THE WORKS OF
ARCHIMEDES

Translated into English, together with
Eutocius' commentaries, with commentary,
and critical edition of the diagrams

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Volume I
The Two Books *On the Sphere and the Cylinder*

CAMBRIDGE
UNIVERSITY PRESS

ON THE SPHERE AND THE CYLINDER, BOOK II

/Introduction/

Archimedes to Dositheus: greetings

Earlier you sent me a request to write the proofs of the problems, whose proposals[1] I had myself sent to Conon; and for the most part they happen to be proved[2] through the theorems whose proofs I had sent you earlier: <namely, through the theorem> that the surface of every sphere is four times the greatest circle of the <circles> in it,[3] and through <the theorem> that the surface of every segment of a sphere is equal to a circle, whose radius is equal to the line drawn from the vertex of the segment to the circumference of the base,[4] and through <the theorem> that, in every sphere, the cylinder having, <as> base, the greatest circle of the <circles> in the sphere, and a height equal to the diameter of the sphere, is both: itself, in magnitude,[5] half as large again as the sphere; and, its surface, half as large again as the surface of the sphere,[6] and through <the theorem> that every solid sector is equal to the cone having, <as> base, the circle equal to the surface of the segment of the sphere <contained> in the sector, and a height equal to the radius of the sphere. Now, I have sent you those theorems and problems that are proved through these theorems <above>, having proved them in this book. And as for those that are found through some other theory,

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1 *Protasis*: see general comments.

2 "Prove" and "write" use the same Greek root.

3 *SC* I.33. 4 *SC* I.42–3.

5 The words "in magnitude" refer to what we would call "volume" (to distinguish from the following assertion concerning "surface").

6 *SC* I.34 Cor.

<namely:> those concerning spirals, and those concerning conoids, I shall try to send quickly.[7]

Of the problems, the first was this: Given a sphere, to find a plane area equal to the surface of the sphere. And this is obviously proved from the theorems mentioned already; for the quadruple of the greatest circle of the <circles> in the sphere is both: a plane area, and equal to the surface of the sphere.

TEXTUAL COMMENTS

Analogously to the brief sequel to the postulates in the first book, so here, again, the introductory material ends with a brief unpacking of obvious consequences. Assuming that Archimedes' original text did not contain numbered propositions, there is a sense in which this brief unpacking can count as "the first proposition:" it is the first argument. It is also less than a proposition, in the crucial sense that it does not have a diagram. This liminal creature, then, helps mediate the transition between the two radically distinct portions of text – introduction and sequence of propositions.

The propositions probably did not possess numberings; the books certainly did not. It is perfectly clear that the titles of treatises, let alone their arrangement as a consecutive pair, are both later than Archimedes. (It is interesting to note that the same arrangement is present in both the family of the lost codex A, and the Palimpsest, even though the two codices differ considerably otherwise in their internal arrangement.) As for Archimedes, he simply produced two unnamed treatises, with obvious continuities in their subject matter, as well as differences in their focus, that he himself spells out in this introduction. There is no harm in referring to them – as the ancients already did – as "First Book on Sphere and Cylinder" or "Second Book on Sphere and Cylinder." We should take this, perhaps, as our own informal title, akin to the manner in which philosophers sometimes refer to "Kant's First Critique," etc.

GENERAL COMMENTS

Practices of mathematical communication

In this introduction, rich in references to mathematical communication, we learn of several stages in the production of a treatise by Archimedes.

First comes the "proposal" – my translation of the Greek word *protasis*. Now, this word came to have a technical sense, first attested from Proclus' *Commentary to Euclid's Elements* I: that part of the proposition in which the general enunciation is made. It is not very likely, however, that this technical sense is what Archimedes himself already had in mind here: in the later, Proclean sense, a *protasis* has meaning only when accompanied by other, non-*protasis* parts

[7] A reference to *SL*, *CS*. (To appear in Volume II of this translation.)

of the proposition, and clearly Archimedes had sent only the *protasis*. What could that be, then? Literally, *protasis* is "that which is put forward," and one sense of the word is "question proposed, problem" – in other words, a puzzle. It was such puzzles, then, that Archimedes sent Conon. ("All right, I give up," came back Dositheus' reply.)

Next comes the proof. As noted in n. 2, "prove" uses the same Greek root as "write," *graph*. This is also closely related to terms referring to the figure (which is a kata*graphe*, or a dia*gram*ma), so we see a nexus of ideas: writing down, drawing figures, proving; all having to do with translating an idea in the mind of a mathematician to a product that is part of actual mathematical communication – answering the three *sine qua non* conditions of Greek mathematical communication – written, proved, drawn.

What is the relation between the idea in the mind of the mathematician and the idea in actual mathematical communication? Archimedes' references to results he already seems to have in some sense – from *SL* and *CS* – are especially tantalizing. Why does he promise to send them "quickly?" He probably knows how all those theorems and problems are proved – for otherwise he would not send out the puzzles concerning them. So why not send them straight away? Perhaps he was still busy proofreading them. (If so, the morass of inconsistent style and abbreviated exposition we know so well by now from Book I, is what Archimedes can show *after* the proofreading stage!) Or perhaps, all Archimedes had, prior to "sending" to Dositheus, were notes – stray wax tablets with diagrams that he alone could interpret as solutions for intricate problems.

Or perhaps, he does not have a perfect grasp on the proofs, yet? "I have sent you those theorems and problems that are proved through these theorems <above>, having proved them in this book. And as for those that are found through some other theory . . ." Things, then, are either *proved through theorems* or *found through theory*. Perhaps, "theory" (a cognate of "theorem," roughly referring, in this context, to the activity of which "theorems" are the product) is a more fuzzy entity, comprising a bundle of unarticulated bits of mathematical knowledge present to the mathematician's mind. Perhaps, it is such knowledge – and not explicitly written down proofs – which is active in the mathematical *discovery*?

Leaving such speculations aside, we ought to focus not on the stage of mathematical *discovery*, but on the stage of mathematical *communication*. The decisive verb in this introduction is not "discovery," not even "prove," but, much more simply, "send." It is the act of sending which gives rise to a mathematical treatise. In this real sense, then, it was the ancient mathematical community – and not the ancient mathematicians working alone – who were responsible for the creation of Greek mathematical writing.

/1/

The second was: given a cone or a cylinder, to find a sphere equal to the cone or to the cylinder.

Let a cone or a cylinder be given, A, (a) and let the sphere B be
Eut. 270 equal to A,[8] (b) and let a cylinder be set out, ΓZΔ, half as large again as the cone or cylinder A, (c) and <let> a cylinder <be set out>, half as large again as the sphere B, whose base is the circle around the diameter HΘ, while its axis is: KΛ, equal to the diameter of the sphere B;[9] (1) therefore the cylinder E is equal to the cylinder K. [(2) But the bases of equal cylinders are reciprocal to the heights];[10] (3) therefore as the circle E to the circle K, that is as the <square> on ΓΔ to the Eut. 272
<square> on HΘ[11] (4) so KΛ to EZ. (5) But KΛ is equal to HΘ [(6) for the cylinder which is half as large again as the sphere has the axis equal to the diameter of the sphere, (7) and the circle K is greatest of the <circles> in the sphere];[12] (8) therefore as the <square> on ΓΔ to the <square> on HΘ, so HΘ to EZ. (d) Let the <rectangle contained> by ΓΔ, MN[13] be equal to the <square> on HΘ; (9) therefore as ΓΔ to MN, so the <square> on ΓΔ to the <square> on HΘ,[14] (10) that is HΘ to EZ, (11) and alternately, as ΓΔ to HΘ, so (HΘ to MN) (12) and MN to EZ.[15] (13) And each of <the lines> ΓΔ, EZ is given;[16]

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[8] We are not explicitly told so, but we are to proceed now through the method of analysis and synthesis, in which we assume, at the outset, that the problem is solved – in this case, that we have found a sphere equal to the given cone or cylinder. We then use this assumption to derive the way by which a solution may be found.

