagorean theorem in Euclid (*Elements* 1.47), the *ekthesis* just identifies a single right-angled triangle, and it is not until the construction that the squares on all three sides are described and the remainder of the "windmill" diagram drawn (figure 11.9). Even so, the proof has to establish both that certain lines are straight (*GAC* and *HAB*) and that certain triangles are similar (*FBC* and *ABD*, and again *BCK* and *ACE*), from which it follows that the two parallelograms into which the square on the hypotenuse has been divided (*BMLD* and *MCEL*) are equal to the squares on the two other sides, so that the theorem has been proved.

Then, for an even more complex stretch of reasoning, I may turn back to the solution Archytas gave for the duplication of the cube. This was a problem, not a theorem, insofar as that distinction is conventionally used to distinguish between constructions to be effected and propositions to be proved. Nor does the reasoning, as reported by Eutocius, stick to the pattern set out by Proclus. Yet there could be no more striking example of mathematical reasoning depending on constructions

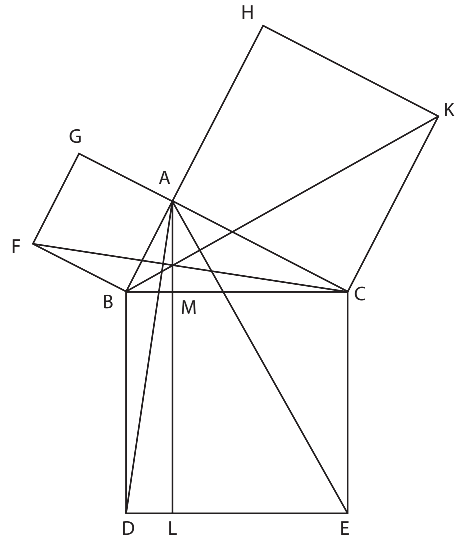

Figure 11.9. The construction used in *Elements* 1.47 to prove Pythagoras's theorem.

involving surfaces and solids in revolution, the generation of a curve, and the identification of a point by considering the intersection of that curve and the cone.

Thus far I have taken cases from Greek geometry. But arithmetic proofs similarly involved sequences of steps. Thus the famous proof of the infinity of primes (Euclid, *Elements* 9.20) proceeds by a reductio. It is first assumed that prime numbers form a finite set. Euclid assumes that the members of such a set can be multiplied together, and to the number so formed he adds the unit. If that number is prime, then he has shown that the starting assumption is false. But if it is not prime, then by 7.31 it is measured by some number. If that number is itself a prime, he has again disproved the initial assumption. But if it is not, it is divisible by some number, and that again gives a new prime. For if it were in the original set of primes, it would measure both that set and the remainder, the unit added to the product of its members, and that is absurd (*atopon*).

Finally, both arithmetic and geometry use iterative procedures that in some cases can be continued indefinitely. In the misnamed "method of exhaustion," applied to determine π, for example, increasingly many-sided regular polygons are inscribed in the circle. As the number of sides

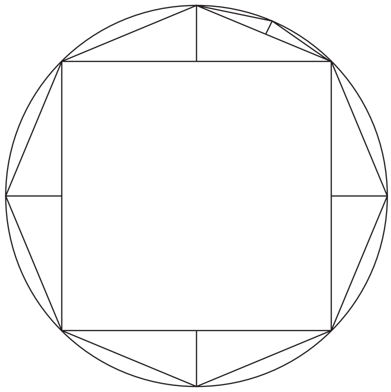

Figure 11.10. Inscribed regular polygons used to approximate the area of a circle. (From Heath, A *History of Greek Mathematics*, 1:221.)

of the polygon increases, the difference between its area and that of the circle diminishes (figure 11.10). While the circle is never exhausted (the Greeks did not contemplate an infinite-sided polygon), the difference between the rectilinear area and the circle can be made as small as one wishes.[17] But this means that the procedure of increasing the sides of the polygon has to be continued indefinitely.

Again, one may cite parallels from elsewhere in Greek mathematics. Fowler Fowler (1999) especially has studied *anthyphairesis*, the reciprocal subtraction algorithm. One application, in Euclid, *Elements* 10.2, was to establish which magnitudes are and which are not commensurable with one another. "If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable." A similar method, using so-called side and diagonal numbers, generally associated with pre-Aristotelian Pythagoreans, was used to get successively closer approximations to the value of irrational square roots, starting with the square root of two. Again, the fundamental point, from the perspective of our concerns here, is that the mathematician proceeds through a sequence of steps, indefinitely prolonged if necessary, to arrive at her results.[18]

We can thus see both where a stretch of mathematical reasoning resembles and where it differs from a narrative. Narratives, as I have remarked, deal with events that have a chronological sequence, whether or not the narrative itself follows that sequence. In mathematical reasoning, time in the sense of chronology is not relevant, since the truths revealed are indeed timeless. On the other hand, the reasoning does involve a sequence of steps that are essential to reveal, or as Aristotle would say to actualize, the truths that are there in potentiality in the geometric figures or the quantities discussed. In the sense that the proof depends on a construction or procedures that are carried out at some point *after* the statement of what is to be shown, in that sense the mathematical reasoning shares the sequentiality, if not the temporality, of narrative. An actuality, Aristotle claimed, is prior to a potentiality in definition and in substantiality. It is even prior in time, by which he meant that in biological generation, for example, it is an actually existing pair of animals that produces the new member of the species. But a final remark in *Metaphysics* (1051a32f) concedes that, even in the case of mathematical thought, the individual actualization (namely, by a particular mathematical reasoner on a particular occasion) is posterior in coming to be to the corresponding potentiality. At this point, rather exceptionally, the lived experience of the reasoner is allowed to enter the picture, for once qualifying the usual claims for the timelessness of the items she reasons about.

