Bryson's squaring of the circle: a discussion of Becker's *Eudoxus-Studien II*

I. Bryson's theory (Alexander's interpretation)

a. Themistius (4th century) on Aristotle's *Posterior Analytics*, p. 19.7-11, 13 (Wallies)
χρῆται γὰρ ἀξιώματι ἀληθεῖ μὲν κοινῷ δέ· τοιοῦτον γὰρ **ὧν τὰ αὐτὰ μείζω καὶ ἐλάττω, ἐκεῖνα εἶναι ἴσα ἀλλήλοις.** ... ὁ κύκλος, φησί, τῶν ἐγγραφομένων πολυγώνων ἁπάντων μείζων ἐστί, τῶν περιγραφομένων δ'ἐλάττων· ὁμοίως καὶ τὸ πολύγωνον τὸ μεταξὺ γραφόμενον τῶν τε ἐγγραφομένων και τῶν περιγραφομένων τῷ κύκλῳ· τῶν αὐτῶν ἄρα μείζων τε καὶ ἐλάττων ἐστὶν ὅ τε κύκλος καὶ τουτὶ τὸ πολύγωνον, ὥστε καὶ ἀλλήλοις ἴσα διὰ τὸ ἀξίωμα τὸ εἰρημένον.

"For [Bryson] makes use of an axiom that is true but common. This is: **things than which the same things are larger and smaller are equal to one another**. ... The circle, he says, is greater than all inscribed polygons and smaller than all circumscribed polygons. Similarly with the polygon drawn between the polygons inscribed and circumscribed about the circle. The circle and this polygon are greater and smaller than the same, so that they are equal to one another due to the stated axiom." (tr. Mendell, slightly emended)

b. Alexander of Aphrodisias (early 3rd century), cited by John Philoponus (6th century), on Aristotle's *Posterior Analytics*, p. 111.21ff. (Wallies)
παντὸς ἐγγραφομένου ἐν τῷ κύκλῳ εὐθυγράμμου σχήματος μείζων ἐστὶ ὁ κύκλος, τοῦ δὲ περιγραφομένου ἐλάττων· ... ἀλλὰ καὶ τὸ μεταξὺ τοῦ τε ἐγγραφομένου καὶ περιγραφομένου εὐθυγράμμου γραφόμενον εὐθύγραμμον σχῆμα τοῦ μὲν περιγραφομένου ἐστὶ ἔλαττον τοῦ δὲ ἐγγραφομένου μεῖζον· **τὰ δὲ τοῦ αυτοῦ μείζονα καὶ ἐλάττονα ἴσα ἀλλήλοις ἐστίν·** ὁ κύκλος ἄρα ἴσος ἐστὶ τῷ μεταξὺ γραφομένῳ εὐθυγράμμῳ τοῦ τε ἐγγραφομένου και περιγραφομένου. ἔχομεν δὲ παντὶ δοθέντι εὐθυγράμμῳ ἴσον τετράγωνον συστήσασθαι· τῷ κύκλῳ ἄρα ἴσον τετράγωνον ἔστι ποιῆσαι.

"The circle, [Bryson says,] is larger than every inscribed rectilinear figure, but is less than every circumscribed one. ... But also the rectilinear figure drawn between the circumscribed and inscribed figure is smaller than the circumscribed and larger than the inscribed figure. **Things larger and smaller than the same are equal to one another.** Therefore, the circle is equal to the rectilinear figure drawn between the inscribed figure and the circumscribed figure. But we can construct a square equal to every given rectilinear figure. Therefore it is possible to produce a square equal to the circle." (tr. Mendell, slightly emended)

c. Euclid, *Elements* I.1
τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.
"Things equal to the same are equal to one another." (tr. Mendell)

II. Bryson's axiom and the Dedekind cut

a. Dedekind's continuity axiom
Suppose that the set of points of a line *l* is a disjoint union of two nonempty subsets *S* and *T*, such that no points of either subset is between two points of the other. Then there is a unique point *O* on *l* such that one of the subsets is equal to a ray of *l* with vertex *O*, and the other subset is equal to the complement.

b. Becker 1933, p. 371
Es besagt dann im wesentlichen, daß alles, was dieselbe Einteilung (denselben Dedekindschen "Schnitt") in einer linear geordneten Menge von Größen hervorbringt, einander gleich ist. Mit andern Worten: es wird die Eindeutigkeit, aber nicht die Existenz der einen Schnitt hervorbringenden Größe gefordert.