[9] This construction is a straightforward application of *SC* I.34 Cor., as explained in Steps 6–7.

[10] *Elements* XII.15. This is recalled in the interlude of the first book, but no such reference needs to be assumed in this, second book, and in general I shall not refer in this book to the interlude of the first book.

[11] *Elements* XII.2. [12] *SC* I.34 Cor.

[13] ΓΔ is given, and it is therefore possible (through *Elements* I.45) to construct a parallelogram on it – therefore also a rectangle – equal to a given area, in this case equal to the square on HΘ. It is then implicit that MN is *defined* as the second line in a rectangle, contained by ΓΔ, MN, which is equal to the square on HΘ.

[14] Compare VI.1, ". . . parallelograms which are under the same height are to one another as the bases," and then the square on ΓΔ and the rectangle contained by ΓΔ, MN can be conceptualized as lying both under the height ΓΔ, with the bases ΓΔ, MN respectively (so ΓΔ:MN::the square on ΓΔ:the rectangle contained by ΓΔ, MN); and then the rectangle contained by ΓΔ, MN has been constructed equal to the square on HΘ.

[15] A complex situation. We have just seen (Steps 9–10) that A. ΓΔ:MN::HΘ:EZ, which, "alternately" (*Elements* V.16), yields B. ΓΔ:HΘ::MN:EZ. On the other hand, the construction at Step d, together with *Elements* VI.17, yields C. ΓΔ:HΘ::HΘ:MN. Archimedes starts from A, and then says, effectively, "(Step 11:) alternately C (Step 12:) and B." This is very strange: the "alternately" should govern B, not C. Probably Step 11 should be conceived as if inside parenthesis – which I supply, as an editorial intervention in the text, in Step 11.

[16] I.e., they are determined by the "given" of the problem, namely the cone or cylinder A (see Step b in the construction). Note, however, that they are given only as a couple. Both together determine a unique volume, but they may vary simultaneously (the one

(14) therefore HΘ, MN are two mean proportionals between two given lines, ΓΔ, EZ; (15) therefore each of <the lines> HΘ, MN are given.[17]

So the problem will be constructed[18] like this:

So let there be the given cone or cylinder, A; so it is required to find a sphere equal to the cone or cylinder A.

(a) Let there be a cylinder half as large again as the cone or cylinder A,[19] whose base is the circle around the diameter ΓΔ, and its axis is
Eut. 272 the <axis> EZ, (b) and let two mean proportionals be taken between ΓΔ, EZ, <namely> HΘ, MN, so that as ΓΔ is to HΘ, HΘ to MN and MN to EZ, (c) and let a cylinder be imagined, whose base is the circle around the diameter HΘ, and its axis, KΛ, is equal to the diameter HΘ. So I say that the cylinder E is equal to the cylinder K.

(1) And since it is: as ΓΔ to HΘ, MN to EZ, (2) and alternately,[20] (3) and HΘ is equal to KΛ [(4) therefore as ΓΔ to MN, that is as the <square> on ΓΔ to the <square> on HΘ,[21] (5) so the circle E to the circle K],[22] (6) therefore as the circle E to the circle K, so KΛ to EZ. [(7) Therefore the bases of the cylinders E, K are reciprocal to the heights]; (8) therefore the cylinder E is equal to the cylinder K.[23] (9) But the cylinder K is half as large again as the sphere whose diameter is HΘ; (10) therefore also the sphere whose diameter is equal to HΘ, that is B,[24] (11) is equal to the cone or cylinder A.

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growing, the other diminishing in reciprocal proportion) without changing that volume. To say that "each of them is given" is, then, misleading. We may in fact derive a solution for the problem, regardless of how we choose to set the cone ΓZΔ, since what we are seeking is for *some* cylinder or cone satisfying the equality: one among the infinite family of such cylinders and cones, their bases and heights reciprocally proportional.

[17] A single mean proportional of A, C is a B satisfying A:B:B:C. Two mean proportionals satisfy A:B::B:C::C:D, where B and C are the two mean proportionals "between" A and D.

[18] Greek "*sunthesetai*," "will be synthesized." The word belongs to the pair *analysis/synthesis*, perhaps translatable as "deconstruction/construction," literally something like "breaking into pieces," "putting the pieces together." As we saw above, Archimedes (as is common in Greek mathematics) did not introduce in any explicit way his *analysis*; but the *synthesis* is introduced by an appropriate formula.

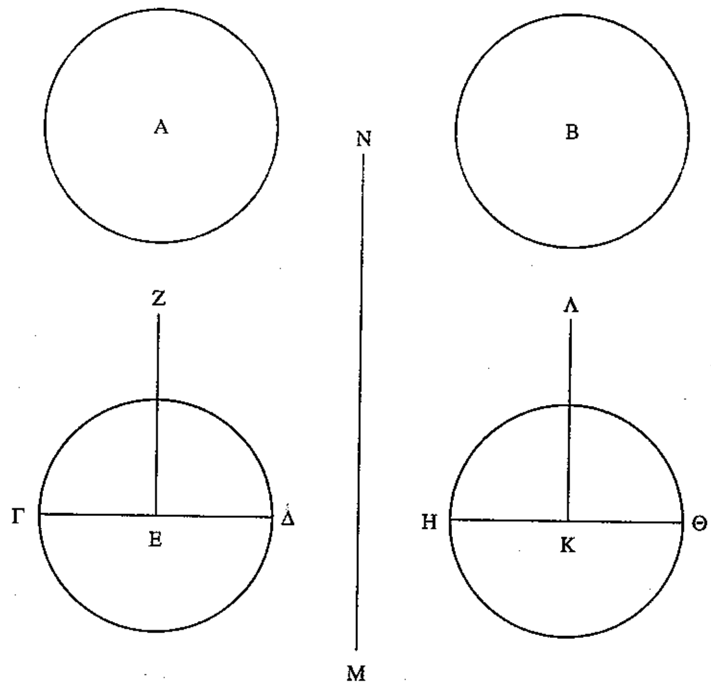
[19] See Eutocius for this problem, which is essentially relatively simple (it requires one of several propositions from *Elements* XII, e.g. XII.11 or 14).

[20] *Elements* V.16, yielding the unstated conclusion: ΓΔ:MN::HΘ:EZ.

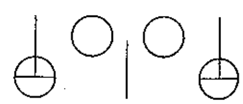
[21] From *Elements* V. Deff. 9–10, and the stipulation that the lines ΓΔ, HΘ, MN, EZ are in continuous proportion (which is an equivalent way of saying that HΘ, MN are two mean proportionals between ΓΔ, EZ).

[22] *Elements* XII.2. [23] *Elements* XII.15.

