ARCHIMEDES

BY

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Summmis ingeniis dux et magister fuit

(Heiberg, *Archimedis opera omnia* III,
Prolegomena XCV)

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CHAPTER IX.

ON THE EQUILIBRIUM OF PLANES
OR
CENTRES OF GRAVITY OF PLANES

Book I.

1. The treatise on the equilibrium of planes occupies a place apart in the work of Archimedes. In fact, whereas in all his mathematical treatises he builds on foundations long ago established, in this work he concerns himself with an investigation into the very foundations; moreover he leaves the domain of pure mathematics for that of natural science considered from the mathematical point of view: he sets forth certain postulates on which he bases a chapter from the theory of equilibrium, and he is thus the first to establish the close interrelation between mathematics and mechanics, which was to become of such far-reaching significance for physics as well as mathematics.

Through this extension of the field of his activity Archimedes comes into contact directly with the fundamental difficulties inherent in the foundation of mechanics on postulates. The way in which he solves these difficulties, in so far as they are manifested in his subject, even to our own day gives rise to differences of opinion between his commentators and critics: the same argument (in Prop. 6) which is fundamental for the further development, not only of the work itself, but also of two of the purely mathematical treatises still to be discussed, is rejected by some as a paralogism and accepted as correct by others. In the following pages we shall have to define our standpoint with regard to this matter; since, however, the discussion concerns not only the course of the disputed argument itself, but also the structure of the underlying system of postulates, we shall first submit to the reader's attention without any interruption all the postulates and those propositions which precede the theorem in question.

Postulates.

I. *We postulate that equal weights at equal distances are in equilibrium, and that equal weights at unequal distances are not in equilibrium, but incline towards the weight which is at the greater distance.*

II. *that if, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium, but incline towards that weight to which something has been added.*

III. *similarly that, if anything be taken away from one of the weights, they are not in equilibrium, but incline towards that weight from which nothing has been taken away.*

IV. *when equal and similar figures are made to coincide, their centres of gravity*[1]) *likewise coincide.*

V. *in figures which are unequal, but similar, the centres of gravity will be similarly situated*[2]).

VI. *if magnitudes at certain distances be in equilibrium, other [magnitudes] equal to them will also be in equilibrium at the same distances.*

VII. *in any figure whose perimeter is concave in the same direction the centre of gravity must be within the figure.*

This having been postulated:

Proposition 1.

Weights which are in equilibrium at equal distances are equal.

For, if they were unequal, by taking away from the greater the weight by which it exceeds the lesser, we should disturb the equilibrium on account of postulate III, whereas because of postulate I there would precisely have to be equilibrium in the new position.

Proposition 2.

Unequal weights at equal distances are not in equilibrium, but will incline towards the greater weight.

If from the greater weight is taken away the weight by which it exceeds the lesser, there will be equilibrium (postulate I). If then the original position is restored, the truth of the proposition becomes apparent with the aid of postulate II.

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[1]) We shall revert presently to the question as to the meaning of this term.

[2]) To this it is added: *we say that points are similarly situated in relation to similar figures if straight lines drawn from these points to the equal angles make equal angles with the homologous sides.*

Proposition 3.

Unequal weights can [only] be in equilibrium at unequal distances, the greater [weight] being at the lesser [distance][1]).

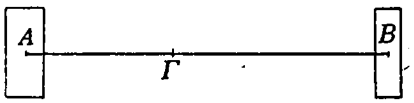

Fig. 116.

In Fig. 116 let the weights A and B ($A > B$) be in equilibrium at the distances $A\Gamma$ and ΓB. It has to be proved that

$$A\Gamma < \Gamma B.$$

Proof: If the weight $A - B$ is taken away from A, the weights have to incline towards B (postulate III). This, however, is impossible, if $A\Gamma = \Gamma B$ (postulate I) and also when $A\Gamma > \Gamma B$ (postulate I). Consequently $A\Gamma$ has to be less than ΓB.

It is obvious that also weights which are in equilibrium at unequal distances are unequal, the greater [weight] being at the lesser [distance].

This is not elucidated any further. Apparently this enunciation contains nothing but a contraposition of the second part of postulate I[2]).

Proposition 4.

If two equal magnitudes have not the same centre of gravity, the centre of gravity of the magnitude composed of the two magnitudes will be the middle point of the straight line joining the centres of gravity of the magnitudes.

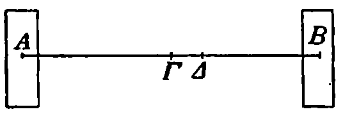

Fig. 117.

In Fig. 117 let A and B be successively the centres of gravity of the magnitudes A and B, Γ the middle point of the line segment AB. If now Γ is not the centre of gravity of the magnitude composed of A and B, let it be some other point Δ of AB. A and B therefore are in equilibrium when Δ is held. This, however, is impossible because of postulate I. Consequently Γ is the centre of gravity.

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[1]) This proposition is formulated very elliptically: τὰ ἄνισα βάρεα ἀπὸ τῶν ἀνίσων μακέων ἰσορροπησοῦντι, καὶ τὸ μεῖζον ἀπὸ τοῦ ἐλάσσονος.

[2]) Postulate I, 2nd part says: if weight A_1 = weight A_2 and distance $l_1 \neq$ distance l_1, there is no equilibrium. Contraposition: if there is equilibrium and also $l_1 \neq l_2$, then $A_1 \neq A_2$. It then follows from the first part of Prop. 3 that the weight at the lesser distance must be the greater of the two.

Proposition 5.

If of three magnitudes the centres of gravity are on a straight line and the magnitudes have equal weight, and if also the straight lines between the centres are equal, the centre of gravity of the magnitude composed of all the magnitudes will be the point which is also the centre of gravity of the middle magnitude.

This is recognized by noting that according to Prop. 4 the centre of gravity of the system of the extreme magnitudes coincides with that of the middle magnitude.

In Porism I the theorem is extended to any odd number of magnitudes the centres of gravity of which are on a straight line at equal distances, while any two magnitudes which are equidistant from the middle point of the line segment joining the two extreme centres are of equal weight.

In Porism II it is stated that if from the system described above the middle magnitude be omitted, the centre of gravity of the system will be the same point as before.

This is followed by the famous and much disputed proposition in which the so-called lever principle is enunciated.

Proposition 6.

Commensurable magnitudes are in equilibrium at distances reciprocally proportional to the weights.

Let the commensurable magnitudes be A and B, of which A and B are the centres, and let $E\Delta$ be a given distance, and let the distance $\Delta\Gamma$ be to the distance ΓE as A to B. It has to be proved that the centre of gravity of the magnitude composed of A and B is Γ (Fig. 118).

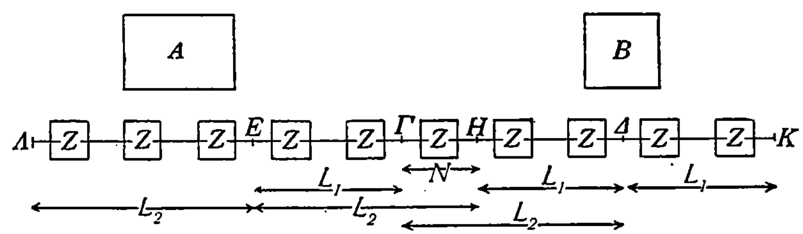

Fig. 118.

