Imagination and Layered Ontology in Greek Mathematics

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Abstract

This essay is a study in the routine use of imagination in Greek mathematical writings. By "routine" is meant both that this use is indeed very common, and that it is ultimately mundane. No flights of fancy, no poetical-like imaginative licenses are at stake. The issue rather is the systematic use made of the ability to imagine a virtual presence, and to refer to this virtual presence as if it were on equal footing with the real. This process is not merely routine in Greek mathematics: it may well be considered one of its chief characteristics. In this essay, I describe some features of this practice in detail, and then briefly offer some interpretative comments on the history and philosophy of this practice. The emphasis, however, is on the description, and I concentrate mainly on one key element—the use of the Greek verb *noein*. Thus the structure of the essay is as follows: part 1 is an analysis of *noein* in Greek mathematical writings; part 2 is a less-detailed description of other practices of Greek mathematics that involve "imagination" or, in general, a "layered" reality; and part 3 is a tentative interpretation.[1]

1. At this point, the reader might imagine a set of definitions: What is the real? What is the imaginary? What is the virtual? Such chimeras are to be banished. There is no profit in discussing such abstract generalities and a lot to be gained from understanding particular practices: this essay elucidates, in detail, just one such practice of imagination. I return to such semiotic generalities, contrasting briefly the role of the imaginary in science and in "fiction," in the "Conclusions" section.

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Introduction: *Noein* in Greek Mathematical Writings

We start, appropriately, from the elevated. In the following example, the text seems to "ascend," literally, from a geometrical plane. This is Euclid's *Elements*, the construction stage of proposition XI.12. The task is to set up a perpendicular from a given point on a given plane. Euclid has chosen an arbitrary plane (not given by any diagrammatic letters) and an arbitrary point in it, A, so that the task now is to set up a perpendicular to the given plane, from A. Here begins the construction:

> *nenoēsthō ti sēmeion meteōron to B* (Let any elevated point B be conceived).[2] (In my own system of translation, this would come out as "Let some elevated point be imagined, B.")[3]

The verb *nenoēsthō* is the third-person, perfect passive imperative of the verb *noein*. Its meaning, to begin with, can very well be translated by either "imagined" or "conceived" (we will come gradually to see why I prefer, after all, the English verb *imagine*). As pointed out in my *The Shaping of Deduction in Greek Mathematics*, it is not immediately obvious why one would need to point out that a given component in the diagram is conceived or imagined; after all, is this not true of all diagrammatic objects?[4] Indeed, since it now appears from various studies that ancient diagrams were "schematic" in character (with, for example, equal lines drawn to represent unequals, and so on), clearly everything about the depicted object is "imagined" rather than directly "drawn."[5]

In fact, it is not too difficult to see what is strange or "imaginary" about this particular point, in this particular proposition. Thomas Heath's (the editor and translator of *The Thirteen Books of Euclid's Elements* cited above) printed diagram differs from the manuscripts' version in that he supplied a visible reference plane; the manuscripts

2. Thomas Little Heath, ed. and trans., *The Thirteen Books of Euclid's Elements*, 2nd ed., vol. 3 (Cambridge: Cambridge University Press, 1926), p. 294.

3. The differences between the two translations are either a mere matter of style or cannot clearly be settled by the evidence of the Greek. I do not claim any evident superiority of my translation over Heath's.

4. Reviel Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Cambridge: Cambridge University Press, 1999), pp. 52–53.

5. See especially Ken Saito, "A Preliminary Study in the Critical Assessment of Diagrams in Greek Mathematical Works," *Sciamvs* 7 (December 2006): 81–144; and Reviel Netz, *The Works of Archimedes: Translated into English Together with Eutocius' Commentaries, with Commentary, and Critical Edition of the Diagrams*, vol. 1 (Cambridge: Cambridge University Press, 2004).

have merely two parallel lines drawn side by side. While we cannot fully rely on the manuscripts for Euclid's original version of the diagram, it remains true that the absence of letters referring to the plane make it very likely that the same was true of the original. (Had Euclid drawn a plane, why did he not label it the way he seems in general to do with all diagrammatic objects?) If so, the "imagined" or "conceived" acquires a clear meaning, referring not so much to point B as to its adjective "elevated." The diagram seems to present a fairly standard two-dimensional configuration of parallel lines; to demand that point B is elevated is already to ask us to go beyond the diagram in an extra effort of imagination, assuming an added dimension at which the diagram does not as much as hint. (Could the right translation after all be "Let some point be imagined elevated, [namely] B"?)

Another example of the same verb comes from earlier in the *Elements* (book IV.12). Euclid studies systematically the inscription and circumscription of regular polygons. One step of the construction goes as follows:

> *nenoēstho tou engegrammenou pentagōnou gōniōn sēmeia ta A, B, G, D, E . . .* (Let angle points of the inscribed pentagon be imagined, [namely] A, B, G, D, E).

The use of *noein* here seems to be differently motivated: there is no three-dimensionality involved, and one certainly could draw the actual points in the plane of the figure. However, here the evidence of the manuscripts must be taken into consideration, and it seems to suggest a plausible reason for the verb: the diagram does not in fact draw a pentagon and merely has the five points marked (this in turn is a reasonable choice: we do not really need the lines of the pentagon for the argument, and these would clutter the diagram to the point that its drawing becomes very difficult indeed). We are therefore requested to put in an extra effort of visualization, inserting, in our mind's eye, the missing scaffold of a pentagon to support the visible angles. Note, incidentally, that once again, it is not the pentagon directly that we are required to imagine; rather, we are required to imagine the *points* (which are, after all, the grammatical subject of the verb) *as angle points of a pentagon*. The verb *noein* acts, in such cases as we have seen, as a "seeing-as" operator, demanding that a point be "seen-as" elevated or "seen-as" angle point in a pentagon. This, of course, derives in part from the logical character of the construction in which we engage: every step in the construction brings into existence an object, and the grammatical subject of the verb will be the object brought into existence. That such an object may possess further properties, not seen in the diagram, should then be implied by some kind of modifier dependent on the main verb.

Yet another usage of the verb *noein*, not attested in Euclid himself (simply due to the subject matter) is the *physical*. Archimedes' *Method* provides a typical example:

> *noeisthō zugos ho ΓΘ* (Let a balance be imagined, ΓΘ).

This is a key step of *Method* 1. Archimedes already has established a certain geometrical setting: the points Γ and Θ are already fixed in position, so that the meaning of verb cannot be to bring a certain *geometrical* object into existence. However, we can easily see the import of the verb based on the parallels seen elsewhere: we are once again asked to perform a certain "seeing-as," endowing an object with a new meaning, or perhaps superimposing an object on the one previously given. We had first a geometrical network of points and lines; now, the same letters ΓΘ refer not only to points and their line, but also to a balance, which, of course, is not visualized in any way beyond its linear representation.[6]

We have seen three contexts where the diagram is in some sense deficient, calling for the mind's eye to complete it. One is the three-dimensional, where the plane image is not enough to represent the projected three-dimensional configuration. Another is the complex configuration, where a certain object just does not fit into the constraints of the drawing and therefore has to be imagined instead. The third and final is the physical, where a certain physical significance of the points and lines drawn cannot be encoded by the geometrical configuration itself.

There are, indeed, many occurrences of the verb in this diagrammatic context, and all can be classified—if occasionally with some difficulty—into one of these three categories. I present in Table 1 the main data on *noein* in its mathematical-diagrammatic, impersonal imperative forms.[7]

6. It may be significant—at any rate it is worth noting—that the form in such cases is *noeisqō zugos ho* ΓΘ and not *noeisqō zugos hē* ΓΘ. In other words, the text does not say "let the line ΓΘ be imagined as a balance," but literally asks us to imagine a new object, a balance, superimposed upon that previously given.

7. In practice, we look for exactly four forms: *noeisthō*, *noeisthōsan*, *nenoēsthō*, *nenoēsthōsan*. The realities of Thesaurus Linguae Gracae electronic searches are such that I performed the *noeisthō* search (504 hits) apart from the *nenoēsthō* search (108 hits), and one can note several interesting properties of the tenses: the perfect is almost without exception in the strict mathematical-diagrammatic sense, while the aorist is about evenly divided between the mathematical-diagrammatic and the nontechnical meaning of "conceived"/"imagined" (on which more below). This, however, mainly points to the fact that the perfect impersonal imperative is not a genuine form of

Table 1 Mathematical-Diagrammatic Uses of the Verb Noein in the TLG

Author	3-D	Complexity	Physical	Total
Eudemus/Archytas	1			1
Euclid	13	1		14
Archimedes	33	11	21	65
Apollonius	3			3
Ars Eudoxi			2	2
Philo Mechanicus			1	1
Theodosius	2		1	3
Hipparchus			1	1
Hypsicles		1		1
Strabo			2	2
Theon Phil.			1	1
Hero	8	7	18	33
Ptolemy		1	34	35
Galen			1	1
Alexander			3	3
Porphyry	1	1		2
Serenus	1			1
Pappus	13	8	21	42
Theon	4	2	34	40
Proclus		2	8	10
Eutocius	1	18	3	22
Scholia in Euclidem			6	6
Simplicius			2	2
Olympiodorus	1			1
Anthemius	1			1
"Euclid XV"	5	1		6
Ps. David		1		1
Hero of Byzantium			1	1
Pachymeres	1	1		2
Nicephor	1			1
Total	89	55	160	304

Notes: 1) The TLG (Thesaurus Linguae Graecae) is a research center and digital library of Greek literature published by the University of California, Irvine; and 2) these authors are arranged chronologically.