[24] This sphere B – the real requirement of the problem – has not been constructed at all at the synthesis stage. Archimedes offers two incomplete arguments that only taken together provide a solution to the problem. See general comments to this and following problems, for the general question of relation between analysis and synthesis.



II.1
Codex A had the slightly different lay-out of the thumbnail (clearly the difference is that codex C has two columns of writing in the page, while codex A probably had only one: with wider space available, A adopted a shorter arrangement. Late ancient writing would tend to have two columns, answering to the narrow column of the papyrus roll, hence I prefer the layout of C). Codices DH4 do not have the point M extending to below the lower circles, perhaps representing codex A. Once again, I follow codex C. Codices DG had KΛ greater than EZ. Codex G had the two circles A, B (equal to each other) greater than the circles ΓEΔ, HKΘ (also roughly equal to each other); circle HKΘ somewhat lower than circle ΓEΔ. Codex 4 permutes M/N.



TEXTUAL COMMENTS

Heiberg brackets Step 2 in the analysis, as well as the related Step 7 in the synthesis, presumably for stating what are relatively obvious claims: but this being the very beginning of the treatise, we may perhaps imagine Archimedes being more explicit than usual. Steps 6–7 in the analysis, on the other hand, are very jarring, in repeating, in such close proximity, the claim of Step c: they seem most likely to be a scholion to Step 5, interpolated into the text.

Steps 4–5 in the synthesis are more difficult to explain. They make relevant and non-obvious claims. They are problematic only in that their connector is wrong: the "therefore" at the start of Step 4 yields the false expectation, that the claim of Steps 4–5 taken together is somehow to be derived from the preceding steps. I can not see why this mistaken connector should not be attributed to Archimedes, as a slip of the pen.

GENERAL COMMENTS

Does "analysis" *find* solutions?

The pair of analysis and synthesis is a form of presenting problems, whose intended function has been discussed and debated ever since antiquity. In the comments to this book, I shall make a few observations on the details of some arguments offered in this form.

A basic question is whether the analysis in some sense "finds" the solution to the problem. In this problem, the solution can be seen quite simply (arguably, the problem is simpler than the synthesis/analysis approach makes it appear), and it is therefore a useful case for answering this question.

We may conceive of the problem of finding a sphere equal to a given cone or cylinder, as that of transformation: we wish to transform the cone or cylinder into a sphere. Consider a cylinder. Given any cylinder, we may transform it into a "cubic" cylinder (where the diameter of the base equals the height), by conserving (new circle):(old circle)::(old height):(new height). (This is not a trivial operation, and it already calls for two mean proportionals, involving as it does a proportion with both lines and areas.) The sphere obtained inside this "cubic" cylinder would be, following *SC* I.34 Cor., $\frac{2}{3}$ the cylinder itself. We may therefore enlarge this new sphere by a factor of $\frac{3}{2}$, by enlarging its diameter by a factor of $\sqrt[3]{\frac{3}{2}}$. This new sphere, with its new diameter, would now be the desired sphere; but it is obviously simpler to enlarge the original cylinder by a factor of $\frac{3}{2}$ (no need to specify how, but the simplest way is by enlarging its height by the same factor, following *Elements* XII.14). Then all we require to do is to transform this new, enlarged cylinder into a "cubic" cylinder, which is done through two mean proportionals.

Thus the solution to the problem has two main ideas. One is to use *SC* I.34 Cor. to correlate a sphere and a "cubic" cylinder; the second is to make this correlation into an equality, by enlarging the given cylinder in the factor $\frac{3}{2}$. The second idea is an *ad-hoc* construction, which does not emerge in any obvious way out of the conditions stated by the problem. And indeed, it is not anything we derive in the course of the analysis: to the contrary, this is a stipulated construction, occurring as Step b of the analysis. Thus this second aspect of the solution clearly is not "found" by the analysis.

But neither is the first one. To begin with, the main idea is derived not from the analysis process, but from *SC* I.34 Cor. itself. But this obvious observation aside, it should be noticed that the idea of using two mean proportionals – arguably, the most important point of the analysis – is, once again, not a direct result of the analysis as such. Once again, it has to be stipulated into the analysis by an *ad-hoc* move – that of Step d, where the line MN is stipulated into existence (with several further manipulations, this line yields the two mean proportionals). Nothing in the analysis necessitates the introduction of this line, which was inserted into the proposition, just like the auxiliary half-as-large cylinder, because Archimedes already knew what form the solution would make.

In other words: in this case, there is nothing "heuristic" about analysis. Here we see analysis not so much a format for finding solutions, but a format for presenting them.

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Every segment of the sphere is equal to a cone having a base the same as the segment, and, <as> height, a line which has to the height of the segment the same ratio which: both the radius of the sphere and the

EUTOCIUS' COMMENTARY TO *ON THE SPHERE AND THE CYLINDER* II

Now that the proofs of the theorems in the first book are clearly discussed by us, the next thing is the same kind of study with the theorems of the second book.

First he says in the 1st theorem:

Arch. 188 "Let a cylinder be taken, half as large again as the given cone or cylinder." This can be done in two ways, either keeping in both the same base, or the same height.[1] And to make what I said clearer, let a cone or a cylinder be imagined, whose base is the circle A,[2] and its height AΓ, and let the requirement be to find a cylinder half as large again as it.

(a) Let the cylinder AΓ be laid down, (b) and let the height of the cylinder, AΓ,[3] be produced, (c) and let ΓΔ be set out <as> half AΓ; (1) therefore AΔ is half as large again as AΓ. (d) So if we imagine a cylinder having, <as> base, the circle A, and, <as> height, the line AΔ, (2) it shall be half as large again as the <cylinder> set forth, AΓ; (3) for the cones and cylinders which are on the same base are to each other as the height.[4]

(e) But if AΓ is a cone, (f) bisecting AΓ,[5] as at E, (g) if, again, a cylinder is imagined having, <as> base, the circle A, and, <as>

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[1] There are infinitely many other combinations, of course, as Eutocius will note much later: his comment is not meant to be logically precise, but to indicate the relevant mathematical issues.

[2] Eutocius learns from Archimedes to refer to a circle via its central letter. This is how ancient mathematical style is transmitted: by texts imitating texts.

[3] This time AΓ designates "height," not "cylinder:" no ambiguity, as the Greek article (unlike the English article) distinguishes between the two.

[4] *Elements* XII.14.

[5] This time AΓ is a line, not a cone; again, this is made clear through the articles.

height – AE, (4) it will be half as large again as the cone AΓ; (5) for the cylinder having, <as> base, the circle A, and, <as> height, the line AΓ, is three times the cone AΓ,[6] (6) and twice the cylinder AE; (7) so that it is clear that the cylinder AE, in turn, is half as large again as the cone AΓ.

So in this way the problem will be done keeping the same base in both the given <cylinder>, and the one taken. But it is also possible to do the same with the base coming to be different, the axis remaining the same.