But is this just a feature of Greek mathematical reasoning, as discussed by Aristotle, as practiced in, for example, Euclid's *Elements*, and as formally analyzed by writers such as Proclus? We have only to consider some of the evidence for Chinese mathematics (for example) to see that the answer to that question must be no. As Chemla and Martzloff, among other scholars, have shown,[19] Chinese arithmetic depends crucially on certain procedures carried out on counting boards. To effect a division, for example, certain quantities must first be displayed on the boards, and then they have to be manipulated. The sequentiality of the procedures is precisely analogous to what I have discussed in the construction of Greek geometric diagrams and in the steps involved in their arithmetic reasonings. Notions of proving, or of showing certain statements to be correct, certainly differ in a variety of ways between ancient Greeks and Chinese.[20] But mathematical reasoning in both those ancient societies generally exhibits just that aspect that mathematics as a whole shares with narrative. Such at least is the suggestion I wish to propose in this note.

NOTES

1. Plato, *Republic* 527ab, ed. Paul Shorey, Loeb Classical Library 276 (Cambridge, MA: Harvard University Press, 1935). Translation adapted from this source.

2. Plutarch, *Marcellus*, chap. 17, in *Plutarch's Lives*, ed. Bernadotte Perrin, vol. 5, Loeb Classical Library 87 (Cambridge, MA: Harvard University Press, 1917). Translation based on this edition.

3. Elsewhere (*Table-talk* 718e, Perrin), Plutarch adds Menaechmus to the list of those whom Plato reproached.

4. Eutocius, *Commentary on Archimedes On The Sphere and Cylinder*, II: 3.84.13–88.2, Heiberg-Stamatis.

5. In figure 11.1, from Heath (1921, 1:247), *AC* and *AB* are the two straight lines between which two mean proportionals are to be found. *AC* is the diameter of a circle and *AB* a chord in it. Draw a semicircle with *AC* as diameter, but in a plane at right angles to the plane of the circle *ABC*. Imagine this semicircle to revolve about a straight line through A perpendicular to the plane of *ABC*, thus describing half a torus with inner diameter zero. Next draw a right half-cylinder on the semicircle *ABC* as base: this will cut the half-torus in a certain curve. Finally, let *CD*, the tangent of the circle *ABC* at point C, meet *AB* produced at D, and suppose the triangle *ADC* to revolve about *AC* as axis. This will generate the surface of a right circular cone. The point B will describe a semicircle *BQE* at right angles to *ABC* with its diameter at right angles to *AC*, and the surface of the cone will meet, at some point P, the curve that is the intersection of the half-cylinder and the half-torus.

6. Plutarch, *Marcellus*, chap. 14. The last remark concerning "bodies" and "laborious manual work" (*phortike banausourgia*) certainly does not apply to the mathematical proof reported by Eutocius but would have some relevance if Archytas attempted the difficult task of making a scale model of his construction.

7. For Plato, mathematical intermediates share intelligibility with the forms, but unlike them (but like perceptible particulars) they are plural, not singular.

8. Several passages in Aristotle state that mathematical objects are not, in themselves, subject to change: *On the Movement of Animals* 698a25–26, *Physics* 193b34, *Metaphysics* 989b32–33 (where he is dealing with Pythagorean ideas).

9. In figure 11.2, from Heath (1949, 73), *DB* is the *orthe* drawn at right angles to the diameter *AC*. Euclid (*Elements* 3.21) supplies the theorem to show that angle *ADC* and angle *AEC* are equal; in other words, both are right angles.

10. Figure 11.3, from Heath (1949, 72), gives the diagram used in Euclid (*Elements* 3.31). On the second interpretation, the *orthe* drawn from the center of the circle would be any radius, as here *AE*.

11. Our evidence for this comes from Proclus, *Commentary on Euclid Elements Book I* 379.2–16. The line drawn parallel to the base of the triangle is *AC*, as in figure 11.4.

12. As in figure 11.5. Burnyeat et al. (1984, 150–51), however, point out that a shift in the depiction of the triangle, as in figure 11.6, could meet that objection.

13. See Heath (1949, 216), who cites *Categories* 14a39 and *Metaphysics* 1014a36 for this sense, though neither text is unambiguous.

14. He did, however, go on to observe that "to make the construction intelligently . . . is to see the proof, and Aristotle at once passes to this" (Ross 1924, 2:268).

15. See his *Commentary on Euclid Elements Book I* 203.1–15.

16. I have somewhat simplified Netz's presentation which scrupulously indicates terms that have to be understood in Euclid according to normal Greek mathematical terminology. Figure 11.7, from Heath (1921, 1:382), sets out the diagram that Euclid used for the proof.

17. Objections to any idea that such procedures terminate are common. Antiphon in the fifth century BCE is reported to have claimed to have squared the circle on the grounds that the side of the inscribed polygon will at some point coincide with the circumference, an assumption that was held to be in breach of the continuum assumption (see Knorr 1986, 25–27, citing Simplicius, *Commentary on Aristotle's Physics* 54.20ff.). Similarly, when in the *Collection* III Pappus reports the method used by an anonymous geometer to obtain two mean proportionals, he objects that the iterative procedure adopted does not yield the result claimed, even though successive approximations get closer and closer to it. See Cuomo (2000, 130ff.).

18. Fowler (Fowler (1999), chap. 9) sets out the subsequent history of continued fractions; see also Knorr (Knorr (1975), chap. 8, Knorr (1986), chap. 5).

19. See, for example, Chemla and Guo (2004, 15–20) and Martzloff (1997, chaps. 13, 14).

20. The varieties of proof procedures used in different mathematical traditions are the subject of Chemla (forthcoming).

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