Greek, and is instead mostly an artificial form kept alive within mathematical usage. Euclid, who has a kind of mathematician's mathematical vocabulary, prefers the perfect, while Archimedes, who keeps closer to natural Greek, uses only the aorist. My impression is that Hero uses *noeisthō* exclusively in physical contexts, *nenoēsthō* exclusively in geometrical contexts. (As if strict Euclidean terminology is appropriate for geometry, but not for its physical application?) Such questions call perhaps for a separate study of the fine structure of Greek mathematical language, but they are irrelevant for the history of imagination in mathematics. In the analysis below, I do not distinguish between the tenses.

The first observation is the widespread of the use of the verb, going through the range of ancient mathematical authors. This verb is clearly entrenched within the Greek mathematical style as such (and not just within the sub-style of a certain group of authors). It is interesting that no uses are found from Aristotle or the Aristotelian corpus, but this may reflect not a late entry of the verb, only the accident of Aristotle's limited need for such a verb: his geometrical configurations are typically simple, and within his style and overall discursive language, diagrams referring to extra-mathematical objects, such as pieces of machinery or the body, are taken "at face value" and are not taken to be "imaginery." Indeed, the accidents of different contents seem to drive the overall distribution of the verb. Euclid has little use for the physical, and his geometry is mostly two-dimensional—and simple. When he does foray into the three-dimensional (in the later books of the *Elements*), the use of *noein* becomes very noticeable. (Apollonius, we may note, is a very two-dimensional author whose complexity arises from more abstract proportions, not from concrete diagrams; his few three-dimensional propositions give rise to a few uses of *noein*.)

The opposite is the case of Archimedes: his wide range of complex, three-dimensional, and physical passages makes him one of the more versatile users of the verb *noein*, and, fittingly, the most significant one in sheer quantitative terms. The works of Archimedes are truly imaginative. Similar considerations help explain why the physical is somewhat more common. A great many of the extant works in the Greek mathematical-diagrammatic style are not in pure geometry, but in mixed genres—the astronomical and mechanical traditions. Once again, it is natural that "complexity" should play a role only occasionally: after all, with careful planning, one may always find a way of packaging information into the actual diagram.[8] All in all, my impression is that the "focal meaning" of *noein* should not be traced to this or that domain, or that its origins should be sought in this or that author; instead, we should look for the elements that are common for the usage across the various authors and genres.

We note further that the verb is quite common: sixty-five uses in Archimedes (whose corpus takes some 100,000 words) are just under 1 in 1,000, which is very respectable for a word type. Euclid,

8. Many of the cases counted under "complexity" derive from the commentators, especially Eutocius, and these are often cases where the complexity is truly unavoidable: this is when a commentator asks us to imagine an object within the *given diagram of the original*. Even if the commentator adds his new diagram, the new inserted object is necessarily invisible and therefore "imagined" relative to the old diagram.

of course, presents a much lower ratio (for reasons given above), but the Archimedean ratio is approached in a few other major authors of the genre—in particular, Hero, Ptolemy, and Pappus. Since the verb is typically used no more than once or twice in a given proposition, and since an average proposition takes a few hundred words, what we find is that among the major authors, when the context at all allowed this, the verb *noein* was used once for every few propositions. This underlines the "routine" nature of the verb.

The Verb *Noein*: The Nonmathematical Context

The verb is indeed fairly routine in Greek as a whole. I did not even attempt a Thesaurus Linguae Graecae (TLG) search for the many various *no*-forms, but it is striking that many could be found—more than three hundred outside mathematical texts—for *noeisthō* alone (*nenoēsthō*, as noted above, is nearly a uniquely mathematical form). Many, as will be seen, do come from a context influenced by mathematics, and there is no doubt that the form *noeisthō* did acquire a specific mathematical undertone. Certainly, these are overwhelmingly in technical writing, which is obvious given the nature of the impersonal imperative. They present a great variety of forms, but two seem especially consistent: the allegorical, and the paradigmatic-explanatory.

A good example of the allegorical usage is from Origen's commentary on Matthew (XVII 5.43):

> *Pas Israēl sōthēsetai. Noeisthō de ho Israēl ouch ho kata sarka* (All Israel will be saved [but Israel should be *understood to mean* not "in the flesh"]).

Here, I translate *noeisthō* by "understood to mean." The meaning, however, does not differ very much from the "imagine" used in my translations from Greek mathematics. The main difference has to do with the different subject matter: *noein* ranges in Greek mathematics across diagrammatic objects, while in Christian allegorical reading, it ranges across verbal meanings. In the Greek mathematical context, you see a certain diagrammatic configuration and train your mind's eye to see beyond the visible (which I translate with "imagine"); in the Christian allegorical context, you read a certain group of words and train your mind to understand in them a meaning different from that uttered. The fundamental sense is the same seeing-beyond.

At an even more conceptual level stands the paradigmatic-explanatory usage. This is often used in many expository works, such as, for instance, in handbooks of astrology. Thus Vettius Valens, with twenty-five, is among the most frequent users of the form *noeisthō* (which often take the following form: after a discussion of a certain topic concerning a given object, Valens explains that the same should be understood, *noeisthō*, for yet another object—for example, "the same *should be understood* for Mars").[9] Perhaps, in astrology, the example of mathematical usage is near enough to influence the author's choice of this verb.

It is interesting to quote the following example, which shows a rather similar effect of proximity to mathematics. Sextus Empiricus, in the course of his overall critique of scientific practice, has shown the absurdity of the assumptions of infinity and continuity, for instance: that a line should be bisected (say, the line has an odd number of points: did we then "cut," impossibly, the "middle" point?). Now he proceeds to assert:

> *Ho de autos kai epi tou kuklou logos noeisthō* (The same argument should be understood to hold in the case of the circle [for here, too, division collapses under natural assumptions about the number of points in the figure]).[10]

In the paradigmatic-explanatory usage, a certain detailed account is made to be a paradigm, which can then be applied to another case. The detailed account, in this case, is left implicit, and the reader is somehow meant to imagine for him- or herself, in his own understanding, what form such an account might actually take. Once again, then, one is to "see beyond."

Of course, the meaning of a Greek verb is not to be found in its third-person imperative form: this is undoubtedly a marked, technical usage. One could have left it at that and suggested that the mathematical *nenoēsthō* (or the more widely used, though still "technical" *noeisthō*) are forms unto themselves, already set apart from the nontechnical meaning of *noein*. While possible in principle, I do not think we need to postulate such a break between the technical and nontechnical uses of *noein*, nor should I now pursue a full-blown semantic study of the term, which would be otiose, following the magnificent study by von Fritz.[11] In a series of studies from 1943 onward, this German classicist and scholar of ancient philosophy

9. Vettius Valens II.2.77.

10. Sextus Empiricus, Adv. Math. IX.284 l.1–2.

11. K. von Fritz, "NOOS and NOEIN in the Homeric Poems," *Classical Philology* 38 (1943): 79–93; von Fritz, "NOOS, NOEIN and Their Derivatives in Pre-Socratic Philosophy (Excluding Anaxagoras), Part 1: From the Beginning to Parmenides," *Classical Philology* 40 (1945): 223–242; von Fritz, "NOOS, NOEIN and Their Derivatives in Pre-Socratic Philosophy (Excluding Anaxagoras), Part 2: The Post-Parmenidean Period," *Classical Philology* 41 (1946): 12–34.

developed an analysis of the terms *nous* and *noein* within the framework of Bruno Snell's history of ideas: an archaic, more concrete concept, gradually evolving into the more abstract and sophisticated form of classical thought. If Snell contrasted Homer with the lyrical poets, than von Fritz contrasted Homer with the pre-Socratic philosophers.