Proof: Since A and B are commensurable, so are $\Delta\Gamma$ and $E\Gamma$. Let N be a common measure of these two distances. Make $\Delta H =$

$\Delta K = E\Gamma$ and $E\Lambda = \Delta\Gamma$. Apparently EH is also equal to $\Delta\Gamma$. Since $H\Lambda = 2.\Delta\Gamma$ and $HK = 2.E\Gamma$, we also have

$$A:B = \Lambda H:HK .$$

Now let the magnitude Z be contained as many times in A as the distance N in ΛH, whence also as many times in B as N in HK. Divide ΛH and HK each into equal parts N, A and B each into equal parts Z. Place on each of the line segments N a magnitude Z, so that in each case the centre of gravity of Z is the middle point of N, then the centre of gravity of all the magnitudes Z placed on the parts of ΛH will be the point E (Prop. 5, Porism II), while in the same way the centre of gravity of all the magnitudes Z placed on the parts of HK will be the point Δ. *Now therefore A will be at E and B at Δ. There will now be equal magnitudes on a straight line, the centres of gravity of which are equidistant from one another and the number of which is even. It is now obvious that of the magnitude composed of all the magnitudes the middle point of the straight line bounded by the centres of the middle magnitudes will be the centre of gravity. So that the centre of gravity of the magnitude composed of all the magnitudes is the point Γ. If therefore A is at E and B at Δ, they will be in equilibrium about Γ.*

It has thus been proved that the inverse proportionality of force and arm is a sufficient condition for the equilibrium of a lever supported in its centre of gravity under the influence of two weights on either side of the fulcrum. It is not proved by Archimedes that this condition is also necessary; the proof of this might have been given as follows:

Let weights A and B be in equilibrium at distances ΓA and ΓB. If it were not true that

$$A:B = \Gamma B:\Gamma A ,$$

there would have to exist a weight Δ, different from B, such that

$$A:\Delta = \Gamma B:\Gamma A .$$

This weight Δ would be in equilibrium with A by the proved Proposition 6. If then the original situation is restored, the equilibrium will be disturbed (on account of the postulates II and III), which is contrary to the supposition that A and B are in equilibrium.

It is to be considered a flaw in the treatise on equilibrium that Archimedes omits to give this proof; indeed, later on (see Chapter XII) he repeatedly applies the condition of inverse proportionality as a necessary condition of equilibrium.

2. Let us now study the proof of Proposition 6 somewhat more closely. Its real nucleus appears to consist in the manner in which the magnitudes A and B are placed at the points E and Δ; this is done by dividing each of them into equal parts Z, which are suspended on the segments of the straight line ΛK in such a way that the centres of gravity of the two systems come to lie successively at E and Δ. On the lever are thus suspended, not the originally given magnitudes A and B, but two systems of magnitudes which are successively of the same weight as A and B and whose centres of gravity are at the points E and Δ, which are looked upon successively as positions of A and B. This implies that the influence exerted on the equilibrium by a body suspended on a lever is judged exclusively by the gravity of the body and the place of its centre of gravity, and that the shape is immaterial.

We now have to ask ourselves: was Archimedes aware of the fact that the conclusion he draws is based on this premiss? If so, has he expressly formulated this premiss as such? And if so again, does he account for it any further?

This threefold question (which is often wrongly not split up into three members) has been answered in various ways, which testify to great differences in appreciation of the argument of Archimedes.

Let us listen in the first place to the opinion of a well-known writer on the history of Mechanics, Ernst Mach[1]), whose views enjoy great authority. We are giving his words *in extenso*:

"So überraschend uns nun auf den ersten Blick die Leistung von Archimedes ... erscheint, so steigen uns bei genauer Betrachtung doch Zweifel an der Richtigkeit derselben auf. Aus der bloszen Annahme des Gleichgewichts gleicher Gewichte in gleichen Abständen wird die verkehrte Proportion zwischen Gewicht und Hebelarm abgeleitet! Wie ist das möglich?

Wenn wir schon die blosze Abhängigkeit des Gleichgewichts vom Gewicht und Abstand überhaupt nicht aus uns herausphilosophieren konnten, sondern aus der Erfahrung holen muszten, um

[1]) Ernst Mach, *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt.* 7e Auflage (Leipzig 1912). p. 14.

wieviel weniger werden wir die Form dieser Abhängigkeit, die Proportionalität, auf spekulativem Wege finden können.

Wirklich wird von Archimedes und allen Nachfolgern die Voraussetzung dasz die (gleichgewichtstörende) Wirkung eines Gewichts P im Abstand L von der Achse durch das Produkt $P.\ L$ (das sogenannte statische Moment) gemessen sei, mehr oder weniger versteckt oder stillschweigend eingeführt. Zunächst ist klar, dasz bei vollkommen symmetrischer Anordnung das Gleichgewicht unter Voraussetzung irgendeiner beliebigen Abhängigkeit des gleichgewichtstörenden Moments von L, also $P.f(L)$, besteht; demnach kann aus diesem Gleichgewicht unmöglich die bestimmte Form $P.L$ abgeleitet werden. Der Fehler der Ableitung musz also in der vorgenommenen Transformation liegen, und liegt hier auch. Archimedes setzt die Wirkung zweier gleicher Gewichte unter allen Umständen gleich der Wirkung des doppelten Gewichts mit dem Angriffspunkte in der Mitte. Da er aber einen Einflusz der Entfernung vom Drehpunkt kennt und voraussetzt, so darf dies nicht von vornherein angenommen werden, wenn die beiden Gewichte ungleiche Entfernung vom Drehpunkt haben. Wenn nun ein Gewicht, das seitwärts vom Drehpunkt liegt, in zwei gleiche Teile geteilt wird, welche symmetrisch zu dem ursprünglichen Angriffspunkt verschoben werden so nähert sich das eine Gewicht dem Drehpunkt so viel, als sich das andere von dem selben entfernt. Nimmt man nun an, dasz die Wirkung hierbei dieselbe bleibt, so ist hiermit schon über die Form der Abhängigkeit des Moments von L entschieden, denn dies ist nur möglich bei der Form $P.\ L$, bei Proportionalität zu L. Dann ist aber jede weitere Ableitung überflüssig. Die ganze Ableitung enthält den zu beweisenden Satz, wenn auch nicht ausdrücklich ausgesprochen und in anderer Form, schon als Voraussetzung."

To Mach's view, therefore, Archimedes in the first place was not aware that his argument is based on the premiss that a lever equilibrium is not disturbed, if one of the suspended magnitudes is made to change its shape while preserving its weight and centre of gravity, and in the second place the whole proof becomes superfluous as soon as this premiss is consciously adopted, because it already implies the theorem to be proved (*viz.* that inverse proportionality of two weights to the lengths of the arms on which they are suspended constitutes a sufficient condition for the equilibrium

of a lever). The second part of the objection may also be worded as follows in present-day formulation: once we have accepted the fact that the function $P.f(L)$ satisfies the functional equation[1])

$$\tfrac{1}{2}P.f(L+h)+\tfrac{1}{2}P.f(L-h) = P.f(L),$$

it is superfluous to prove that $P.f(L)$ has the form $P.L$, for the functional equation in question exists only if $f(L)$ has the form L.