For let there be again a cone or cylinder, whose base is the circle ZH, and <its> height the line ΘK. Let it be required to find a cylinder half as large again as this, having a height equal to ΘK. (a) Let a square, ZΛ, be set up on the diameter of the circle ZH, (b) and, producing ZH, let HM be set out <as> its half, (c) and let the parallelogram ZN be filled; (1) therefore the <parallelogram> ZN is half as large again as the <square> ZΛ, (2) and MZ <is half as large again> as ZH. (d) So let a square equal to the parallelogram ZN be constructed,[7] namely <the square> ΞΠ, (e) and let a circle be drawn around one of its sides, <namely> ΞO, as diameter. (3) So the <circle> ΞO shall be half as large again as the <circle> ZH; (4) for circles are to each other as the squares on their diameters.[8] (f) And if a cylinder is imagined, again, having, <as> base, the circle ΞO, and a height equal to ΘK, (5) it shall be half as large again as the cylinder whose base is the circle ZH, and <its> height the <line> ΘK.[9]

(g) And if it is a cone, (h) similarly, doing the same,[10] and constructing a square such as ΞΠ, equal to the third part of the parallelogram ZN, (i) and drawing a circle around its side ΞO, (j) we imagine a cylinder on it, having, <as> height, the <line> ΘK; (5) we shall have it half as large again as the cone put forth. (6) For since the parallelogram ZN is three times the square ΞΠ, (7) and <it is> half as large again as ZΛ, (8) the <square> ZΛ shall be twice the <square> ΞΠ, (9) and through this the circle, too, shall be twice the circle (10) and the cylinder <twice> the cylinder.[11] (11) But the cylinder having, <as> base, the circle ZH, and, <as> height, the <line> ΘK, is three times the cone <set up> around the same base and the same height;[12] (12) so that the cylinder having, <as> base, the circle ΞO, and a height equal to ΘK, is in turn half as large again as the cone put forth.

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[6] *Elements* XII.10. [7] *Elements* II.14.

[8] *Elements* XII.2. [9] *Elements* XII.11.

[10] Refers to Steps (b–d) in this argument (not to (e–g), (4–7) in the preceding argument).

[11] *Elements* XII.11. [12] *Elements* XII.10.

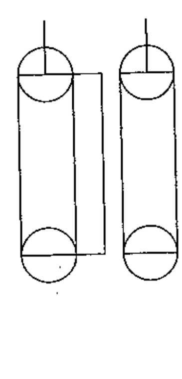
but we have come across writings by many famous men that offered this very problem (of which, we have refused to accept the writing of Eudoxus of Cnidus, since he says in the introduction that he has found it through curved lines, while in the proof, in addition to not using curved lines, he finds a discrete proportion and uses it as if it were continuous,[15] which is absurd to conceive, I do not say for Eudoxus, but for those who are even moderately engaged in geometry). Anyway, so that the thought of those men who have reached us will become well known, the method of finding of each of them will be written here, too.[16]

As Plato[17]

Given two lines, to find two mean proportionals in continuous proportion.

Let the two given lines, whose two mean proportionals it is required to find, be ΑΒΓ, at right <angles> to each other. (a) Let them be produced along a line towards Δ, E,[18] (b) and let a right angle be constructed,[19] the <angle contained> by ZHΘ, (c) and in one side, e.g. ZH, let a ruler, KΛ, be moved, being in some groove in ZH, in such a way that it shall, itself <=KΛ>, remain throughout parallel to HΘ. (d) And this will be, if another small ruler be imagined, too, fitting with ΘH, parallel to ZH: e.g. ΘM; (e) for, the upward surfaces[20] of ZH, ΘM being grooved in axe-shaped grooves (f) and knobs being made,

(*cont.*) so closely the original layout of codex A, has the two diagrams consecutive *on the same page*. I edit here the first of the two diagrams; the second is largely identical, with the exception that Φ was omitted in codex A, and M is omitted in codex H. Codex D adds further circles to the rectangles: see thumbnail. Codices DH have genuine circles, instead of almond shapes, at Φ, TY; codex D has them also at Γ, A. Codex G has all base lines on the same height; D has all on the same height except for TY which is slightly higher; H has A at the same height as Π, both higher than ΛN, in turn higher than TY; B has the figures arranged vertically, rather than horizontally. Perhaps the original arrangement cannot be reconstructed. The basic proportions, however, are remarkably constant between the codices. Codex E has X (?) instead of Λ.



[15] That is, instead of a:b::b:c::c:d, all the pseudo-Eudoxus text had was a:b::c:d.

[16] In paraphrase: "although strictly speaking I merely write a commentary on Archimedes, here I have come across many interesting things that are less well known and, to make them better known, I copy them into my new text." It is interesting that Eutocius' bet came true: his own text, because of its attachment to Archimedes, survived, whereas his sources mostly disappeared.

[17] It is very unlikely that Plato the philosopher produced this solution (if a mathematical work by Plato had circulated in antiquity, we would have heard much more of it). The solution is either mis-ascribed, or – much less likely – it should be ascribed to some unknown Plato. In general, there are many question marks surrounding the attributions made by this text of Eutocius: Knorr (1989) is likely to remain for a long time the fundamental guide to the question. In the following I shall no more than mention in passing some of these difficulties.

[18] For the time being, Δ, E are understood to be as "distant as we like." Later the same points come to have more specific determination.

[19] The word – *kataskeuasthō* – is not part of normal geometrical discourse, and already foreshadows the mechanical nature of the following discussion. Notice also that we have now transferred to a new figure.

[20] "Upward surfaces:" notice that the contraption is seen from above (otherwise, of course, there is nothing to hold KΛ from falling).

fitting KΛ to the said grooves, (1) the movement of the <knobs>[21] KΛ shall always be parallel to HΘ. (g) Now, these being constructed, let one chance side of the angle be set out, HΘ, touching the <point> Γ,[22] (h) and let the angle and the ruler KΛ be moved to such a position where the point H shall be on the line BΔ, the side HΘ touching the <point> Γ,[23] (i) while the ruler KΛ should touch the line BE on the <point> K, and on the remaining side[24] <it should touch> the <point> A,[25] (j) so that it shall be, as in the diagram: the right angle <=of the machine, namely ΘHK> has <its> position as the <angle contained> by ΓΔE, (k) and the ruler KΛ has <its> position as EA has;[26] (2) for, these being made, the <task> set forth will be <done>. (3) For the <angles> at Δ, E being right, (4) as ΓB to BΔ, ΔB to BE and EB to BA.[27]

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[21] The manuscripts – not Heiberg's edition – have a plural article, which I interpret as referring to the knobs.

[22] Imagine that what we do is to put the contraption on a page containing the geometrical diagram. So we are asked to put the machine in such a way, that the side KΛ touches the point Γ. This leaves much room for maneuver; soon we will fix the position in greater detail.

[23] The freedom for positioning the machine has been greatly reduced: H, one of the points of HΘ, must be on the line BΔ, while some other point of HΘ must pass through Γ. This leaves a one-dimensional freedom only: once we decide on the point on BΔ where HΘ stands, the position of the machine is given. Each choice defines a different angle ΓΔB. (Notice also that it is taken for granted that HΘ is not shorter than BΓ.)

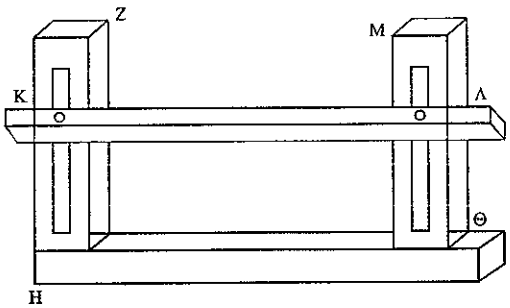
[24] "The remaining side" means somewhere on the ruler KΛ, away from K and towards Λ, though not necessarily at the point Λ itself.