Let me say immediately that von Fritz is, in my view, at his best when studying Homer; there, he is under no pressure to show an "evolution" and can concentrate instead on a synchronic semantic analysis. The later claims for the pre-Socratic philosophers seem forced in comparison; in particular, one has the impression that von Fritz finds new contexts for the *application* of the term, and declares them to be developments of the meaning of the term itself. Obviously, Homer and Democritus talk about different topics. The thematic range of the Homeric poems provides scant opportunity for the discussion of the opposition between sense perception and logical reasoning—just as, at least in the fragments we possess, the pre-Socratics never mention a situation where one suddenly recognizes the evil intent of an opposing hero (which is the most standard usage of *noein* in Homer).

This, once again, does not take away from the value of von Fritz's discussion of Homer. He fits the verb into Snell's discussion of *idein* and *gignōskein*, and his statement is well worth quoting in some detail:

> The term *idein* covers all the cases in which something comes to our knowledge by the sense of vision, including the case in which this object remains indefinite: for instance . . . a brown patch. . . . The term *gignōskein*, on the other hand, designates specifically the recognition of this object as something definite: for instance . . . a human being. . . . This recognition implies, of course, the classification of the object under a general concept. . . . The term *noein*, then, signifies a further step in the recognition of the object: the realization, for instance, that this brown patch is not only a human being but an enemy lying in ambush.[12]

Those are three kinds of seeing: seeing a *token*, seeing a *type*, and *seeing-as*. (Wittgenstein was thinking toward the *Philosophical Investigations* at about the same time that von Fritz was studying the uses of *noein*.) I see no reason to depart from this interpretation, even for our Hellenistic and later sources of the verb *noein* in the impersonal imperative form in mathematical and other technical writings. Throughout, the operation involved is that of seeing-as: looking at a point and imagining it elevated; looking at a line and

12. Von Fritz, "NOOS and NOEIN in the Homeric Poems," p. 88.

imagining it as a balance; looking at the blank space defined by five points and imagining there a pentagon; or considering an account as if it extended to another case, or considering the word "Israel" as if it did not mean "Israel in the flesh." In all cases, one looks at one thing and sees in it something else, beyond what meets the eye.

This puts into relief, first of all, one major feature of what one may call the "philosophical grammar" of the verb *noein*: that it is not about bringing an object into existence. In Greek mathematics, one does not have, say, *noeisthō kuklos* (let a circle be imagined) going on to discuss the circle in the absence of a labeled diagram. The meaning of the verb *noein* is not to bring about the *existence* of the circle, but to make something else *which already exists independently*—say, a certain visible trace—be seen-as, say, a circle.

Why do I prefer, then, the English verb *imagine* (which, after all, does suggest the existential meaning of bringing an object into existence in our mind's eye)? Indeed, in many cases, the English verb *understand*, in the sense akin to "construe" (number 5 in the shorter *Oxford English Dictionary*), is another attractive translation (and the one I have naturally resorted to in my translations of the nonmathematical *noeisthō*). For one thing, of course, the kind of object taken by the mathematical *noeisthō*—traces and their intended objects—does not lend itself to the verb *understand* with its highly abstract connotations. But the issue is deeper than that. What is so striking about the use of *noein* in the mathematical context and that does prompt me to prefer, after all, the verb *imagine*, is its very grammatical context: the imperative form. When a Homeric hero has the *noein*-experience of recognizing another hero as an enemy, this can only be accomplished with regard to a hero who is actually an enemy (unless, that is, the act of *noein* is false). But a mathematical *noein*-experience does not stand in any relation of truth to its object: the line surely is not a balance, which, however, does not leave the act of *noein* as false. Quite simply, the *noein* in question is not an indicative but an imperative: one does not "come to recognize that the line is in fact (although one could not see this immediately) a balance"; rather, one deliberately enters into the agreement that the line is *to be taken as if it were a balance*. This deliberate character of a make-believe seeing-as is best captured, in my view, by the English verb *imagine*. That this particular type of imagination is so routine in Greek mathematics must be telling for Greek mathematical ontology.

When to Imagine?

One key question left unanswered is why the verb is not even more routine. As explained above, there are good reasons for using the verb pretty much anywhere in Greek mathematics. To say that it is used when the make-believe is especially egregious does not really help in defining the sense of the word. What we need to find are places where the verb appears at first glance appropriate and yet is not used. Let me take two examples that I consider typical: one three-dimensional, and the other physical. (I do not see the point for looking for such examples from the "complex"—cases where a diagrammatic object is omitted and yet it is not "imagined," but is instead drawn by the text in some direct form. Obviously, the natural explanation in such cases, which, of course, do show up, is that the absence from the diagram is no more than diagrammatic textual corruption.)

My first three-dimensional example is from Archimedes' *Sphere and Cylinder* (hereafter *SC*).[13] Having measured the surface of a segment of a sphere smaller than a hemisphere, Archimedes quickly establishes the parallel result for a segment greater than a hemisphere. Here is how the construction works out:

> For let there be a sphere, and in it a great circle, and let it be imagined cut (*noeisthō tetmēmenē*) at AΔ by a right plane [i.e., one perpendicular to the great circle].

We can immediately see why the cut is "imagined": the cutting plane is inherently nondiagrammed, as it is perpendicular to the plane of the drawing. But why is the sphere itself not imagined in the first place? Isn't it equally three-dimensional, equally extruding beyond the diagrammable? The diagram of a cutting plane is just a line—and the diagram of a sphere is just a circle. Why can we let the sphere "be" and ask the plane to be "imagined"?

Let us now add a further contrast. It is quite easy to find parallels where an object is cut by a plane perpendicular to the drawing, the verb used being the unmarked *cut*. Why could we not have the sphere simply "cut" at AΔ by a right plane? Why "*imagined* cut"? Here is how proposition 37—a mere few pages back—begins:

> Let a sphere be cut (*tetmēsthō*) by a plane not through its center, and in it a great circle, AEZ, cutting at right [angles] the cutting plane . . .

The configuration is precisely the same as proposition 43 (indeed, the two propositions belong to the same line of investigation of the segment of the sphere). The cutting plane, in both cases, projects into space (as does, of course, the sphere). And yet, proposition 37

13. Netz, *Works of Archimedes* (above, n. 5), p. 1:43.

uses the simple verb *to cut* (*tetmēsthō*), as against the more complex expression "imagined cut" (*noeisthō tetmēmenē*) in proposition 43. I cannot recognize any meaningful difference in the geometrical setting as such. While such judgments have to be subjective, I believe the difference has to do not with the geometrical, but with the discursive context. The idea is not as fanciful as it may sound. A Greek proposition, especially in its construction phase, engages in a certain implicit dialogue with its reader—"please do that," "now have this," "now produce that"—and the very same action may acquire different meanings depending upon its different positions in this implicit dialogue. Now, proposition 37 establishes a field for discussion—how we go about studying the segment of the sphere, focusing at first on the segment smaller than the hemisphere. Having established that, proposition 43 is much more dependent in character and has, as well, a decidedly "hurried" character: now that the result is known for the "smaller than hemisphere" case, it is quickly transferred to the "greater than hemisphere" case. The diagram is perfunctory, and the entire discussion is accomplished through seven steps of argument (including two doubted by Heiberg). In this context, the imagined cut acquires a certain sense: while we bother, in some sense, actually to cut the sphere in proposition 37, in 43, we merely contemplate its being imagined cut, as we quickly move along to derive our result. In proposition 37, we envisaged the sphere in more fullness; in 43, we merely gesture toward its possibility.

If correct, this is highly suggestive. For, of course, proposition 37 was no less "contemplative" or "imaginative." In neither case did we actually take a perfectly rounded ball and cut it with a perfectly smooth plane. Proposition 43 gives up momentarily on the suspension of disbelief, and the imaginary character of the cut is conceded. However—and this is surely a major feature to which we shall have to return—this absent suspension of disbelief makes no difference, as an imaginary sphere whose imaginary nature is conceded works just fine for the sake of the argument, and indeed it cannot be distinguished from the one whose imaginary character is somehow suspended away. As long as the object is possible, it does not really matter that it is purely imaginary—mathematics is the art of the possible.

Let us now look at another, physical example. In proposition 3 of Archimedes' *Floating Bodies* (hereafter *FB*), *I*, it is shown that objects of the same weight as the liquid in which they are immersed remain floating just underneath the surface. The object is put in the water and is assumed to extend above it. Now the discussion proper begins with an extra construction:

> So, let a certain plane be imagined produced (*noeisthō ekbeblēmenon*) through both: the centre of the earth and the water; as well as the solid magnitude [the one immersed in the water], and let the section of surface of the water be ABGD.