Neither of Mach's objections here summarized seems to us to be wholly valid. In fact, we found Archimedes explicitly declaring that owing to the position of the parts Z on the line segments N the common centre of gravity of the parts of A lies at E and that of the parts of B at Δ, and that *consequently* the magnitude A is placed at E and the magnitude B at Δ, whereas actually neither A nor B are present in their original form. From this it is already clearly apparent that he judges the influence of a magnitude solely by the weight and the position of the centre of gravity. But in addition to this he has expressed in postulate VI that when magnitudes at given distances are in equilibrium, magnitudes equal to the former will be in equilibrium at the same distances. Upon first consideration this may seem to be a perfectly superfluous tautology. As soon as we assume, however, that by "magnitudes at the same distances" he understands "magnitudes the centres of gravity of which lie at the same distances from the fulcrum", we have conferred a reasonable meaning on this postulate and at the same time have found the explicit formulation of the premiss applied in Prop. 6, which Mach deems to be lacking. The first of his objections may thus be considered to have been refuted[2]).

And as regards the second: its untenability is evident as soon as we pay attention to the mathematical formulation given of it above.

[1]) In fact, this equation expresses that the influence of the equilibrium of a weight P at a distance l is equivalent to the combined influences of weights $\tfrac{1}{2}P$ at points which are symmetrically situated in relation to the point where P was first suspended. It is naturally also based on the supposition that the influences in question can be combined additively.

[2]) The conception of the postulate VI here described is due to O. Toeplitz, whom I thank for the oral and written explanation of his point of view. The ideas of Toeplitz have been elaborated in a very careful study by W. Stein, *Der Begriff des Schwerpunktes bei Archimedes.* Quellen und Studien zur Geschichte der Mathematik, Physik und Astronomie. Abt. B: Quellen. I (1930), 221–244.

Indeed, if it is not necessary to solve the functional equation in this case, when is there any sense in doing so? And if in general the drawing of a conclusion Q from a group of premisses P may be called superfluous when P applies only if Q is true, one may just as well reject as superfluous all mathematical proofs, for if Q is not true, it is certain that P will not hold!

Now we also found Mach having recourse to the general argument that from the existence of equilibrium with symmetrical loads one can never draw a conclusion about the nature of the function $f(L)$, because this equilibrium would exist with any form of this function. To this we can only reply that Archimedes does not draw his conclusion from the mere existence of this equilibrium, but that this is only one among the premisses on which his argument is based. This argument can be judged only by considering the whole system of postulates, definitions, and propositions already proved which are used in the course of it. It is the really weak spot in Mach's view of the question that he has neglected this consideration.

The same defect in a somewhat different form is also found in the author of the German Archimedes translation, A. Czwalina[1]). The latter remarks that if the lever principle were to state the inverse proportionality of the force and the square of the arm as a condition of equilibrium, the propositions 1–5 would stand unchanged, whereas Prop. 6 would be incorrect, from which he concludes that Prop. 6 does not therefore result from Props 1–5, and that consequently the derivation of Prop. 6 is not correct.

By this argumentation the proof of Euclid I, 29 (equality of alternating interior angles as a necessary condition of parallelism) and the proof of I, 32 based on this (the sum of the angles of a triangle is 180°) would not be correct either; in fact, if the sum of the angles of a triangle were to be less than 180°, the propositions 1–27 would stand unchanged, whereas 29 and 32 would not be true. The analogy between the two cases is perfect. It is true that the propositions 29 and 32 do not result from 1–27, but from the latter supplemented with the fifth postulate; in the same way the sixth proposition of the treatise on the equilibrium of planes does not result from the propositons 1–5, but from the latter, combined with the postulate VI that has meanwhile been introduced. As soon as

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[1]) A. Czwalina, *Ueber das Gleichgewicht ebener Flächen* (see p. 45, Note 6).

we accept this postulate in the above interpretation, we can no longer find any fault with the proof of Prop. 6.

A slightly different point of view in relation to the question under consideration is taken by O. Hölder[1]). The latter agrees with Mach in considering the proof of Prop. 6 insufficient, because he, too, is of opinion that Archimedes neglects to postulate or prove the admissibility of combining or splitting up weights while maintaining the position of the centre of gravity. In his view, however, the proof can be corrected (and will then acquire value) if only we succeed in supplementing the gap presumed to have been left by Archimedes. It would seem to us, however, that the way in which he attempts to do this (by superposition of positions of equilibrium, involving, *inter alia*, the concept of the reactive force exerted by a fulcrum) fits in very little with the character of Greek mechanics, while moreover there is not the slightest evidence for it in the text.

We have now got to the point where we can answer the first two members of the above question with some certainty: Archimedes was fully aware of the fact that the proof of the sixth proposition is based on the premiss that the influence of a weight suspended on a lever on the equilibrium depends on the gravity of the body and the position of its centre of gravity; and he has explicitly formulated this premiss in the sixth postulate, in which he postulates, apparently tautologically, that the lever equilibrium is not disturbed if the weights suspended on it are replaced by other weights, which are equal to the first and are suspended in the same places.

We are now left with the question whether he accounts for this statement or not, a question which is related with another, *viz.* what he understands by the centre of gravity of a body and how he conceives this concept to be introduced. In fact, it is striking that he always refers to the centre of gravity as if it were a perfectly familiar thing: the term is used, starting with postulate IV, without any explicit definition, and in the proofs of the propositions 4 and 6 it is identified in the case of a loaded lever, without any

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[1]) O. Hölder, *Die Mathematische Methode. Logisch-erkenntnistheoretische Untersuchungen im Gebiete der Mathematik, Mechanik und Physik* (Berlin, 1924) § 12. *Der Hebelbeweis des Archimedes*, p. 39 *et seq.*

motivation, with the point in which this lever has to be supported in order to be in (indifferent[1])) equilibrium.

With regard to the questions thus raised we may take either of the following two standpoints:

a) it is possible that Archimedes, when writing the treatise on the equilibrium of planes, could assume the theory of the centre of gravity to be familiar to a certain extent, because this theory had already been developed either by earlier students of mechanics or by himself in a treatise now lost[2]).

b) it is possible that the work on the equilibrium of planes is an entirely autonomous treatise, and that the definition of the concept of centre of gravity is to be conceived of as being implied in the postulates on which this work is built.

Both these points of view have been defended: the former by G. Vailati[3]) and the latter by Toeplitz and Stein[4]). We shall first discuss the latter view. According to this, all the terms to be found in the postulates of the work, in so far as they relate to statics (such as *βάρος*, weight; *ἰσορροπεῖν*, to be in equilibrium; *κέντρον τοῦ βάρεος*, centre of gravity), are to be considered so many unknowns, for the finding of which the postulates serve as equations, while other similar equations can be found by explicitly formulat-

[1]) It is possible to ask with W. Stein (*l.c.*, p. 228) whether in his considerations of equilibrium Archimedes is really thinking exclusively of indifferent equilibrium or whether he also admits the possibility of stable equilibrium. This amounts to asking whether the condition that the lever is supported in its centre of gravity is considered as sufficient only or as necessary as well. If this question is to be answered, the view of Stein that in the latter case the Prop. 4 is only a trivial consequence of Postulate I, from which he concludes that Archimedes does not postulate it as a necessary condition of equilibrium that the lever should be supported in the centre of gravity, seems to us to carry little conviction: in Greek mathematics there are numerous instances of trivial consequences which nevertheless constitute the subject of a separate proposition. It would seem more important that in **Q.P.** Archimedes invariably considers really stable equilibria, so that it does not after all seem to be his intention to identify the centre of gravity and the fulcrum.