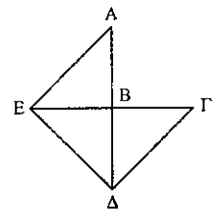
[25] The point K must be on BE, while some point of the ruler KΛ must be on the point A. Once again, a one-dimensional freedom is left (there are infinitely many points on the line BE that allow the condition). Each choice of point on BE, once again, defines a different angle AEB. Thus the conditions of Steps h and i are parallel. They are also inter-dependent: AE, ΓΔ being parallel, each choice of point on BΔ also determines a choice on BE. Of those infinitely many choices, the closer we make Δ to B, the more obtuse angle ΓΔE becomes, and the further we make Δ from B, the more acute angle ΓΔE becomes. Thus, by continuity, there is a point where the angle ΓΔE is right, and this unique point is the one demanded by the conditions of the problem – none of the above being made explicit.

[26] Now – and only now – Δ and E have become specific points.

[27] Note also that the lines AE, ΔΓ are parallel, and also note the right angles at B (all guaranteed by the construction). Through these, the similarity of all triangles can be easily shown (*Elements* I.29 suffices for the similarity of ABE, ΓBΔ. Since Δ, E are right, and so are the sums BΓΔ+BΔΓ, BAE+BEA (given *Elements* I.32), the similarity of ΓEΔ with the remaining two triangles is secured as well). *Elements* VI.4 then yields the proportion.

A general observation on the solution: it uses many expressions belonging to the semantic range of "e.g., such as, a chance". This can hardly be for the sake of signaling generalizability. Rather, the hypothetical nature of the construction is stressed. Further, the main idea of the construction is to fix a machine on a *diagram*. So the impression is





As Hero in the *Mechanical Introduction* and in the *Construction of Missile-Throwing Machines*[28]

Let the two given lines, whose two mean proportionals it is required to find, be AB, BΓ. (a) Let them <=the two given lines> be set out, so that they contain a right angle, that at B, (b) and let the parallelogram BΔ be filled, and let AΓ, BΔ be joined [(1) So it is obvious, that they <=AΓ, BΔ> are equal, (2) bisecting each other; (3) for the circle drawn around one of them will also pass through the limits of the other, (4) through <the property that> the parallelogram is right-angled].[29] (c) Let ΔΓ, ΔA be produced [to Z, H], (d) and let a small ruler be imagined, as ZBH, moved around some knob fixed at B, (d) and let

Catalogue: Plato
I avoid a full edition of this diagram. It is almost unique in the Archimedean corpus in offering a detailed three-dimensional perspective. Study of the nature of this three-dimensional representation will require attention to precise details of angles, which are very difficult to convey, and many lines can be named only by cumbersome expressions. To complicate further, scribes often had to erase and redraw parts of the diagram, making it much more complicated to ascribe anything to codex A. A facsimile of all figures, with discussion, is called for. The diagram printed follows, for each line-segment drawn, the majority of codices, which is usually either the consensus of all codices, or the consensus of all codices but one. For the geometrical structure ABΓΔE: codex E has the line-segments in "correct" proportions (BΓ>BΔ>BE>BA) and, since codex E is on the whole the most conservative visually, it may perhaps be preferable. Codex D has the geometrical

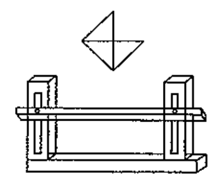
that this is a *geometrical* flight of fancy, momentarily more realistic with the reference to the axe-shaped grooves, but essentially a piece of geometry. This is a geometrical toy, and the language seems to suggest it is no more than a *hypothetical* geometrical toy: for indeed – for geometrical purposes – imagining the toy and producing it are equivalent.

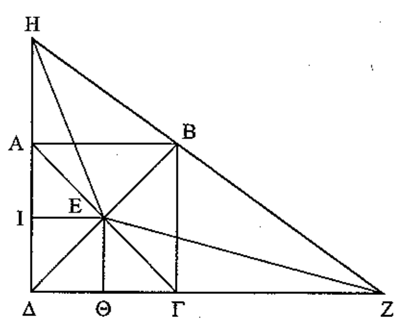
[28] One version of this, that of the *Mechanical Introduction*, is preserved in Pappus' *Collection* (Hultsch [1886] I. 62–5, text and Latin translation). *The Construction of Missile-Throwing Machines* is an extant work (for text and translation, see Marsden [1971] 40–2). The following text agrees with both, though not in precise agreement; the differences are mainly minor, and the phenomenon is well known for ancient quotations in general. Hero was an Alexandrine, probably living not much before the year AD 100. Relatively many treatises ascribed to him are extant; some readers might feel too many. While a coherent individual seems to emerge from the writings (a competent but shallow popularizer of mathematics, usually interested in its more mechanical aspects), little is known about that individual, and perhaps no work may be ascribed to him with complete certainty.

[29] *Elements* III.22. Heiberg square-brackets Steps 1–4 here, as well as several other passages in this proof, because of their absence in the "original" of Hero. There are many possible scenarios (say, that we have here, in fact, the true original form of Hero, corrupted elsewhere; or that Hero had more than one version published . . . or that such questions miss the nature of ancient publication and quotation).

it be moved, until it cuts equal <lines drawn> from E, that is EH, HZ. (e) And let it <=the ruler> be imagined cutting <the lines> and having <its> position <as> ZBH, with the resulting EH, EZ being, as has been said, equal. [(f) So let a perpendicular EΘ be drawn from E on ΓΔ; (5) so it clearly bisects ΓΔ. (6) Now since ΓΔ is bisected at Θ, (7) and ΓZ is added, (8) the <rectangle contained> by ΔZΓ together with the <square> on ΓΘ is equal to the <square> on ΘZ.[30] (9) Let the <square> on EΘ be added in common; (10) therefore the <rectangle contained> by ΔZΓ together with the <squares> on ΓΘ, ΘE is equal to the <squares> on ZΘ, ΘE. (11) And the <squares> on ΓΘ, ΘE are equal to the <square> on ΓE,[31] (12) while the <squares> on ZΘ, ΘE are equal to the <square> on EZ];[32] (13) therefore the <rectangle contained> by ΔZΓ together with the <square> on ΓE is equal to the <square> on EZ. (14) So it shall be similarly proved that the <rectangle contained> by ΔHA, too, together with the <square> on AE, is equal to the <square> on EH. (15) And AE is equal to EΓ, (16) while HE <is equal> to EZ; (17) and therefore the <rectangle contained> by ΔZΓ is equal to the <rectangle contained> by ΔHA [(18) and if the <rectangle contained> by the extremes is equal to the <rectangle contained> by the means, the four lines are proportional];[33] (19) therefore it is: as ZΔ to ΔH, so AH to ΓZ. (20) But as ZΔ to ΔH, so ZΓ to ΓB (21) and BA to AH [(22) for ΓB has been drawn parallel to one <side> of the triangle ZΔH, namely to ΔH, (23) while AB <has been drawn> parallel to <another,> ΔZ];[34] (24) therefore as BA to AH, so AH to ΓZ and ΓZ to ΓB. (25) Therefore AH, ΓZ are two mean proportionals between AB, BΓ [which it was required to find].

Plato (*cont.*) structure inside the mechanism, as in the thumbnail.