It should be explained why this use of the verb *noein* is at all what I call "physical." Let us first note that there is nothing difficult, geometrically, about producing a plane through a given triad of points (that the cutting plane is to be produced "through the solid magnitude" is perhaps vague as a geometrical expression, but good enough for the purposes of this proposition; we merely need the planar section to cut whichever two-dimensional trace of the solid magnitude); nor is the requirement difficult as far as the visualization goes, because, after all, the resulting plane is precisely the plane of the drawing. Rather, the "imagination" acquires its meaning from the preceding physical context: we started out from earth, water, and a solid magnitude, and within this setting we are now to introduce a fully geometrical plane, which therefore calls for imagination. The direction of the imaginary is different from the imagined balance we saw above in the *Method*, where we have inserted an imaginary physical balance into a more fully conceived geometrical configuration. Here, we insert an imaginary geometrical plane into a more fully conceived physical configuration; but even though the direction is different, the main point is the juxtaposition of the physical and the geometrical, explaining the use of "imagined."

Once again, however, we may contrast this with other uses. I now turn back one proposition only, to the second proposition of the same book. The text edited by Heiberg is in Latin, but we now have, in the Archimedes Palimpsest, the original Greek that I translate.[14] The object in this case is to show that the surface of a liquid at rest is spherical:

> For let the liquid be imagined (*noeisthō*), set in such a way that it remains immobile, and let its surface be cut (*tetmasthō*) by a plane through the center of the earth, and let the center of the earth be the [point] K, and the section of the surface—the [line] ΑΒΓΔ.

The liquid is imagined, because, after all, the papyrus in front of Archimedes' readers was solid enough. One had to imagine that the curved figure enclosed inside ΑΒΓΔ stood for a liquid object (in itself understood primarily as a highly conceptual representation of the ocean). However, why could one simply "cut" it by a plane? Isn't this plane, once again, an intrusion of the geometrical into the physical? Once

14. The text is that of the Archimedes' *Palimpsest*, 88r12–18.

again: Why is the plane simply cut (*tetmasthō*) in proposition 2, while it is "imagined produced" (*noeisthō ekbeblēmenon*) in proposition 3?

I am not sure that there is an absolute geometrical difference in this case, no more than there was between *SC* I.37 and I.43. Instead, I think we should look for a contextual difference that is, once again, fairly obvious: in proposition 2, but not 3, the verb *noein* is already used for an object other than the plane; it appears, therefore, that the use of *noein* for the liquid, in proposition 2, immediately preceding the cutting of the plane, would make it much less felicitous to use *noein* again for describing the cut. This is not so much a stylistic as a conceptual point. We see that imagination is a relative term: within the overall context of the imaginary in Greek mathematics, the author sometimes chooses to focus on the imaginary character of a certain act. Such emphasis would lose its meaning had it become ubiquitous (and for this reason, it is indeed rare to have more than one or two uses of *noein* in any given proposition). Emphasis is always relative.

We can sum up the ground covered so far: in some diagrams, some of the information may not be directly represented so that one must not "see it," but "see-it-as (something else)" (but this does not really matter, because diagrams are not direct representations of anything anyway); only in some of those cases it might be pointed out explicitly by the text that objects are "imagined" (but this does not really matter, because everything is imagined anyway, so the verb *to imagine* does not suggest any weaker ontological status). Why do we say so explicitly in some cases? Because it is sometimes helpful to point out that a certain piece of the proposition is even "more imaginary"; for example, because we only glance at it rapidly (*SC* I.43), or because we call for an extra ontological effort in the combination of the physical and the geometrical (*FB* I.2, 3; *Method* 1), or because the diagram is even more rudimentary than usual (*Elements* IV.12, and in a sense also XI.12). The grounds for applying the verb *noein* seem to fragment under close inspection, but the various grounds share a common theme, of the relative or the layered. Some parts of the configuration are "more shown" in *Elements* IV.12, some are "more imagined"; some spheres are considered cut in a fuller detail in *SC* I.37, some are considered cut in a more abstract fashion in *SC* I.43. The liquid is "imagined" in *FB* I.2, and, relative to it, the plane is concretely cut (i.e., the plane is no "more imaginary" than the liquid and therefore one does not need the verb *noein*).

We end up with a truly curious observation: that our texts seem to operate within a multi-layered ontology, with some objects being "more real" and others being, relatively, "more imaginary"—an ontological contrast that keeps being done and undone, because, having made the ontological contrast, the text goes on to assume the reality, equally, of all of its objects, and this for the very good reason that in mathematics, possibility—that is, imaginability—is just as good as reality itself.

More on Layered Ontology

Let us look at a passage in Archimedes' *SC* I.40—the same discussion we have seen already of the segment of the sphere. HZ is said to be greater than ΔΞ (Fig. 1):

> for if we join (*ean gar epizeuxōmen*) KZ, it will be parallel to ΔA

(and then a simple argument with similarity of triangles gives rise to the required result). Why "*if* we join"?—that is, why not simply have a line be joined as usual, in the imperative, with no conditional? Apparently because *no line is joined*. The diagram is telling in this regard: it has no trace to fit the line KZ, and this, combined with the curious expression, suggests that the text expressly acknowledges its reference to a missing line, one which can only be imagined though cannot be seen. (This would be a "complexity"-type imagination, partly because the figure is indeed cluttered, but mostly because we may have here a second-order, "commentary"-like arrangement—likely enough a phrase coming from a late scholiast who therefore simply had to make do with the diagram as given.)

In this case then, an "imagination" of exactly the same character we have seen so far is at work—only that, of course, a TLG search for *noeisthō* would not pick this one up. The realm of the imaginary is bigger than that of *noein*. Having said as such, let us now concentrate not on the rare, marked moments of the "if we join" variety. There are much more central practices that we may reasonably associate with imagination in the sense of the verb *noein*. I shall briefly discuss three: *dunaton*, the future, and the imperative.

Layered Ontology and the Possible

Greek mathematical propositions can usually be divided into *theorems* and *problems*, the theorems stating truths and the problems showing how tasks may be achieved.[15] The first problem in Archimedes' *Spiral Lines* (titled by Heiberg "Proposition 3") has the following enunciation:

15. I have presented my own account of how theorems and problems differ (in their different emphases on the claim and on its grounds) in "Why did Greek Mathematicians Publish their Analyses?" in *Ancient and Medieval Traditions in the Exact Sciences,* eds. P. Suppes et al., (SLI: Stanford, 2000: 139–157).

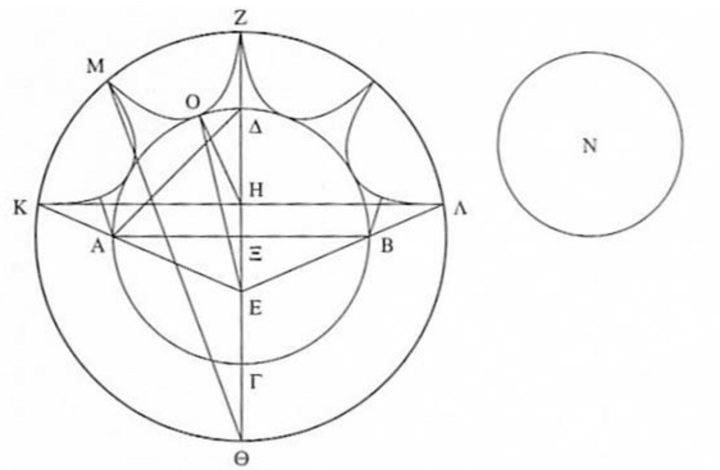

Figure 1. Archimedes' *Sphere and Cylinder,* I.40.

> Given circles, however many in number, it is possible (*dunaton estin*) to take a [straight] line which is greater than the circumferences of the circles.

This is followed by a brief account (which does not even require a diagram) explaining how this may be accomplished: the circumference of any circumscribed polygon is greater than that of its circle (for which Archimedes produced a rigorous argument in *SC* I), so that the task is indeed obvious.

Now, this kind of operation is so standard that we may even fail to notice how strongly it relies on "imagination," in the sense delineated in the previous section. Indeed, no figure is drawn; even if it were, the text would go, necessarily, beyond it. For Archimedes' point is not just to produce a particular straight line greater than a certain set of circumferences; he wishes to make a more general case, that one could always, in principle, no matter which circles are given, produce ("take," in his expression) a certain line. Thus his "it is possible," which refers to an imaginary act, merely gestured at: one glances at the *possibility* of some set of circles, and then nods at the direction of how those circles *might* be circumscribed, and acknowledges that then, indeed, a greater line *would* be found. No need to set up circles, polygons, lines: the possibility, the imaginary, is quite enough. A layered ontology of the imaginary springs up from the real.