[2]) This might then have been one of the works quoted by Pappus and Simplicius, which we mentioned in Chapter II in the enumeration of the lost treatises sub 3).

[3]) G. Vailati, *Del concetto di centro di gravità nella Statica d'Archimede.* Atti della R. Accad. d. Scienze di Torino. 32 (1896–97), 742–758.

[4]) See Note 2 of pag. 293.

ing the assumptions which are tacitly made in the proofs of the propositions. The system of postulates thus completed then comprises implicitly the definitions of all the terms in question, and it is then in particular no longer necessary to ask for an explicit definition of the meaning of the term "centre of gravity".

The investigation defined above has been made very carefully by W. Stein, and undoubtedly has considerably clarified our insight into the statics of Archimedes. It would, however, seem doubtful whether this method will enable us actually to follow his line of thought. Not because the idea of a definition being implicit in a system of postulates was entirely alien to the Greeks: when in the *Elements* Euclid uses the concept of a "straight line", he never refers to the property of length without breadth, which is given as the characteristic of this concept in the definitions, but exclusively to the properties of its being determined by two points and of the unlimited extensibility of any line segment mentioned as characteristics in the postulates. To that extent it may be said that he uses an implicit definition of the concept of a straight line, if not *ex confesso*, at least *de facto*. But this is practically the only instance of this method of definition in the *Elements*: it is not used for other terms, where it might have been applied, such as the area of a figure or the volume of a solid; the meaning of these terms is apparently considered to be intuitively known, and the postulates given about them (congruent figures have equal areas, the whole is greater than the part), though they do constitute a step in the direction of the implicit definition (which would be given by a complete system of postulates), do not yet testify to the conscious desire to define the concepts in question entirely in this way. This cannot be altered by the fact that it is possible afterwards, by formulating so-called tacit assumptions (which are not yet assumptions or conscious suppositions from the point of view of the author), to complete the underlaying system of postulates in such a way that it becomes sufficient for implicit definition.

Now we ask ourselves whether it is likely that, whereas in the elementary geometry of the Greeks the conscious implicit definition by means of a system of postulates was as yet so little used, this method of definition would have been carried so far in Greek mechanics that not only one term, as in the geometrical examples, but seven terms at a time were defined in this way. Is it really to be believed that Archimedes forced himself, when using the words "to be in equilibrium", "incline", "weight", to think of nothing but the relations established between them in the postulates? Is it not much more probable that he constantly had in mind an idealized lever, which he mentally saw inclining or remaining in equilibrium under the influence of weights (in the form of planimetric figures) suspended on it, that the postulates contain nothing but the formulation of the results of the simplest observations he was able to make, and that the clarity of the images thus obtained completely prevented the desire for the abstract definition of the words used from arising?

If this is true, the term "centre of gravity", which is introduced just as unemphatically as the other terms mentioned, must have had an intuitive meaning that was plain to all. When we read the postulates and the propositons with an unbiassed mind, we do get the impression that the author is thinking of a lever supported in its fulcrum (the lever being idealized into a straight line), on which thin plates (idealized into planimetric figures) are attached in their centres of gravity (as is clear from the frequent use of the same letter to refer to the suspended figure, the centre of gravity of that figure, and the point of the lever indicating the position of that figure). Since, however, the term "centre of gravity" cannot possibly have had a meaning intuitively as clear as "be in equilibrium" or "incline", there is every reason to assume with Vailati that this term could be supposed to be familiar to the readers of the work in view of the advance knowledge they might be expected to have; thus we arrive at the first of the possibilities referred to above, and we therefore have to ascertain what data are available to us in the history of Greek natural sciences about an elementary theory of statics in which the term "centre of gravity" may have been introduced.

It is mainly a group of remarks in Heron's *Mechanica*[1]) and a coherent exposition devoted to the subject by Pappus in the *Collectio*[2]) which may be considered for this purpose. Both Heron and

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[1]) Of this work a complete text has only been preserved in Arabic, while in Greek there are fragments. Both are to be found with a German translation in: Heronis *Opera* II, 1. The passages in question are: Book I, Cap. 24; Book II, Cap. 35 *et seq.* (II, 35 also in Greek).

[2]) Pappus, *Collectio* VIII, 5; 1030 *et seq.*

Pappus are authors who came long after Archimedes, and it may therefore at first view seem a little illogical to quote them in this connection. The former, however, writes in such an elementary way and the latter is so encyclopaedic that it does not appear very risky to consult them on those points which may have remained beneath the threshold of exposition in the work of their great predecessor, whose level neither of them attains; moreover, Greek mechanics was arrested so much in the stage of elementary principles that in authors of the first and third centuries A.D. we must not by any means expect any further development of the subject than may already have been reached in the third century B.C.

Apparently Heron and Pappus drew from the same source in their treatment of the subject "centre of gravity"; since Heron, however, writes about it sketchily and indistinctly, we shall mainly give Pappus' words, only occasionally quoting Heron for confirmation.

In the eighth book of the *Collectio*, in which Pappus speaks about mechanics, the theory of the centre of gravity (*κέντρον τοῦ βάρους*) is called the starting point and element of the barycentric theory (*ἀρχὴ καὶ στοιχεῖον τῆς κεντροβαρικῆς πραγματείας*) because after the exposition of this the other parts of the theory automatically become clear. This is followed by an explicit definition: *We say that the centre of gravity of any body is a point within that body which is such that, if the body be conceived to be suspended from that point, the weight carried thereby remains at rest and preserves its original position*[1]).

Heron expresses himself in approximately the same way when he defines a point which is rendered in the German translation of the available Arabic text by *Aufhängepunkt*, which point, according to him, was distinguished by Archimedes and his adherents from the centre of gravity[2]). For the centre of gravity, however, he quotes a definition of the Stoic Poseidonius, who again says approximately the same thing in a somewhat careless manner[3]), so that it does not

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[1]) *ibidem* line 11: *λέγομεν δὲ κέντρον βάρους ἑκάστου σώματος εἶναι σημεῖόν τι κείμενον ἐντός, ἀφ' οὗ κατ' ἐπίνοιαν ἀρτηθὲν τὸ (βάρος) ἠρεμεῖ φερόμενον καὶ φυλάσσει τὴν ἐξ ἀρχῆς θέσιν.* Here *βάρος* is to be considered synonymous with *σῶμα.*

[2]) *Heronis Opera* II, 1. p. 64.