Catalogue: Hero Codices DE have AB greater than BΓ. Codex B omits I as well as the line IE.

[30] *Elements* II.6.
[31] *Elements* I.47. Original word order: "to the squares . . . is equal the square."
[32] *Elements* I.47. [33] *Elements* VI.16. [34] *Elements* VI.2.

As Philo the Byzantine[35]

Let the two given lines, whose two mean proportionals it is required to find, be AB, BΓ. (a) Let them be set out, so that they will contain a right angle, that at B, (b) and, having joined AΓ (c) let a semicircle be drawn around it, <namely> ABEΓ, (d) and let there be drawn: AΔ, in right <angles> to BA, (e) and ΓZ, <in right angles> to BΓ, (f) and let a moved ruler be set out as well, at the <point> B, cutting the <lines> AΔ, ΓZ (g) and let it be moved around B, until the <line> drawn from B to Δ is made equal to the <line> drawn from E to Z, (1) that is <equal> to the <line> between the circumference of the circle and ΓZ. (h) Now, let the ruler be imagined having a position as ΔBEZ has, (i) ΔB being equal, as has been said, to EZ. I say that AΔ, ΓZ are mean proportionals between AB, BΓ.

(a) For let ΔA, ZΓ be imagined produced and meeting at Θ; (1) so it is obvious that (BA, ΘZ being parallel) (2) the angle at Θ is right, (b) and, the circle AEΓ being filled up, (3) it shall pass through Θ, as well.[36] (4) Now since ΔB is equal to EZ, therefore also the <rectangle contained> by EΔB is equal to the <rectangle contained> by BZE.[37] (5) But the <rectangle contained> by EΔB is equal to the <rectangle contained> by ΘΔA ((6) for each is equal to the <square> on the tangent <drawn> from Δ)[38] (7) while the <rectangle contained> by BZE is equal to the <rectangle contained> by ΘZΓ ((8) for each, similarly, is equal to the <square> on the tangent <drawn> from Z);[39] (9) so that, in turn, the <rectangle contained> by ΘΔA is equal to the <rectangle contained> by ΘZΓ, (10) and through this it is: as ΔΘ to ΘZ, so ΓZ to ΔA.[40] (11) But as ΘΔ to ΘZ, so both: BΓ to ΓZ, and ΔA to AB; (12) for BΓ has been drawn parallel to the <side> of the triangle ΔΘZ, <namely> ΔΘ (13) while BA <has been drawn> parallel to <its side> ΘZ;[41] (14) therefore it is: as BΓ to ΓZ, ΓZ to ΔA and ΔA to AB; which it was set forth to prove.

And it should be noticed that this construction is nearly the same as that given by Hero; for the parallelogram BΘ is the same as that taken

[35] Philo of Byzantium produced, in the fourth century BC, a collection of mechanical treatises, circulating in antiquity, but surviving now only in parts. Those parts reveal Philo as an original and brilliant author, probably one of the most important ancient mechanical authors. It appears that the solution quoted here was offered in a part of the work now lost. See Marsden (1971) 105–84.

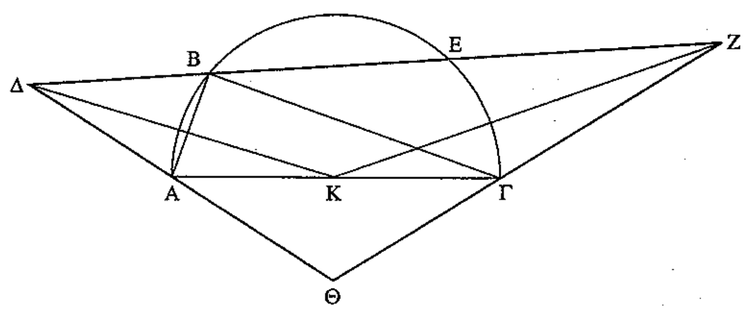
[36] *Elements* III.31.

[37] *Elements* VI.1. That EΔ=BZ is a result of the construction ΔB=EZ (EB common).

[38] *Elements* III.36. [39] *Elements* III.36. [40] *Elements* VI.16.

[41] And then apply *Elements* VI.2 in addition to VI.16, to get Step 11.

in Hero's construction, as are the produced lines ΘA, ΘΓ and the ruler moved at B. They differ in this only: that there,[42] we moved the ruler around B, until the point was reached that the <lines drawn> from the bisection of AΓ, that is from K, on the <lines> ΘΔ, ΘZ, were cut off by it <=K> <as> equal, namely KΔ, KZ; while here, <we moved the ruler> until ΔB became equal to EZ. But in each construction the same follows. But the one mentioned here[43] is better adapted for practical use; for it is possible to observe the equality of ΔB, EZ by dividing the ruler ΔZ continuously into equal parts – and this much more easily than examining with the aid of a compass that the <lines drawn> from K to Δ, Z are equal.[44]



Catalogue: Philo
Codex H has ΔZ parallel to AΓ.
Codex D has Σ instead of E.

As Apollonius[45]

Let the two given lines, whose two mean proportionals it is required to find, be AB, AΓ (a) containing a right angle, that at A, (b) and with center B and radius AΓ let a circumference of a circle be drawn,

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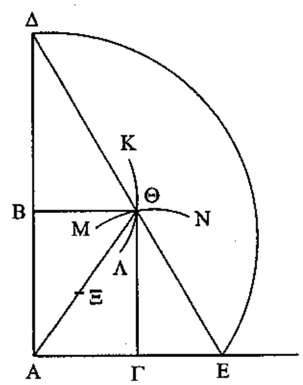
[42] I.e. Hero's solution. [43] I.e. Philo's solution.

[44] The idea is this: we normally have an unmarked ruler, but we can mark it by continuous bisection, in principle a geometrically precise operation. The further we go down in the units by which we scale the ruler, the more precise the observation of equality. Since precise units are produced by continuous bisections from a given original length, there is a great advantage to having the two compared segments measured by units that both derive from the same original length. Hence the superiority of Philo's method, where the two segments lie on a single line, i.e. on a single ruler, or on a single scale of bisections. In other words, absolute units of length measurement were considered less precise than the relative units of measurement produced, geometrically, by continuous bisection.

[45] Apollonius is mainly known as the author of the *Conics* (originally an eight-book work, its first four books survive in Greek while its next three survive in Arabic, as do several other, relatively minor works.) The ancients thought, and the *Conics* confirm, that, as mathematician, he was second to Archimedes alone: not that you would guess it from the testimony included here.

<namely> KΘΛ, (c) and again with center Γ and radius AB let a circumference of a circle be drawn, <namely> MΘN, (d) and let it <=MΘN> cut KΘΛ at Θ, (e) and let ΘA, ΘB, ΘΓ be joined; (1) therefore BΓ is a parallelogram and ΘA is its diameter.[46] (f) Let ΘA be bisected at Ξ, (g) and with center Ξ let a circle be drawn cutting the <lines> AB, AΓ, after they are produced, (h) at Δ, E – (i) further, so that Δ, E will be along a line with Θ – (2) which will come to be if a small ruler is moved around Θ, cutting AΔ, AE and carried until <it reaches> such <a position> where the <lines drawn> from Ξ to Δ, E are made equal.

For, once this comes to be, there shall be the desideratum; for it is the same construction as that written by Hero and Philo, and it is clear that the same proof shall apply, as well.