The layered character is mostly binary: as against the (in itself hypothetical) set of circles, a line *can* be found. The same binary opposition between real and imaginary is most noticeable in yet

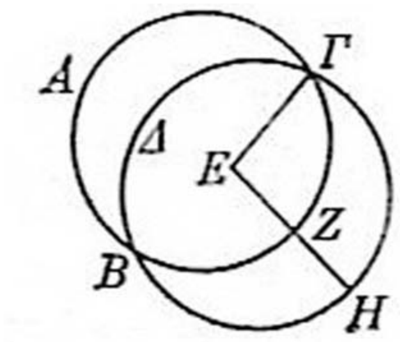

Figure 2. Euclid's Elements, III.5.

another ubiquitous use of the *dunaton*: the argument from contradiction. Thus Euclid, in *Elements* III.5, states that two circles cutting each other do not have the same center (Fig. 2):

> for, if possible (*ei gar dunaton*), let it be E [= center of both circles]. . . . Since the point E is the center of the circle ΑΒΓ . . . again, since the point E is the center of the circle ΓΔΖ.

So is the point E center of both circles or isn't it? Of course, it will ultimately be shown not to be, as such a configuration is indeed impossible (not *dunaton*, after all). But if it *were* possible—well, then it would be *possible* and then we *could* have it, couldn't we? (In mathematics, possibility is tantamount to reality.) And so even the impossible is imaginably possible and therefore can be imagined and locally entertained as reality. Within the terms of the argument, point E simply *is* center of ΑΒΓ, simply is center of ΓΔΖ.

The text does set up two layers of reality: the "normal" one, where the rules of mathematics apply and therefore a point that is center to two mutually touching circles is impossible; and another one, which is so to speak "eccentric," where this rule no longer applies and where the possibility of such a point is entertained. The real of the actual tolerates, briefly, the imaginary of the as-if *dunaton*.

Less counterfactual, but no less complex ontologically, is the case of the indefinitely given object. This, indeed, is one of the most crucial operations in mathematics, and, in a sense, we have seen this already with the line greater than however many circumferences of circle. However many indeed? Five? A million? The text is explicitly silent on this. This becomes much more explicit with the operations related to the argument from exhaustion (i.e., with any application of potential infinity, without which nearly all interesting mathematics becomes impossible). Take the most elementary case, that of Euclid's *Elements* X.1. We wish to show that given two magnitudes, AB greater than Γ, by repeatedly more-than-halving AB (i.e., taking segments that are smaller than a half), one ultimately gets at a segment smaller than Γ. To show this, we take Γ and multiply it; then there is a multiple big enough to make Γ, multiplied by it, greater than AB (this key assumption is more or less included in the axiomatic introduction to book V). And now:

> let it be multiplied, and let it be ΔE.

From which point onward, the proof can indeed flow without too much difficulty. However, what is the status of the magnitude ΔE? Presumably it is the result of taking Γ and multiplying it several times. But just how many times? We are deliberately left in the dark. And so, one is hard pressed to provide this magnitude with a definite meaning. There is no magnitude corresponding to the magnitude ΔE, because we were given no basis from which to say just which multiple of Γ ΔE is. It is merely some indefinite multiple. But there is no such thing as multiplying indefinitely: to multiply is to multiply a certain number of times. What ΔE really is, is a gesture toward a magnitude, a peg holder to stand for the claim that a certain magnitude is possible. It is an imagined object set up relative to the more concretely given AB and Γ.

Layered Ontology and the Conditional

The central operation of Archimedes' first book in *SC*, already briefly mentioned above, involves the imagined rotation of a circle and its associated polygon, giving rise to a sphere and its associated figure composed of truncated cones (Fig. 3); or as Archimedes explains (in, for example, proposition I.23):

> If (*ean*) . . . the circle ABΓΔ is carried in a circular motion (*perienechthēi*) . . . it is clear that its circumference will be carried (*enechthēsetai*) along the surface of the sphere, while the angles . . . will be carried along the circumferences of circles . . . and their diameters will be (*esontai*) . . . and the sides of the polygon will be carried along certain cones . . . so there will be (*estai*) a certain figure inscribed in the sphere . . . whose surface will be (*estai*) smaller than the surface of the sphere.

Technically, of course, this future should not be seen as a simple tense marker: it is rather part of the grammatical marker of a certain conditional, in this case with the *an*/subjunctive *protasis* and a future *apodosis*. Rather than emphasizing a future tense, the verbs chosen stand for a *hypothetical claim*. The text does not state that a

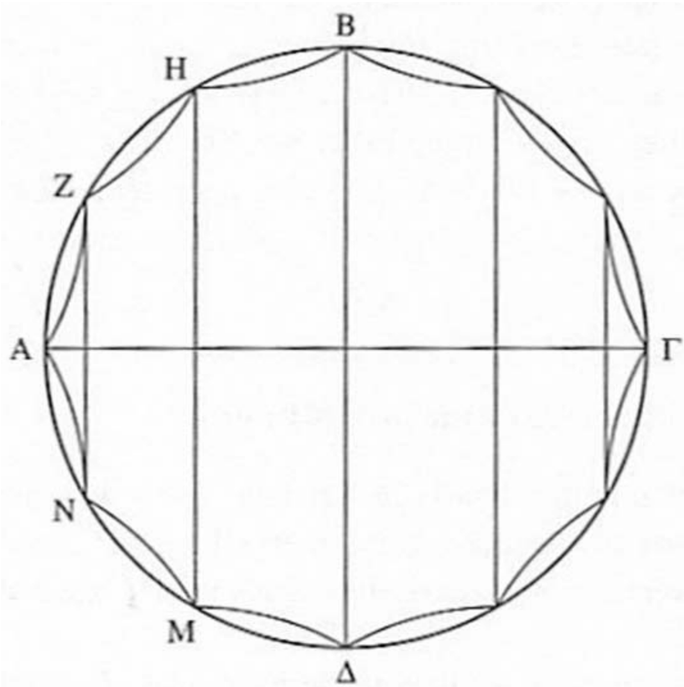

Figure 3. Archimedes' *Sphere and Cylinder,* I.23.

certain circle is rotated, or even would be rotated; it merely states that if indeed such an event would ever come to pass, then the result would follow as well.

Why should Archimedes limit himself to such a hypothetical claim? Instead, he could have asked us to rotate the circle and the polygon, in the usual imperative, and then assert the results in the usual present indicative: for example, "such and such is carried along a circle," "this is greater than that." This, however, would miss the special opportunity of this argument, which is designed to spring a surprise upon us: we have looked already at this circle, this polygon, in propositions 21–22, and now we suddenly realize their relation with the sphere and the cones. The "if the circle is moved" phrase implicitly asks us to look back at the preceding propositions and imagine how they are projected into space through a rotational movement. This, then, is in line with the acts of imagination that we have seen in the preceding sections: against the background of the more stable, given circle of propositions 21–22, we are now to imagine an extension into space that is more "imaginary" in character.

There is another, complementary account: we may explain the use of the structure of an "*ean*/subjunctive → future" conditional as a reflection of the wider practice of Greek enunciations. In these "thought experiment" propositions in *SC* book I, Archimedes does not develop the standard division of the propositions into parts headed by a general enunciation; instead, the propositions start *in medias res* with the construction, and the main claim (that if the rotation is performed, a certain surface is smaller than another) stands, as it were, for the enunciation of the propositions. If so, it is only natural that it should take the form of an "*ean*/subjunctive → future" conditional, since such, after all, is common enough in Greek theorems, starting with the very first theorem of Euclid's *Elements* (book I.4) (Fig. 4):

> *If* (*ean*) two triangles *have* (*echēi*) the two sides equal to two sides respectively etc. . . . they *will* also have (*hexei*) the base equal to the base etc.

Here, the future is very obviously not used in the sense of the tense: the equality of the base is surely co-temporal with the equality of the sides. Even the conditional usage, however, is not obvious: one would expect the present in the *apodosis* to stand for a "general truth" (which one does find occasionally in Greek mathematics—for example, the very first proposition of *SC* book I, in which, if the condition of the *protasis* is met, a certain circumference is greater [*meizōn estin*] than another, in the present tense). Euclid also avoids, in this case, stating the positive claim (which one often finds in modern geometrical textbooks) of the type "in any geometrical object where *X* holds, *Y* holds as well." This is used occasionally in Greek mathematics in the "every" (*pas*) version; for example, in book I.18 of Euclid's *Elements*:

> *Pantos trigōnou hē meizōn pleura tēn meizona gōnian hupoteinei* (In every triangle, the greater side subtends the greater angle).

The alternatives are easier to understand: an enunciation in the "general rule" form, or one eschewing conditional form altogether, referring instead to a general class. And yet the standard form of a Greek enunciation is that of the "*ean*/subjunctive → future" conditional. This conditional is called by the grammar books a "more vivid future."[16] Goodwin cites Euripides: "If you do not restrain your tongue, you will have trouble."[17] The "trouble" involved is imagined

16. The discussion by a modern linguist (Albert Rijksbaron, *The Verb in Classical Greek* [Amsterdam: J. C. Gieben, 1994], p. 67) is slightly different: "The speaker considers fulfillment of the condition very well possible" (!).