[3]) *ibidem*, p. 63: *der Schwerpunkt ist ein solcher Punkt, dasz wenn die*

become quite clear to what distinction he is actually referring. It is, however, significant that he repeatedly uses a term which is rendered by *Schwer- oder Neigungspunkt* in the German translation, and which will probably have been *κέντρον τοῦ βάρους ἢ τῆς φορᾶς* in Greek. In this term *φορά* simply refers to the natural motion of a heavy body; the idea therefore seems to be that, since the gravity is the cause of this motion, the downward tendency may be conceived to be concentrated in the centre of gravity, so that this point can also be styled falling centre (a concept which is analogous to the later *centrum oscillationis* or *centrum percussionis*).

With regard to this centre of gravity or falling centre Heron goes on to say[1] that it is a point through which pass all the verticals of the points of suspension; here point of suspension refers, in contrast with the *Aufhängepunkt* used above, to any point in which the body may be suspended, and the verticals in question are those lines of the body which in the position of equilibrium coincide with the verticals of the points of suspension. Heron also observes that the centre of gravity or falling centre may also be situated outside the substance of the body, for example with rings or wheels.

In order to determine the centre of gravity Pappus now imagines a vertical plane $\alpha\beta\gamma\delta$, on whose horizontal upper edge $\alpha\beta$ the body is so placed as to be in equilibrium. Now the plane $\alpha\beta\gamma\delta$, when extended, will divide the body into two parts balancing each other about this plane as plane of support[2], *i.e.* into parts of equal apparent weight. The body is now placed again on $\alpha\beta$ in a different position, so that there is once more equilibrium. It is again divided into two parts of equal apparent weight by the plane. The two intersections which the plane $\alpha\beta\gamma\delta$ has determined in the body in the two positions successively taken up by the body will have to

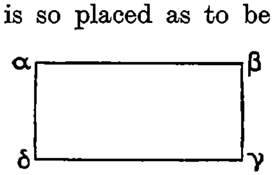

Fig. 119.

Last in demselben aufgehängt wird, sie in zwei gleiche Teile geteilt wird; where "gleich" apparently was to have been understood as "keeping each other in equilibrium", though, as will appear, it is often "of equal weight" which is meant.

[1]) *ibidem*, p. 36.

[2]) Pappus, *Collectio* VIII, 5; 1030, line 26. *τεμεῖ τὸ ἐπικείμενον σῶμα εἰς ἰσόρροπα δύο μέρη, οἷον περὶ ἄρτημα τὸ ἐπίπεδον ἰσορροποῦντα.*

intersect each other. In fact, if these two intersections were parallel, the same parts would at the same time be of equal apparent weight and not of equal apparent weight, which is absurd[1]).

The body will also be in equilibrium when it rests on the straight line, which the two intersections have in common, as on a support[2]). If we now find another straight line, which may also serve as support, this line, when produced, will have to meet the one first mentioned. Indeed, if this were not the case, it would be possible to draw through the two straight lines two parallel planes, both of which would divide the body into parts which would be of equal apparent weight when considered in one way, and not of equal apparent weight when considered in another way[3]).

From this it follows that all the supporting lines obtained in the way described above pass through one point, *viz.* the centre of gravity as defined above[4]). In fact, any plane through this point divides the body into two parts balancing each other when supported in said plane[5]).

According to Pappus, this is the most essential part (*τὸ μάλιστα σύνεχον*) of the barycentric theory. For the elements of what can be proved with the aid of this he refers to the work of Archimedes on the equilibrium of planes and to Heron's mechanics[6]). He himself discusses, as an application, a planimetric theorem which does

[1]) Pappus, *Collectio* VIII, 5; 1032, line 2. *εἰ γὰρ μὴ τεμεῖ, τὰ αὐτὰ μέρη καὶ ἰσόρροπα καὶ ἀνισόρροπα γενήσεται ἀλλήλοις, ὅπερ ἄτοπον.*

[2]) It appears to be meant that the body is supported in the lowest point of intersection of its surface with the said straight line. The body can therefore be kept in equilibrium both by supporting it along a horizontal straight line (or in two points of the latter) and by supporting it in a point. This is what is probably meant by Heron (II, 1; p. 64) when he quotes from Archimedes: *Lasten neigen sich nicht auf einer Linie und auf einem Punkte.* Here *sich neigen* = falling. The passage thus implies: the body can be prevented from falling by being supported along a straight line or in a point.

[3]) This conclusion is based on the assumption that any plane through a supporting line obtained as above divides the body into two parts of equal apparent weight.

[4]) Pappus *Collectio* VIII, 5; 1032, line 26.

[5]) Ibidem, line 30.

[6]) Ibidem 1034; lines 1–4. Pappus therefore does not in any case consider the work of Archimedes as an entirely independent treatise, but as an application of the theory of the centre of gravity.

not interest us here as such[1]), but which is significant on account of the way in which he is found to determine the centre of gravity of a triangle on the basis of the above considerations. Indeed, he observes (Fig. 120) that if the triangle $\alpha\beta\gamma$ is placed (in the horizontal position) with the median $\alpha\delta$ on the upper edge of a vertical supporting plane, the figure will be in equilibrium, because the areas of the triangles $\alpha\beta\delta$ and $\alpha\gamma\delta$ are equal. The same applies to the median $\beta\varepsilon$, and the point of intersection ζ of $\alpha\delta$ and $\beta\varepsilon$ is therefore the centre of gravity of the triangle.

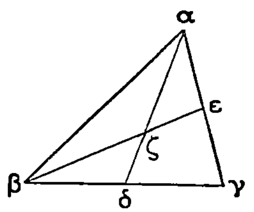

Fig. 120.

The fragment from Pappus which we have rendered here is of interest for the history of mechanics in more than one respect; in fact, the characteristic feature of the evolution of this branch of physics consists in the very early tendency of dealing with the theory of motion and equilibrium in an axiomatic manner. With regard to the last-mentioned subject, that of statics, this tendency is already very plain: on the basis of a number of physical experiences concerning the equilibrium of bodies which are supported by a narrow horizontal beam, a deductive treatment of the theory of equilibrium is given very soon, in which the conditions are idealized to the same extent as was the case with the deductive treatment of geometry. The supporting beam becomes a horizontal straight line; for the supported body is taken a planimetric figure; no attention is paid to the degree of stability of the equilibrium; it is not considered what procedure would have to be followed to support any body in its centre of gravity; probably we should not imagine the experience gained as greatly varied either; it is sufficient to think of rectangular blocks or of thin plates of a rectangular or triangular form.

It seems moreover that the process of idealization here involved has indeed been considered theoretically more in detail, to wit by Archimedes himself. At least, Heron observes[2]) that it is naturally

[1]) The theorem is as follows: if on the sides $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ of a triangle $\alpha\beta\gamma$ be situated the points η, ϑ, $\varkappa$ successively, in such a way that

$$\alpha\eta:\eta\beta = \beta\vartheta:\vartheta\gamma = \gamma\varkappa:\varkappa\alpha\,,$$

the triangles $\alpha\beta\gamma$ and $\eta\vartheta\varkappa$ have the same centre of gravity.

[2]) *Heronis Opera* II, 1. p. 62.

only possible to speak of gravity and natural motion in the case of physical bodies, but that Archimedes has made it sufficiently plain in what sense a centre of gravity may also be assigned to solid or plane mathematical figures.