Catalogue: Apollonius Codices BD have a quadrant for an arc. This is badly executed in codex D, where the arc falls short of E, falling instead on the line ΔE itself. Codex D has BΘ greater than ΓΘ. Codex G has ΔB equal to BA. Codex B has removed the continuation of line AE, and has added lines ΞΔ, ΞE. Codex A had Ψ instead of E (corrected in codex B). Codex D omits Ξ.

As Diocles in *On Burning Mirrors*[47]

In a circle, let two diameters be drawn at right <angles>, <namely> AB, ΓΔ, and let two equal circumferences be taken off on each <side> of B, <namely> EB, BZ, and through Z let ZH be drawn parallel to AB, and let ΔE be joined. I say that ZH, HΔ are two mean proportionals between ΓH, HΘ.

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[46] By joining the lines ΘA, BΓ we can prove the congruity, first, of ΘBΓ, BΓA (*Elements* I.8), so the angle at Θ is right as well as that at A; and by another application of *Elements* I.8, we get the congruity of ΘAΓ, ΘAB, hence the angle at B = the angle at Γ, and ΘBAΓ must be a parallelogram.

[47] A work surviving in Arabic (published as Toomer [1976]) – Diocles' only work to survive. Probably active in the generation following Apollonius, Diocles belongs to a galaxy of brilliant mathematicians whose achievements are known to us only through a complex pattern of reflections.

(a) For let EK be drawn through E parallel to AB; (1) therefore EK is equal to ZH, (2) while KΓ <is equal> to HΔ. (b) For this will be clear once lines are joined from Λ to E, Z; (3) for the <angles contained> by ΓΛE, ZΛΔ will then be equal,[48] (4) and the <angles> at K, H are right; (5) therefore also all <=sides and angles> are equal to all,[49] (6) through ΛE's being equal to ΛZ;[50] (7) therefore the remaining ΓK, too, is equal to the <remaining> HΔ. (8) Now since it is: as ΔK to KE, ΔH to HΘ,[51] (9) but as ΔK to KE, EK to KΓ; (10) for EK is a mean proportional between ΔK, KΓ;[52] (11) therefore as ΔK to KE, and EK to KΓ, so ΔH to HΘ. (12) And ΔK is equal to ΓH, (13) while KE <is equal> to ZH, (14) and KΓ < is equal> to HΔ. (15) Therefore as ΓH to HZ, so ZH to HΔ and ΔH to HΘ. (c) So if equal circumferences – MB, BN – are taken on each side of B, (d) and, through N, NΞ is drawn parallel to AB, and ΔM is joined, (16) NΞ, ΞΔ, again, will be mean proportionals between ΓΞ, ΞO.

Now, producing in this way many continuous parallels between B, Δ; and, at the side of Γ, setting <circumferences> equal to those taken, by these <parallels>, from B; and joining lines from Δ to the resulting points (similarly to ΔE, ΔM) – then the parallels between B, Δ will be cut at certain points (in the diagram before us, at the <points> O, Θ), to which we join lines (by the application of a ruler <from one point to its neighbor>) – and then we shall have a certain line figured in the circle, on which: if a chance point is taken, and, through it, a parallel to AB is drawn, then: the drawn <parallel>, and the <line> taken by it <=the parallel> from the diameter (in the direction of Δ), will be mean proportionals between: the <line> taken by it <=the parallel> from the diameter (in the direction of the point Γ); and its <=the parallel's> part from the point in the line <=the line produced by the ruler> to the diameter ΓΔ.[53]

Having made these preliminary constructions, let the two given lines (whose mean proportionals it is required to find) be A, B, and let there be a circle, in which <let there be> two diameters in right <angles> to each other, ΓΔ, EZ, and let the line <produced> through the continuous points be drawn, as has been said, <namely> ΔΘZ, and

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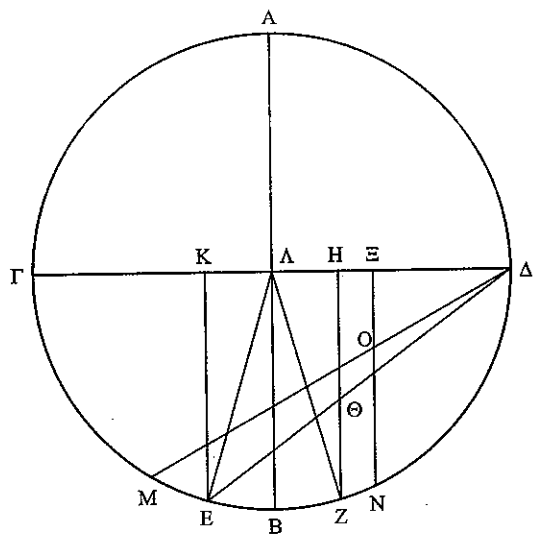
[48] *Elements* III.27. [49] Referring to the triangles ΛKE, ΛHZ. *Elements* I.26.
[50] Two radii. [51] *Elements* VI.2. [52] *Elements* VI.8 Cor.
[53] This formulation is at least as opaque in the Greek as it is in my translation, and it is readable only by translating its terms to diagrammatic realities. This translation is effected in the ensuing proof. What must be understood at this stage is that we have repeated, virtually, the operation of the preceding argument a certain number of times (perhaps, infinitely many times), producing a line connecting many (or all) of the points of the type O, Θ.



Catalogue: Diocles
Codex A had the letter O on the intersection of MΔ/HZ (corrected in codices BDG).
Codex E has H (?) instead of N.

let it come to be: as A to B, ΓH to HK,[54] and joining ΓK and producing it, let it cut the line at Θ, and let ΛM be drawn through Θ parallel to EZ; (1) therefore, through what has been proved above, MΛ, ΛΔ are mean proportionals between ΓΛ, ΛΘ. (2) And since it is: as ΓΛ to ΛΘ, so ΓH to HK,[55] (3) and as ΓH to HK, so A to B (a) if we insert means between A, B in the same ratio as ΓΛ, ΛM, ΛΔ, ΛΘ, e.g. N, Ξ,[56] (4) we shall have taken N, Ξ, mean proportionals between A, B, which it was required to find.

See diagram on following page.

As Pappus in the *Introduction to Mechanics*[57]

Pappus put forth as his goal "to find a cube having a given ratio to a given cube," and while his arguments, too, proceeded towards such a goal, it is still clear that, finding this, the problem before us will be

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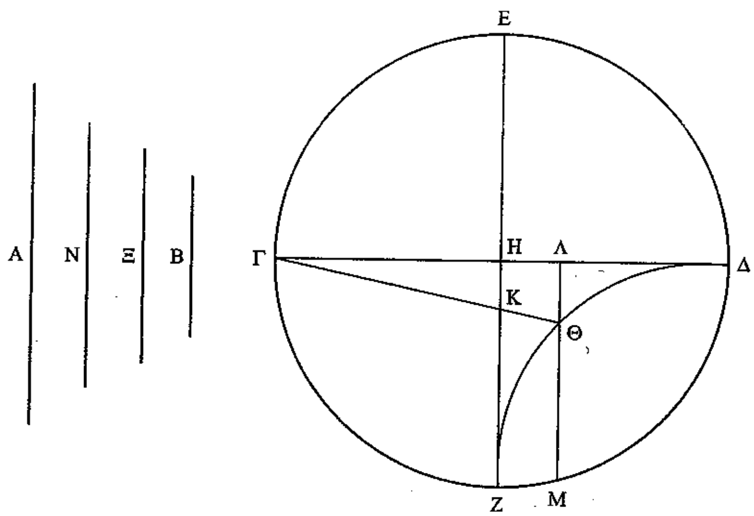
54 This defines the point K. *Elements* VI.12.