17. W. W. Goodwin, *A Greek Grammar* (London: Macmillan and Co., 1894, p. 300), (fr. 5).

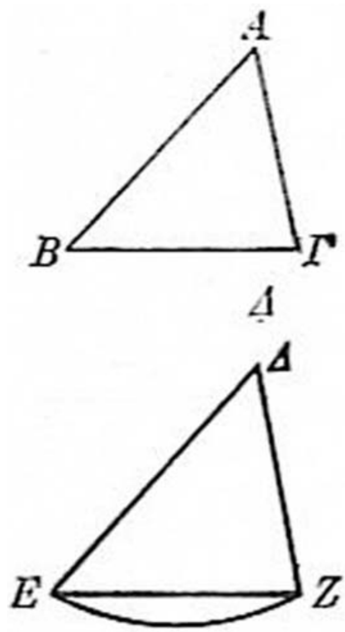

Figure 4. Euclid's Elements, I.4.

vividly, hence the use of the subjunctive/future indicative combination (rather than an optative/optative combination). One is nearly at a loss to see how this applies in Euclid's case. Perhaps our account of the "imaginary," so far, is of some help; Euclid acknowledges the necessary link: if the condition, then the result. But he is not asserting this as a fact, but instead as a projected possibility: relative to the given of a certain condition, then, a certain result would undoubtedly come to pass as well, not in the sense that the *apodosis* follows the *protasis* temporally, but in the sense that the *apodosis* has a qualified, dependent ontological status.

Layered Ontology and the Imperative

One has to be somewhat speculative with the interpretation of the use of the conditional in Greek mathematics: this usage has become formulaic, and so its precise meaning cannot be directly derived from its usage elsewhere. The same is undoubtedly true of one of the most marked formulaic features of the Greek mathematical style—namely, the use of the impersonal imperative. Why would Greek mathematicians ask us, time and again, to "let *X* be?"—a phrase that can hardly even be translated into most other languages! (The English expression is an ad hoc formula with which we make do, as the Greek form has no genuine English equivalent.) While habituated now to its mathematical use, let us remember that this modern expression derives from a Greek original and has no other cultural parallels. The most standard way of making the same kind of request in the course of mathematical constructions is through *personal* imperatives and indicatives. Thus Høyrup's translation of the Babylonian clay tablet VAT 8390 1, chosen quite at random:

> a surface I have built [1st person indicative] . . . 10´ the surface posit [2nd person imperative].[18]

Such is the style associated with the many non-Greek mathematical traditions, which are also often called "algorithmic." A large part of why the operations taken by those traditions are felt to be more algorithmic and less theoretical has to do with this very use of the personal indicative and the imperative. The statements and commands regarding the constructions appear not as Greek theoretical statements of possibility, but rather as concrete acts and directions. The Babylonian clay tablet appears, on its surface reading, to be much more "serious" about its business (and indeed one cannot rule out the possibility that the reader was actually expected to "posit" a value on a counting board, as his Chinese counterpart likely did).[19] Against this background, the Greeks appear to be much more hands-off. Thus goes the first act of construction in Archimedes' *SC*:

> *Let* a polygon . . . *be circumscribed* (*perigegraphthō*) around a circle.

It is most emphatically not said who should perform the circumscription. Indeed, no teacher–student relationship has been formed as in the Babylonian text (or as in any of its cultural parallels elsewhere). No people are envisaged engaging in the construction. This is the work of what Taisbak memorably calls "The Helping Hand":

> a well-known factotum in Greek geometry, who takes care that lines are drawn, points are taken, circles described . . . never is there any of the commands or exhortations so familiar from our classrooms [or, I might add, from ancient non-Greek classrooms!]. . . . Always *The Helping Hand* is there first to see that things are done, and to keep the operations free from contamination by our mortal fingers.[20]

18. Jens Høyrup, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin* (New York: Springer-Verlag, 2002), pp. 61–62.

19. See, for example, Jean-Claude Martzloff, *A History of Chinese Mathematics*, trans. Stephen S. Wilson (1987; reprint, New York: Springer-Verlag, 1997), pp. 209ff.

20. Christian Marinus Taisbak, *Euclid's Data: The Importance of Being Given* (Copenhagen: Museum Tusculanum Press, 2003), pp. 28–29.

By leaving out the question of the responsibility for the action, the third-person imperative immediately stresses its somewhat virtual character: the action is somehow done, but not by anyone in particular—which, at the very least, is curious. No less curious is the choice of morphological tense. The Greek mathematical imperative varies between aorist and perfect. The aorist is easy to understand (and is easily paralleled elsewhere in Greek), referring to the aspect of the action that—naturally, for a geometrical construction—progresses by punctual steps. But the perfect is much rarer elsewhere, so that its mathematical usage is much more marked. Indeed, this usage may well have developed a specific significance in its mathematical usage. Goodwin (who does not seem to have consulted any mathematical texts) suggests the meaning of a "command that something shall be decisive and permanent," though I am not sure that this is at all the sense operative in the mathematical usage. (Are we to have *nenoēsthō*, decisively and permanently?)[21] The apectual sense of the perfect seems relevant: when Archimedes asks that a certain polygon *gegraphthō*, he really wants it to have been *already* circumscribed. This, of course, does stretch our standard notion of the imperative. If one issues a command, surely its fulfillment is to follow the injunction—not the other way around. Such, however, is the case with full-bodied personal commands, where the sequence of command and execution follows the typical temporal logic of human dialogue. The Greek mathematical imperative, however, does not engage in any *typical* human dialogue; it is a statement that something should have been completed, once again employing the imperative rather than the indicative as a gesture designed to set the ontology at a certain remove. No, the polygon is not drawn in actual fact; instead, its having been drawn is what we take for granted. Every piece of Greek mathematical construction is in this sense a projection, a momentary layering of the ontology.

Conclusions

The imaginary in Greek mathematics is ubiquitous and manyfaceted. The verb *noein* asks us, explicitly, to *see* an object, in our imagination, *as* something else. Other forms of expression make the qualified reality of an object explicit: this one is merely asserted to be "possible"; this one is dependent on a condition; that one is impersonal—that is, merely virtually, made to be. The fundamental feature of all these practices is that one momentarily makes more explicit the ontological status of the objects concerned: no, we do

21. Goodwin (1894:273) (above, n. 18).

not have before us a sphere cut by a plane—we imagine it; no, we do not quite say that the point is the center of the circle—we merely assert, for the sake of the argument, that this is possible; no, we do not say that base is equal to base—we merely say that it would be, given some conditions that we do not guarantee; and no, we do not say that the object is really there—we merely assert that it ought to be taken that it is in existence. This ontological caution runs through the entirety of Greek mathematics: nearly all existence claims are qualified in one form or another. But, curiously enough, we do not end up with a sense that there is anything qualified about the results obtained by Greek mathematics; nor do we feel that the ontology is really all that complicated, a fantastically many-layered, nested pattern of qualification within qualification. Everything is qualified and yet everything is felt to be real, because the imagined, the possible, the conditional, the taken-for-granted are immediately transformed—once they have been established—onto a par with the real.

The structure we have seen throughout was not really many-layered, but binary: against the assumed context of ordinary reality, a moment of more explicit imagination projects an extra layer of reality. Immediately following that, however, the projected, more imaginary reality is absorbed into the assumed context of ordinary reality, its imaginary origins forgotten. It is as if the Greek mathematical text was a rare isotope, imagino-active: it keeps emitting a certain radiation of the imaginary, pulses of the verb *noein*, of the possible, the conditional, the third-person imperative. But these emissions have an extraordinarily short half-life. A sentence following the emission of the imaginary pulse—and the imaginary quantity has already decayed into a lump of ordinary reality (which then, of course, emits further and further pulses of the imagination that once again decay into the real). Thus by continuous pulses of the imagination and their continuous decay into the lumps of ordinary reality is Greek mathematical reality constructed.

It is clear why Greek mathematics ought to rely so heavily on the imaginary, because, after all, its objects are never there for us to study—or more precisely, the extraordinary logical achievement, of a perfectly certain argument, could only have been achieved through the willingness to focus on a universe based wholly on the imaginary. However, the remarkable thing is how little this reliance upon the imagination is present to our minds. The key to the success of the operation is the extremely short half-lives: the fact that whatever is imagined is thereby also taken to be on a par with the real. Thus instead of setting up an elaborate Potemkin village of an imaginary mathematical universe, for which a certain set of arguments is made, the Greek mathematicians always operate within a context that is basically taken for real—its momentary emissions of the imaginary are immediately absorbed into this real background.