Experience will undoubtedly further have taught that a beam or a thin plate, when supported successively along two parallel straight lines of the same plane boundary surface, could not be in equilibrium in both cases. This fact, however (as appears from the indirect reasoning of Pappus on the subject), is at once invested with the character of logical evidence. The relation in which the two parts determined in the body by the supporting plane, when extended, are to each other is looked upon as a relation of equality between two magnitudes, and the parallelism of two such supporting planes is felt as absurd, because, when one plane yields the parts A and B, and the other the parts A_1 and B_1, it is not possible that simultaneously with

$$A > A_1, \quad B < B_1 \tag{1}$$

it is also true that

$$A = B, \quad A_1 = B_1\,. \tag{2}$$

This logical evidence is of course only apparent. Pappus does not see that the equilibrium-disturbing effect exerted by each of the parts of the body is not determined by the weight of that part alone, and that therefore from the absurdity of the simultaneous existence of the relations (1) and (2) for the weights of the separate parts no logical conclusion can be drawn as to the impossibility of the influence on the equilibrium. That he is, however, thinking exclusively of the amounts of the weights, is quite clear from his derivation of the centre of gravity of a triangle. The two parts into which a median divides a triangle, in his view, balance when supported along this median because their areas are equal (the weights being conceived to be proportional to the areas). According to this argument there would also have to be equilibrium if the triangle were supported along any other straight line dividing the area into two equal parts; such a straight line, however, only passes through the centre of gravity, if it also contains an angular point. And moreover, since it is assumed as self-evident that any vertical plane through the centre of gravity divides the body into parts of equal apparent weight, any line through the centre of gravity of a triangle would have to divide the area into two equal parts, which is not true either.

It is this which throws an unexpectedly clear light on the object which Archimedes may have aimed at with his work on the equilibrium of planes, the subsidiary title of which refers to centres of gravity of plane figures: he recognized the erroneous nature of the method for the determination of a centre of gravity just described, which apparently dates further back and was preserved by Pappus with curious thoughtlessness; he understood that parts balancing each other in general do not have equal weight, but that the position of their respective centres of gravity, too, has to be taken into account. He thus came to consider a lever on whose arms the parts of the body were attached in their centres of gravity, and in this way the theory of lever furnished him with the means for determining centres of gravity, because he only had tc ask in what point the lever was to be supported in order to obtain equilibrium. This theory in itself, however, was based on the barycentric theory—probably studied long before his day—, which we saw exposed by Pappus and to which, in spite of palpable logical defects, a certain physically convincing effect cannot be denied. Thus the methodically somewhat complex situation arose that the lever principle could be on the one hand an application and on the other hand the basis of the theory of the centre of gravity.

It has thus likewise been elucidated how Archimedes came to adopt, without any further motivation, the principle—formulated none too clearly in postulate VI—which protects the proof of the sixth proposition, as we now have come to realize, against the objections raised to it by Mach: it is in the centre of gravity or falling centre that from the very beginning the whole downward tendency constituting the essence of gravity has been considered concentrated. Was it not bound to seem evident that the influence which a body could exert on a lever as a result of this tendency did not change as long as its intensity and the place where it resides remained unchanged?

3. The above discussion may be considered to have sufficiently elucidated the questions to which the first part of the treatise *On the Equilibrium of Planes* gave rise, so that we can now continue the discussion of the work itself.

Proposition 7.

However, even if the magnitudes are incommensurable, they will be in equilibrium at distances reciprocally proportional to the magnitudes.

The necessity of a separate discussion of the case that the weights suspended on the lever have no common measure follows from the essential significance of the common measure Z of the magnitudes A and B in the proof of Prop. 6.

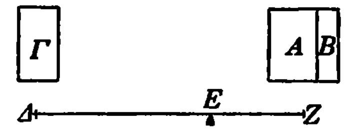

Fig. 121.

The somewhat cursorily written proof may be rendered as follows:

In Fig. 121 let the incommensurable magnitudes be $A+B$ and Γ, the arms of the lever on which they are suspended, EZ and $E\Delta$ respectively. The supposition is then

$$(E\Delta, EZ) = (A+B, \Gamma). \qquad (1)$$

It has to be proved that E is the centre of gravity of $(A+B)$ in Z and Γ in Δ (*i.e.* that the centres of gravity of the magnitudes lie in these points successively), or in other words that the lever, when supported in E, is in equilibrium.

Suppose this is not true, then $(A+B)$ will be either too great or too small for equilibrium. Let $(A+B)$ be too great. Then take away from it such an amount that the remainder is still too great for equilibrium, but commensurable with Γ.

Let this remainder be A. Now we have

$$(A, \Gamma) < (E\Delta, EZ), \qquad (2)$$

so that the lever will incline towards Δ, which is contrary to the supposition that A alone is still too great for equilibrium. In the same way the case that $(A+B)$ is too small can be dealt with.

The proof apparently contains considerable gaps. The possibility of determining the remainder A in such a way that it is too great for equilibrium and at the same time commensurable with Γ is not based either on a postulate or on a theorem. That it follows from the inequality (2) that the lever will incline towards Δ is indeed physically plausible, but logically not justified. It is naturally derived from the consideration that there would be equilibrium if A were replaced by $A' > A$, so that

$$(A', \Gamma) = (E\Delta, EZ), \qquad (3)$$

i.e. by Prop. 6, followed by the application of postulate III. But it appears from (1) that $E\Delta$ and EZ are incommensurable, and it is therefore only possible to conclude from (3) that there is equilibrium, if Prop. 7 has first been proved. In the proof of Prop. 7 it is not permissible to make use of this conclusion.

The proof might be improved, though not quite saved, if the line segment ΔZ were divided in a point H in such a way that

$$(A, \Gamma) = (H\Delta, HZ) .$$

From this it would follow by Prop. 6 that the lever, when loaded with A at Z and with Γ at Δ, is in equilibrium. If upon this it is to be concluded that there can be no equilibrium when the lever is supported in E, it would have to be postulated that a system of bodies has only one centre of gravity, or the consideration used by Pappus, *viz.* that upon the lever being successively supported along two parallel straight lines there cannot be equilibrium in both cases, would have to be adopted as a postulate. For the recognition that if (2) is true, the lever will incline towards Δ, however, another postulate would be required, *viz.* that upon the support being shifted to one side a lever originally in equilibrium will incline towards the other side.

Proposition 8.

If from a magnitude another magnitude be taken away which does not have the same centre as the whole, when the straight line joining the centres of gravity of the whole magnitude and the magnitude taken away be produced towards the side where the centre of the whole magnitude is situated, and when from the produced part of the line joining the said centres a segment be cut off such that it has to the segment between the centres the same ratio as the weight of the magnitude taken away has to the remaining magnitude, the extremity of the segment cut off will be the centre of gravity of the remaining magnitude.

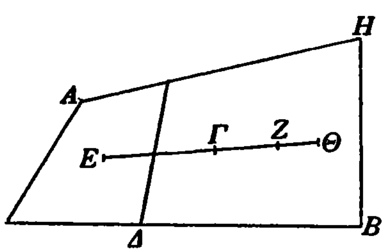

Fig. 122.

In Fig. 122 let the segment $A\Delta$ with centre of gravity E be taken away from the magnitude AB with centre of gravity Γ. On $E\Gamma$ produced Z has been determined so that

$$(Z\Gamma, E\Gamma) = (A\Delta, H\Delta) . \tag{1}$$

It has to be proved that Z is the centre of gravity of ΔH.