[Bryson's theory] says in effect, then, that everything that produces the same division (the same Dedekind cut) in a linearly ordered set of values is equal to one another. In other words: it demands the uniqueness, but not the existence, of a value that produces a cut.

III. Proclus' interpretation

a. Proclus (5th century), cited by John Philoponus, p. 112.21-24
παντός, φησι, τοῦ ἐγγραφομένου εὐθυγράμμου μείζων ἐστὶ ὁ κύκλος, τοῦ δὲ περιγραφομένου ἐλάττων· **οὗ δὲ ἔστι μεῖζον καὶ ἔλαττον, τούτου ἔστι καὶ ἴσον**· ἔστι δὲ μεῖζον καὶ ἔλαττον, τούτου ἔστι καὶ ἴσον· ἔστι δὲ μεῖζον καὶ ἔλαττον εὐθύγραμμον τοῦ κύκλου· ἔστιν ἄρα αὐτοῦ καὶ ἴσον.

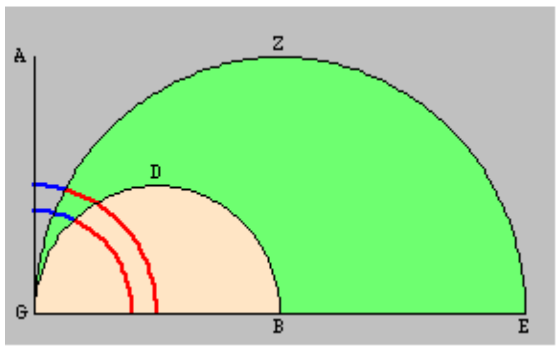
"The circle, he says, is larger than every inscribed rectilinear figure and smaller than every circumscribed figure. **To that for which there is a larger and a smaller, there is also an equal.** There is a rectilinear figure larger and one smaller than the circle. Therefore there is also an equal to it." (tr. Mendell, slightly emended)

b. John Philoponus, p. 112.26-29

οὐδὲ κατεσκεύαζεν ὅλως ἀλλὰ τὸ ἐξ ἀρχῆς ᾐτεῖτο. οὐδὲ γὰρ οἱ τὸν κύκλον τετραγωνίζοντες τοῦτο ἐζήτουν, **εἰ οἷον τέ ἐστι** τῷ κύκλῳ ἴσον τετράγωνον εἶναι, ἀλλ' ὡς οἰόμενοι ὅτι ἐνδέχεται εἶναι ὅυτως ἐπειρῶντο τετράγωνον ἴσον τῷ κύκλῳ **γεννᾶν**.

"... [if Bryson had constructed the squaring of the circle in this way,] he would not at all have constructed it, but one would have begged the question. For those who square the circle did not seek **whether it is possible** for a square to be equal to the circle, but as though they believed it is possible, they attempted in this way to **generate** a square equal to the circle." (tr. Mendell, slightly emended)

IV. Ammonius' counterargument (horn angles)



(http://www.calstatela.edu/faculty/hmendel/Ancient%20Mathematics/Philosophical%20Texts/Bryson/Bryson.html)

a. Ammonius according to John Philoponus, p. 113, 1-4, 6-13; p. 114, 13-17.
... ὅτι ἐπὶ μὲν τῶν ὁμογενῶν ἀληθής ἐστιν ὁ λόγος (ὅτι οὗ ἔστι μεῖζον καὶ ἔλαττον, τοῦτου ἔστι καὶ ἴσον), ἐπὶ μέντοι τῶν ἀνομοιογενῶν οὐκέτι ἀληθὲς τοῦτο. ... τῶν δὲ δύο γωνιῶν τῶν γινομένων ὑπὸ τῆς περιφερείας καὶ τῆς διαμέτρου ... ἡ μὲν ἐκτὸς πάσης ὀξείας γωνίας εὐθυγράμμου ἐλάττων ἐστίν, ἡ δὲ ἐντὸς ... μείζων ἐστί. καὶ ἰδοῦ ἐνταῦθα τῆς αὐτῆς ὀξείας εὐθυγράμμου γωνίας μείζονα καὶ ἐλάττονα δεδειχότες ἴσην εὑρεῖν οὐκ ἂν δυνησώμεθα διὰ τὸ **ἀνομοιογενῆ** εἶναι τὰ μεγέθη ... κακῶς ἄρα ὁ Βρύσων ἐλάμβανεν ... **ἀνομοια** γὰρ κανταῦθα τὰ μεγέθη, λέγω δὴ τὸ εὐθύγραμμον τῷ κύκλῳ, ὥστε οὐδὲν ἴσα ἔσται.