It is not, of course, just in Greek mathematics that imagination is ubiquitous. Any reading, as in Elaine Scarry's memorable title, involves some "dreaming by the book," and more generally, any semiotic act involves a suspension of disbelief: a sign is to be taken for its signified.[22] The ontological compact between author and audience is ubiquitous in all forms of human expression. It takes, however, different forms depending on the structural and historical forces in each individual genre. Seen in this perspective, we need to delimit, finally, the role of the imaginary, contrasting science to literature (a contrast that is sometimes made explicitly to be that of nonfiction to fiction). In literary works, particularly those of a certain modern tradition, suspension of disbelief is invasive: the many individual claims about the figures are each taken to be at some level true, at others false, the entire point of the game being to scrutinize the fiction's believability in its many details. As Catherine Gallagher, in "The Rise of Fictionality," puts it: "Novels seek to suspend the reader's disbelief, as an element is suspended in a solution which it thoroughly permeates. Disbelief is thus the condition of fictionality, prompting judgements, not about the story's reality, but about its *believability*, its plausibility."[23]

The scientific ontological compact we saw was more localized: the author in Greek mathematics relies on a certain fundamental ontological fictionality (which he sometimes wishes to emphasize, hence the use of such forms as *noein*), that this or that object has to be imagined and therefore need not be assumed actually to exist (or need not be assumed actually to possess this or that property). However, the claims that are to follow from the fundamental act of fiction are understood to be entirely independent of the author's power of fictive creation, and are instead taken to be unqualified truths that do follow, as a matter of logic, from the assumptions one takes. Admittedly, this is not entirely unlike the reader's conviction that Anna really has no other option than taking her own life at the end of the novel, though the key difference remains that the author is allowed to ask the question whether the conclusion

22. Elaine Scarry, *Dreaming by the Book* (New York: Farrar, Straus and Giroux, 1999).

23. Catherine Gallagher, "The Rise of Fictionality," in *The Novel*, vol. 1: *History, Geography, and Culture*, ed. Franco Moretti (Princeton, NJ: Princeton University Press, 2006), p. 1:346 (emphasis in original).

really does hold in the novel *Anna Karenina*, and for this reason the work is throughout ontologically ambivalent, the fictional work kept in playful interplay of the imaginary and the plausible. This kind of ontological ambivalence is absent from Greek mathematics: the direction is unique from the imaginary premise to the certain, and real, conclusion. As a consequence of this difference, perhaps the parallel between science and literature should not be pushed too hard: I have shown in this essay that mathematics partakes in the imaginary, but the severe delimitations of the uses to which the imaginary is put are such as to make it, ultimately, very different from the play of reality in works of so-called fiction. The following, however, should be said by way of tentative suggestion for future research in, so to speak, the field of *comparative fictionality*. Gallagher seeks to show how, between the two speech-acts of *truth-telling* and *lying*, the modern novel has constructed the *tertium* of the *contemplation of verisimilitudes*. This *tertium*, however, as we saw, has always been the domain of Greek proofs: How much did the modern novel, in its quasi-logical contemplation of consequences, owe to a tradition whose origins lie with authors such as Euclid?[24] I shall say no more at this general level of semiotic speculation. To conclude, I make two remarks, starting from this phenomenological account: one philosophical (in a narrow sense), and the other historical.

There is not much extant ancient Greek philosophy of mathematics, and some of that (e.g., in Plato and the neo-Platonists) seems to be conducted at a great remove from the ancient mathematical practice. This pattern may reflect the bad luck of the survival of ancient works (we would dearly like to know more about Zeno of Sidon's criticism of mathematics, or indeed more of the Epicurean reception of mathematics).[25] Mostly, however, it appears that the great flowering of Alexandrian mathematics coincided with a certain disengagement of philosophy from it: perhaps this is a matter of Athens diverging from Alexandria (I suggest as much in my book *Ludic Proof* [Cambridge, 2009]). Even so, the traces that do survive suggest both the major philosophical critique of mathematics, as well as its major rebuttal.

Sextus is the most eloquent of surviving critics, devoting an entire, lengthy book of *Against the Professors* to a critique of Greek geometry.The bulk of this book plays out the many variations on the prob-

24 I wish to thank Kathryn Hume for discussions on this topic, which she intends to pursue further in her own research.

25. See David N. Sedley, *Epicurus and the Mathematicians of Cyzicus, Cronache Ercolanesi* (Naples: Macchiaroli, 1976), pp. 6:23–54; and Ian Mueller "Geometry and Scepticism," in *Science and Speculation: Studies in Hellenistic Theory and Practice*, ed. Jonathan Barnes et al. (Cambridge: Cambridge University Press, 1982), pp. 69–95.

lems arising from demanding that the point be zero-dimensional. Take a random argument from section 26: a circle is assumed to be traced by a point rotating at the end of a line; thus its circumference should be without dimension as well, just as its drawing point was—but a nondimensional circumference of the circle is apparently taken by Sextus to be absurd. In other words, Sextus provides us with an arsenal of critiques of certain make-believe moves inherent to Greek mathematics, in particular those having to do with dimensionality. The make-believe having to do with dimensionality is of a special significance: the zero-dimensionality of points, as well as the one-dimensionality of lines and the two-dimensionality of planes, separates them from their correlated physical objects (which are always, obviously, three-dimensional). It appears that the critique of geometry in antiquity most typically revolved around the question of the applicability of geometry to physical objects, on the assumption (shared by all major Hellenistic philosophical schools) that to be true of spatial objects, is to be true of physical objects, so that a study of spatial objects such as geometry ought to be true of physics if it is to be true at all. Aristotle provides the basic account of how such a problem is most naturally avoided: geometry is, of course, true of physical objects inasmuch as (*hēi*) they are studied geometrically. (Considered geometrically, Aristotle would tell Sextus, the circle is indeed bounded by a rotating point.)[26]

This can be taken in at least two ways. One would be that we construct, once and for all, a hypothetical realm of geometrical objects, which we then go on studying independently of physics or indeed any other domain. We may then retrieve the results obtained in this realm as true for physical objects, inasmuch as a certain filter allows the transition. An alternative, or perhaps complementary,

26. The literature on Aristotle's *Metaphysics* M3—the major source for this account—is considerable. For some major references, see: Ian Mueller, "Aristotle on Geometrical Objects," *Archiv für Geschichte der Philosophie* 52 (1970): 156–171; Julia Annas, *Aristotle's* Metaphysics*, Books M and N* (Oxford: Clarendon Press, 1976); Jonathan Lear, "Aristotle's Philosophy of Mathematics," *Philosophical Review* 91 (1982): 161–192; Edward Hussey, "Aristotle on Mathematical Objects," in *PERI TWN MAQHMATWN*, ed. Iab Mueller (*Apeiron* 24:4) (Edmonton: Academic Printing and Publishing, 1992); J. J. Cleary, *Aristotle and Mathematics: Aporetic Method in Cosmology and Metaphysics* (Leiden: E. J. Brill, 1995). I have looked at it myself in "Aristotle's *Metaphysics* M3: Realism and the Philosophy of QUA," *Princeton/Stanford Working Papers in Classics*. 2006. http://www.princeton.edu/~pswpc/pdfs/netz/120602.pdf. To suggest how Aristotle might have replied to Sextus is not all that anachronistic: Aristotle did have in mind specific puzzles, delineated mostly in *Metaphysics* B2, which closely anticipate those of Sextus and that may indeed go back to Protagoras's famous sneer that the ruler does not touch the circle at a point . . . (fr. 7, from Aristotle's *Metaphysics* B2 itself).

interpretation would be that in geometry we keep making ad hoc assumptions that are taken as if they were true of real objects; such ad hoc assumptions are admissible to the extent that they indeed hold. Now, I am not sure that the discussion in Aristotle allows us to distinguish between those two interpretations. Sextus, however, is much more explicit. The very first words of *Against the Professors* (III) are to the effect that geometers, aware of the multitude of the *aporias* besetting them, take refuge in

> *to ex hupotheseōs aiteisthai tas tēs geōmetrias archas* (Demanding the principles of geometry by hypothesis).

The mention of *archai* strongly brings to mind the sense of an axiomatic foundation, while the verb *aiteisthai* clearly points to Euclid's postulates.[27] Thus Sextus clearly has in mind the hypothetical character of Euclid's axiomatic introduction to book I of *Elements*, setting off an hypothetical geometrical realm. This, however, is not the only sense Sextus has in mind. Having started from this description of geometrical practice, he goes on to say (I simplify a bit) that a hypothesis is either true (in which case it should be shown to be true) or it is false (in which case it cannot serve as foundation for an argument). To round off this discussion, Sextus concludes in section 17 by saying that the preceding arguments suffice to show that the geometers do not do well:

> *Ex hupotheseōs lambanontes tas archas tēs apodeixeōs kai hekastou theōrēmatos, epiphthengomenoi to "dedosthō"* (Taking hypothetically the principles of proof and of each proposition, uttering their "let it be given").