Proof: Suppose another point, Θ, to be the centre of gravity of ΔH. Since AB is composed of $A\Delta$ and $H\Delta$, the centre of gravity of AB must be a point O on $E\Theta$, determined by

$$(\Theta O, EO) = (A\Delta, H\Delta) .$$

Therefore Γ is not the centre of gravity of AB, which is contrary to the supposition.

It might be asked why Θ is collinear with E and Γ; in fact, if this is not the case, no contradiction will arise. The proof might therefore be given more effectively as follows:

Since Γ is the centre of gravity of AB, Γ has to lie on $E\Theta$, so that

$$(\Theta\Gamma, E\Gamma) = (A\Delta, H\Delta) .$$

From a comparison with (1) it now appears at once that Θ coincides with Z.

Proposition 9.

The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides of the parallelogram.

Let the parallelogram $AB\Gamma\Delta$ (Fig. 123) be given, in which E and Z are successively the middle points of AB and $\Gamma\Delta$. It has to be proved that the centre of gravity of $AB\Gamma\Delta$ lies on EZ.

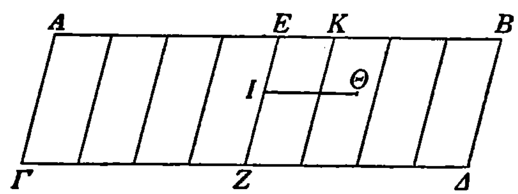

Fig. 123.

Suppose this is not true, but the centre of gravity is a point Θ outside EZ. Then let the straight line through Θ parallel to AB meet the straight line EZ in I. Now apply dichotomy (III; 0.5) to EB until the parts obtained (each equal to EK) are less than ΘI, and through the points of division thus obtained draw lines parallel to EZ. Proceeding in the same way on the other side of EZ, $AB\Gamma\Delta$

is divided into an even number of parallelograms, which are all congruent with KZ. When all these parallelograms are successively applied to KZ, the centres of gravity coincide with that of KZ (postulate IV). All these centres therefore lie on a straight line parallel to AB. By the application of Prop. 5, Corollary 2 we now recognize that the centre of gravity of AB must lie on the line segment having the centres of the central parallelograms for extremities. It cannot therefore be Θ; for $EK < I\Theta$.

Proposition 10.

The centre of gravity of any parallelogram is the point of intersection of its diagonals.

According to Prop. 9 the centre of gravity lies on each of the two straight lines joining the middle points of opposite sides; it is, however, also through the point of intersection of these straight lines that the diagonals pass.

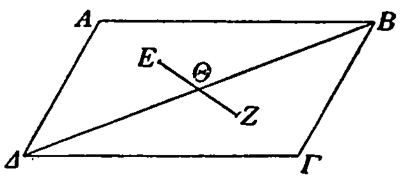

Fig. 124.

A second proof (Fig. 124) is given by considering the triangles into which the diagonal $B\Delta$ divides the parallelogram $AB\Gamma\Delta$. Since these triangles are congruent, the centres of gravity will coincide when the triangles are applied to each other (postulate IV). Now let E be the centre of gravity of $\triangle AB\Delta$, Θ the middle point of ΔB, Z a point on E produced, so that $E\Theta = \Theta Z$. When $\triangle AB\Delta$ is applied to $\triangle \Gamma\Delta B$, E will fall on Z, therefore Z is the centre of gravity of $\triangle \Gamma\Delta B$, and consequently, by Prop. 4, Θ is the centre of gravity of $AB\Gamma\Delta$.

Proposition 11.

If two triangles be similar to each other and within these triangles two points be similarly situated with respect to the triangles, and one point be the centre of gravity of the triangle in which it is situated, the other point will also be the centre of gravity of the triangle in which it is situated.

The meaning of the term "similarly situated" has been explained in the discussion of postulate V. The proof is a *reductio ad absurdum*, it being assumed that another point were the centre of gravity, upon which postulate V is applied; it is thus based on the unambiguousness of the relation of similar situation.

Proposition 12.

If two triangles be similar and the centre of gravity of one triangle lie on the straight line drawn from an angular point to the middle point of the base, the centre of gravity of the other triangle will lie on the straight line similarly drawn.

The proof is based on Prop. 11 in relation to the planimetric theorem that points which divide homologous medians of similar triangles into homologous proportional parts are similarly situated with respect to those triangles.

It appears that this proposition is not applied anywhere.

After these introductory theorems the situation of the centre of gravity of a triangle is found. The chief work is done in

Proposition 13.

In any triangle the centre of gravity lies on the straight line joining any vertex to the middle point of the base.

In Fig. 125 let Δ be the middle point of the base $B\Gamma$ of $\triangle AB\Gamma$. Suppose the centre of gravity Θ of $\triangle AB\Gamma$ not to lie on $A\Delta$. We

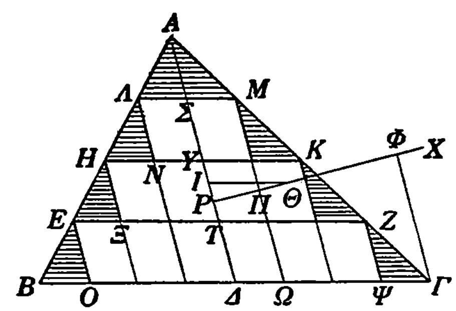

Fig. 125.

then know only that Θ lies within $\triangle AB\Gamma$ (postulate VII). Let the straight line through Θ parallel to $B\Gamma$ meet the straight line $A\Delta$ in I. Now apply dichotomy (III; 0.5) to $B\Gamma$ until the parts thus obtained (each equal to $\Delta\Omega$) are less than ΘI; through the points of division draw straight lines parallel to $A\Delta$, and divide the triangle in the manner indicated in Fig. 125 into parallelograms (MN,

$K\Xi$, etc.) and triangles ($A\Sigma M$, $Z\Psi\Gamma$, etc.). By Prop. 9 the centres of gravity of the parallelograms all lie on $A\Delta$, consequently the centre of gravity P of the figure $\mathbf{X}_\Pi$, consisting of all the parallelograms, also lies on $A\Delta$ (Prop. 6). Join P and Θ; let this straight line meet the straight line ΩM in Π and the straight line drawn through Γ parallel to $A\Delta$ in Φ. Π then lies between P and Θ, and Φ lies outside the triangle.

It is then ascertained how the centre of gravity of the figure $\mathbf{X}_\Delta$, consisting of all the shaded triangles, must be situated with respect to the points P, Θ, and Φ.