"... since the argument is true in the case of things in the same genus, that for which there is a larger and a smaller, there is also an equal, but this is no longer true of things in dissimilar genera. ... Of those two angles formed by the circumference and the diameter ... the outer is smaller than every rectilinear acute angle, the inner ... is greater. And note, when when we have proved that they are larger and smaller than the same acute rectilinear angle, we will not be able to find an equal angle, since the magnitudes are **of dissimilar genera**...

Therefore, Bryson poorly assumed [his axiom]. For the magnitudes in these cases are **dissimilar**, I mean the rectilinear with the circle, so that they will not be equal.” (tr. Mendell, modified)

b. Euclid, *Elements* Book 3 Proposition 16

Ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου, καὶ εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται, καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἁπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἐλάττων.

“The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.” (tr. Heath)

V. Archimedes’ continuity axiom and homogeneity

a. Euclid, *Elements* Book 5 Definition 3 and 4

Λόγος ἐστὶ δύο μεγεθῶν **ὁμογενῶν** ἡ κατὰ πηλικότητά ποια σχέσις.
Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

“A ratio is some relation between two magnitudes **of the same genus** according to size. Magnitudes that are able to exceed one another when multiplied are said to have a ratio to one another.”

b. Becker (1933), p. 382

Während so Proklos und Simplikios die beiden Bedeutungen von Homogenität, die weitere und die engere («archimedische») streng trennen ... verwischen sich die Unterschiede für Ammonios und Philoponos. Simplikios weist mit Recht den übereilten Schluß des Ammonius (den Philoponos unbedenklich mitmacht) von der Unvergleichbarkeit der gemischtlinigen und geradlinigen Winkel auf die Unvergleichbarkeit von Kreis und geradliniger Figur zurück. Denn im ersten Fall handelt es sich ... um ein nicht-archimedisches, im zweiten um ein archimedisches Größensystem...

“While Proclus and Simplicius thus strictly separate the two meanings of homogeneity, the broader and the narrower (“Archimedean”) ... the distinctions blur for Ammonius and Philoponus. Simplicius rightly refutes Ammonius’ hasty movement (with which Philoponus unhesitatingly joins in) from the incommensurability of the mixed and rectilinear angle to the incommensurability of circle and rectilinear figure. In the first instance it refers to a non-Archimedean, in the second to an Archimedean system of values... "[1]

c. Becker (1933), 386-387

Warum also, so fragen wir zum Schluß, haben die Griechen an die Existenz eines dem Kreise flächengleichen Quadrates geglaubt? Und wir erhalten die Antwort: weil die dem Kreis eingeschriebenen und umgeschriebenen Polygone ein «in doppelter Richtung» archimedisches System bilden.
Die Griechen waren der Meinung, daß solche archimedischen Systeme auch im Dedekindschen Sinn stetig seien...

Why then, we ask in conclusion, did the Greeks believe in the existence of a square equal in area to the circle? And we have the answer: because the inscribed and circumscribed polygons form an Archimedean system "in two directions."
The Greeks were of the opinion that such an Archimedean system was also continuous in Dedekind's sense...

Becker, O. "Eudoxos-Studien II: Warum haben die Griechen die Existenz der vierten Proportionale angenommen?," Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik 2 (1933), pp. 369-387

Mendell, H. "Bryson's Squaring of the Circle." http://www.calstatela.edu/faculty/hmendel/Ancient%20Mathematics/Philosophical%20Texts/Bryson/Bryson.html

[1] An Archimedean system is one in which any two elements are comparable and neither is infinitesimal with respect to the other; i.e. there are no infinitely small or infinitely large elements (see Archimedes, *On the Sphere and Cylinder* Axiom V).