Here, the explicit reference to "each proposition"—the present participle that suggests a repeated action, as well as the perfect imperative in the third person[28]—all point to the type of practices we have surveyed in this essay, in particular to those of the impersonal imperative of construction.[29] In other words, Sextus starts out his cri-

27. In extant works, the verb is used by Greek geometers either in the first book of Euclid's *Elements* or, by late commentators, under the influence of Euclid: see Charles Mugler, *Dictionnaire Historique de la Teminologie Géometrique des Grecs* (Paris: Gauthier-Villars et C. Klincksieck, 1958), s.v. aíthma.

28. *Dedosthō*, while in use there, is not a very common verb in Greek mathematical constructions (TLG lists thirteen uses from Archimedes, for instance). I believe it is taken here as a kind of representative of the type of verbs used in the third-person imperative of construction, the meaning of "to be given" being emblematic of the kind of hypothetical operation Sextus has in mind.

29. Sextus ends this sentence saying that with the arguments above, it has been sufficiently "furnished" (*kataskeuastai*) why the mathematicians do not do well. The term

tique of the hypothetical character of mathematics by mentioning Euclid's postulates, but he rounds up his discussion by mentioning the many hypothetical acts of constructing a geometrical object step by step, theorem by theorem. Perhaps he saw Euclid's postulates simply as an especially useful example of a mathematical hypothesis, and had in mind throughout, *even* from the beginning, the issue of the many ad hoc hypotheses.

Be it as it may, Sextus provides clear evidence that the kind of practices discussed in this essay were noticed in antiquity—and noticed for their significance, as *hypothetical* moves. Ancient philosophers who wished to be critical of Greek geometry seized on its hypothetical character—or what I would call now its "imaginary" character. But, of course, it is not as if the Greek geometers naively furnished their opponents with weapons to use against them. To the contrary: they used the hypothetical specifically so as to "take refuge" in it, as Sextus pointed out. The one way to maintain the certainty of Greek geometry was to resort time and again to the fiction of an object precisely answering to a certain description, and then engaging in a discourse where the fictional is always taken on a par with the real. The success of this practice constitutes, for both ancient and modern philosophers, the puzzle of the ontology of mathematics.

The account of this essay—in the way of such phenomenological descriptions—ended up accomplishing a certain estrangement: we start with a mathematical practice that we tend to take for granted, and end up with a strange, dream-like world, an imagino-active substance whose imaginary emissions quickly decay into the real. The brief philosophical context above may already point at the opposite direction and reduce the strange, once again, into the commonplace. The key term in the philosophical context is not the "imaginary," but the "hypothetical," and this is already much easier to grasp: surely, mathematics is a domain where hypothetical existence collapses into actual existence, as mathematical existence is the same as possibility or, at the most, constructability. Regardless of how one brings about a mathematical object—whether by the power of your ruler and your compass or by the power of your imagination—as long as the object is, as a matter of logic, possible (or, more restrictively, constructible), it is as good as real: it just cannot become any more

kataskeuē is enshrined in Proclus' division of the proposition to mean "construction," and in the actual mathematical usage prior to Proclus, it means the overall geometrical construction governed by the imperative, in the setting out as well as the construction (see Reviel Netz, "Proclus' Division of the Mathematical Proposition into Parts: How and Why Was It Formulated?" *Classical Quarterly* 49 [1999]: 282–303). Could Sextus intend a pun here?

real than that, inasmuch as it is a *mathematical* object. All of this may be right as a matter of the contemporary philosophy of mathematics, which I am definitely not going to enter here; my entire point, finally, is to effect an estrangement of *this* logical conclusion.

For, after all, it is not a given that we should engage in a field where reality and possibility are interchangeable. We only do so thanks to a certain historical trajectory whose origins lie in Greek mathematics with its practice of the imaginary. Modern logical theories of the mathematical object are reflections upon a practice whose phenomenology was given in this essay—a practice of the imaginary. Other civilizations do not engage in this practice; other mathematical traditions simply did not engage with the "let it be imagined," the "if a line is extended," let alone the "for, if possible. . . ." We do so because it has proved to be so fruitful: taking the imaginary as the real, one can follow assumptions to their conclusion and gain new ways of understanding. But why start to engage in such a practice? Whence this willingness to revel in the unreal?

Here, of course, one has to be speculative. But Sextus already provides us with some of the answer: the mathematicians "take refuge" in the hypothetical—refuge, that is, from critics such as himself. To admit that a claim is merely hypothetical, merely imaginary, is a way of reducing one's exposure to criticism: "No, I do not say the ruler does touch the circle at a point. But *imagine* it does." Greek mathematics is a practice based on the imaginary, because it was forged under the pressure of relentless criticism. This would have to be only part of the answer, however. Of course, to admit the "merely hypothetical" status of one's claims would be a way of reducing one's exposure to criticism. But why should the price paid in engaging in the unreal be worth paying? Is it worth it to protect oneself from criticism, if one thereby stops talking about what really matters? In a civilization where the imaginary does not count, this would be too heavy a price. In short, we must assume that the civilization that gave rise to the practices of Greek mathematics was already willing to embrace the radical thought experiment, the counter-intuitive, the merely hypothetical.

As, of course, it was. Whence Greek mathematics? From the civilization that contemplates the gods as they would be depicted by Ethiopeans, or by bulls (as does Xenophanes, frr. 15–16), that explains that the way up and down is the same (Heraclitus, fr. 60) and, of course, that shows that there is no change (Parmenides, fr. 8). Following a radical thought experiment to its logical conclusion—whose paradox is not avoided, but relished—is the basic mode of Greek intellectual life. This is also a specifically Greek characteristic:

while thought experiments and paradoxes are present in many cultures, the Greeks are the only culture to rely on the radical thought experiment as their basic intellectual mode. Surely this should be considered as a further background when analyzing the Greek mathematical practice of the imaginary. It is not that "mathematics is the domain where the possible is the real," stated as a trans-historical fact; rather: a certain historical setting—the engagement with the radical thought-experiment characteristic of Greek culture—gave rise to a project where the possible was systematically taken on a par with the real, and the great success of this project gave rise to mathematics as we know it.

We can be more specific as to the historical setting. Geoffrey Lloyd has shown the role of radical debate, one that is willing to question everything, in Greek civilization, and its roots in the unique political experiment of the Greek city-state.[30] Looking specifically at the origins of Greek mathematics, Lloyd has argued essentially that the demonstrative turn of Greek mathematics was a response to the culture of radical criticism: aspiring to attain, within this context, the special status of incontrovertibility.[31] While Lloyd does not refer to the mathematicians "taking refuge" in the hypothetical, his argument can be directly extended to cover this move: as part of their overall effort to attain incontrovertibility, Greek mathematicians make guarded, hypothetical claims.

What I would wish to add to this is another strand of the specifically Greek historical context of Greek mathematics. It is not only that Greeks, being Greeks, made sure that they were guarded in their statements, making them purely hypothetical; more than this, Greeks, being Greeks, were already used to the notion of making a claim that plunges into the unreal, that engages with the radical thought experiment. It was the civilization of radical debate, of a free-for-all where the very fabric of society could be recast. How should one live: in a democracy or an aristocracy? What should the constitution be like? Such questions were open, and as a result, so was the question of how the gods would be drawn by Ethiopeans, by bulls. It was allowed to conceive of an entirely different social fabric,

30. See Geoffrey Lloyd, *Magic Reason and Experience: Studies in the Origin and Development of Greek Science* (Cambridge: Cambridge University Press, 1979); Lloyd, *Demystifying Mentalities* (Cambridge: Cambridge University Press, 1990); and works since, such as Lloyd, *Adversaries and Authorities: Investigations into Ancient Greek and Chinese Science* (Cambridge: Cambridge University Press, 1996); and Lloyd and Nathan Sivin, *The Way and the Word: Science and Medicine in Early China and Greece* (New Haven, CT: Yale University Press, 2002), comparing Greece with China.

31. Lloyd, *Demystifying Mentalities*, chap. 3.

and for this reason, it was conceivable that the road up and down should be the same and that change will be impossible: the seemingly impossible was not shunned. And finally, for the very same reasons, one was willing to contemplate the sphere cut by a plane, the line as a balance, the sides equal to the sides, and the polygon circumscribed around the circle. One was used to the imaginary: after all, it was constantly debated in the agora. In such flights of imagination are to be found, I argue, the origins of mathematics.