For this we first compare the area of $\triangle\ AB\Gamma$ with the sum of the areas of the shaded triangles. By similarity we have:

$$(A\Delta\Gamma, A\Sigma M) = [\mathbf{T}(A\Gamma),\ \mathbf{T}(AM)], \text{ etc.,}$$

whence

$$(A\Delta\Gamma, A\Sigma M + \ldots + Z\Psi\Gamma) = [\mathbf{T}(A\Gamma),\ \mathbf{T}(AM) + \ldots + \mathbf{T}(Z\Gamma)]$$
$$= [\mathbf{T}(A\Gamma),\ \mathbf{O}(AM, A\Gamma)] = (A\Gamma, AM)\,.$$

Likewise

$$(A\Delta B, A\Sigma\Lambda + \ldots + EOB) = (AB, A\Lambda) = (A\Gamma, AM)\,.$$

Therefore

$$(AB\Gamma, \mathbf{X}_\Delta) = (A\Gamma, AM) = (\Delta\Gamma, \Delta\Omega) = (P\Phi, P\Pi) > (P\Phi, P\Theta)$$

and thence *separando*

$$(\mathbf{X}_\Pi, \mathbf{X}_\Delta) > (\Theta\Phi, \Theta P)\,.$$

Now determine a point X on the straight line $P\Phi$ such that

$$(\mathbf{X}_\Pi, \mathbf{X}_\Delta) = (\Theta X, \Theta P)\,,$$

then $\Theta X > \Theta\Phi$, therefore X lies on $P\Phi$ produced.

By Prop. 8, X is now the centre of gravity of the figure $\mathbf{X}_\Delta$, consisting of the shaded triangles, which is impossible, because all these triangles are on the opposite side of the straight line drawn through Γ parallel to $A\Delta$ from X.

The latter conclusion is not, as Heiberg thinks[1]), based on postulate VII, for the perimeter of the figure $\mathbf{X}_\Delta$ is not "concave in the same direction". It should rather be imagined to have been made in view of the consideration that, if the centre of gravity is found of a figure whose component parts all lie on the same side of a straight line by combining with the aid of Prop. 6 two parts, combining their combination with a third part, etc., the centre of gravity of the whole figure must lie on the same side of the straight line on which all the parts lie.

[1]) *Opera* II, 155, Note 3.

In a second proof of Prop. 12 (Fig. 126) Archimedes joins the supposed centre of gravity Θ to A, B, and Γ, and through the

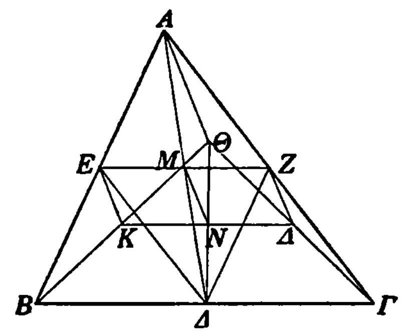

Fig. 126.

middle points E of AB and Z of $A\Gamma$ draws straight lines parallel to $A\Theta$, which meet $B\Theta$ and $\Gamma\Theta$ successively in K and Λ. In the similar triangles $AB\Gamma$ and $EB\Delta$, Θ and K are similarly situated points, therefore (Prop. 11) K is the centre of gravity of $\triangle\ EB\Delta$, and likewise Λ is the centre of gravity of $\triangle\ Z\Gamma\Delta$. Since further the areas of the triangles $EB\Delta$ and $Z\Gamma\Delta$ are equal, the centre of gravity of their combination lies at the middle point of $K\Lambda$, *i.e.* the point of intersection N with $\Theta\Delta$. In the parallelogram $AE\Delta Z$, M is the centre of gravity; the centre of gravity of $\triangle\ AB\Gamma$ therefore lies on MN, and cannot thus be Θ. Therefore Θ cannot lie outside $A\Delta$.

Proposition 14.

In any triangle the centre of gravity is the point in which the straight lines of the triangle joining the vertices to the middle points of the sides meet.

This follows at once from Prop. 13.

To conclude Book I, the centre of gravity of a trapezium is determined.

Proposition 15.

In any trapezium having two parallel sides[1]) *the centre of gravity lies on the straight line joining the middle points of the parallel sides, in such a way that the segment of it having the middle point of the smaller of the paralllel sides for extremity is to the remaining segment as the sum of double the greater plus the smaller is to the sum of double the smaller plus the greater of the parallel sides.*

In Fig. 127, in the trapezium $AB\Gamma\Delta$ $(A\Delta \parallel B\Gamma)$ let the non-parallel sides produced meet in H; Z and E are successively the

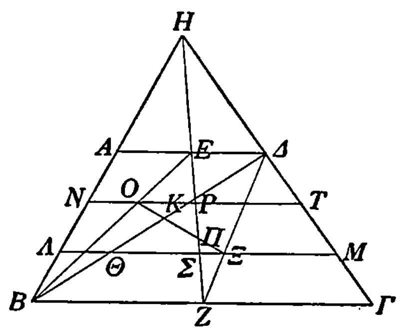

Fig. 127.

middle points of $B\Gamma$ and $A\Delta$. The centres of gravity of the triangles $HB\Gamma$ and $HA\Delta$ by Prop. 13 lie on HZ, therefore (Prop. 8 and postulate VII) the required centre of gravity of $AB\Gamma\Delta$ lies on the line segment EZ.

Now divide $B\Delta$ into three equal parts by means of the points K and Θ, and through these points draw NT and ΛM parallel to $B\Gamma$. Now the point of intersection Ξ of ΛM and ΔZ is the centre of gravity of $\triangle\, \Delta B\Gamma$, and likewise the point of intersection O of $N\Gamma$ and BE is the centre of gravity of $\triangle\, BA\Delta$. The centre of gravity of the trapezium therefore is the point of intersection of EZ and $O\Xi$. Now we have by Prop. 6 or 7:

$$(\Delta B\Gamma, BA\Delta) = (O\Pi, \Xi\Pi) = (P\Pi, \Sigma\Pi)\,,$$

[1]) Trapezium without any further indication refers to a quadrilateral.

and consequently also

$$(B\Gamma, A\Delta) = (P\Pi, \Sigma\Pi)\,,$$

whence

$$(B\Gamma, P\Pi) = (A\Delta, \Sigma\Pi) = (2B\Gamma + A\Delta, 2P\Pi + \Sigma\Pi)$$
$$= (B\Gamma + 2A\Delta, P\Pi + 2\Sigma\Pi)\,.$$

From this it follows that

$$(2B\Gamma + A\Delta, \Pi E) = (B\Gamma + 2A\Delta, Z\Pi)\,,$$

which is equivalent to that which was to be proved.

CHAPTER X

THE METHOD OF MECHANICAL THEOREMS

1. We shall now, deviating from the order in which Archimedes' works appear in Heiberg's edition of the text, first discuss the treatise *The Method of Mechanical Theorems, for Eratosthenes*, to be briefly designated as *The Method*. In fact, when we know this work, it is easier to understand the *Quadrature of the Parabola*, the contents of which in turn are assumed as known in Book II of *On the Equilibrium of Planes*.

The discovery and decipherment of the manuscript of the *Method* has already been discussed in Chapter II. The object of the work becomes clear from the introductory letter to Eratosthenes, which we first give in translation:

Archimedes to Eratosthenes greeting!

On an earlier occasion I sent you some of the theorems found by me, the propositions of which I had written down, urging you to find the proofs which I did not yet communicate at the time. The propositions of the theorems I sent were the following:

firstly: if in a right prism having a square[1]) *for its base a cylinder be inscribed which has its bases in the squares facing each other and its sides in the other faces of the prism*[2]), *and a plane be drawn through*

[1]) The word used is παραλληλόγραμμον, but it is clear from the context that a square is meant.

[2]) The meaning is that the curved surface of the cylinder touches the vertical faces.