55 *Elements* VI.2.

56 This is not a *petitio principii*. Through *Elements* VI.12, it is possible to find the analogues of the series ΓΛ, ΛM, ΛΔ, ΛΘ, starting from the terms A, B – in fact, this is a mere change of scale.

57 A reference to what we know as "Book 8" of Pappus' "Mathematical Collection," much of which is extant – his only surviving work in Greek (more may survive in Arabic, if we believe in the attribution of a commentary to Euclid's *Elements* Book X).



Catalogue: Diocles (second diagram) Codices GH had Ξ instead of Z (corrected later in Codex G).

found as well; for, given two lines, if the second of the two required means is found, the third will thereby be given as well.[58]

For let a semicircle ABΓ be drawn (as he says himself, word by word),[59] and, from the center Δ, let ΔB be drawn at right <angles>,[60] and let a small ruler be moved around the point A, so that while one of its ends will be set to revolve around some small peg standing at the point A, the other end will be moved, between B and Γ, as around the small peg.

Having constructed these, let it be demanded to find two cubes having to each other a demanded ratio.

Let the <ratio> of BΔ to ΔE be made the same as this <demanded> ratio, and joining ΓE let it be produced to Z. So let the ruler be moved along, between B and Γ, until the <moment> when its part taken

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58 The idea is the following. To double a cube, it is necessary to find *one* of the middle terms in a four-terms geometrical progression. That is, if the side of the original cube is 1, and you want a cube with volume 2, the side of the new cube should be an A satisfying 1:A::A:X::X:2. So in a sense you do not need X, all you need is A – as far as doubling the cube is concerned. However, as Eutocius points out, A being given, X is already there: all we need to do is (for instance) to find the mean proportional of A and 2 (a simple Euclidean problem: *Elements* VI.13). Confusingly, Eutocius' ordinals here refer to the sequence of *four* terms in geometrical proportion. Hence the two mean proportionals are not "first and second" but "second and third."

59 The meaning of this is that Eutocius has before him the original Pappic text (and not some second-hand report). That the quotation is not word-for-word is clear (and is to be expected given ancient practices), though the discrepancies are indeed minor.

60 To the base of the semicircle, the line AΓ: the only straight line so far, hence a clear reference.

off between the lines ZE, EB becomes equal to the <part taken off> between the line BE and the circumference BKΓ (for this we will do easily by trial and error as we move the ruler). So let it have come to be and let it have a position <as> AK, so that HΘ, ΘK are equal. I say that the cube on BΔ has to the cube on ΔΘ the demanded ratio, that is the <ratio> of BΔ to ΔE.

(a) For let the circle be imagined completed, (b) and joining KΔ let it be produced to Λ, (c) and let ΛH be joined. (1) Therefore it <=ΛH> is parallel to BΔ (2) through KΘ's being equal to HΘ, (3) and KΔ's being equal to ΔΛ.[61] (d) So let both AΛ and ΛΓ be joined. (4) Now since the <angle contained> by AΛΓ is right ((5) for it is in a semicircle),[62] (6) and ΛM is a perpendicular, (7) therefore it is: as the <square> on ΛM to the <square> on MA, that is ΓM to MA,[63] (8) so the <square> on AM to the <square> on MH.[64] (9) Let the ratio of AM to MH be added as common; (10) therefore the ratio composed of both the <ratio> of ΓM to MA and the <ratio> of AM to MH, that is the ratio of ΓM to MH, (11) is the same as the <ratio> composed of both the ratio of the <square> on AM to the <square> on MH and the <ratio> of AM to MH. (12) But the ratio composed of both the <ratio> of the <square> on AM to the <square> on MH and the <ratio> of AM to MH is the same as the ratio which the cube on AM has to the <cube> on MH; (13) therefore the ratio of ΓM to MH, too, is the same as the ratio which the cube on AM has to the <cube> on MH. (14) But as ΓM to MH, so ΓΔ to ΔE,[65] (15) while as AM to MH, so AΔ to ΔΘ;[66] (16) therefore also: as BΔ to ΔE, that is the given ratio, (17) so the cube on BΔ to the cube on ΔΘ. (18) Therefore ΔΘ is the second of the two mean proportionals which it was required to find between BΔ, ΔE.

And if we make, as BΔ to ΔΘ, ΘΔ to some other <line>, the third will also be found.

And it must be realized that this sort of construction, too,[67] is the same as that discussed by Diocles, differing only in this: the other one <=Diocles'> draws a certain line through continuous points between the <points> A, B; on which <line> H was taken (ΓE being produced to cut the said line);[68] while here H is found by the

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[61] Radii in a circle. 1 derives from 2 and 3 through *Elements* VI.2.

[62] *Elements* III.31. [63] *Elements* VI.8 Cor.

[64] The angle at A is right, for the same reason that the angle at Λ is: both subtend a diameter (*Elements* III.31). Hence through *Elements* VI.8, 4, the claim follows.

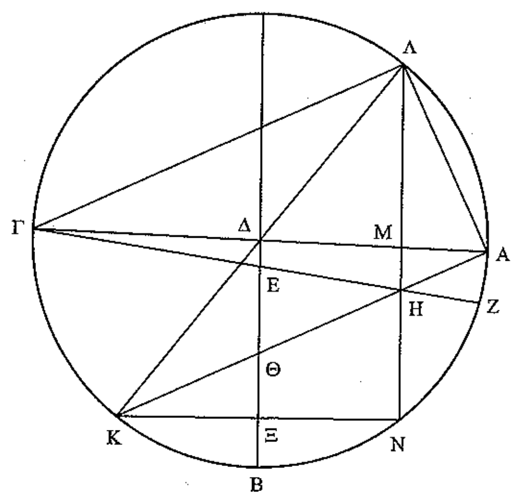
[65] *Elements* VI.2. [66] *Elements* VI.2.

[67] "Too:" i.e., the relation we see between Pappus and Diocles is the same as we saw above for the relation between Hero, Philo, and Apollonius.

[68] In an interesting move, Eutocius translates Diocles' argument to Pappus' diagram.

ruler AK's being moved around A. For we may learn as follows, that H is the same, whether it is taken by the ruler (as in here) or whether <it is taken> as Diocles said: (a) producing MH towards N (b) let KN be joined. (1) Now since KΘ is equal to ΘH, (2) and HN is parallel to ΘB, (3) KΞ is also equal to ΞN.[69] (3) And ΞB is common and at right <angles>; (4) for KN is bisected, and at right <angles>, by the <line drawn> through the center;[70] (5) therefore base is equal to base, too,[71] (6) and through this the circumference KB <is equal> to the <circumference> BN, too.[72] Therefore H is on Diocles' line.

And the proof, too, is the same. For Diocles has said that (1) it is: as ΓM to MN, so MN to MA and AM to MH.[73] (2) And NM is equal to MΛ; (3) for the diameter cuts it at right <angles>;[74] (4) therefore it is: as ΓM to MΛ, so ΛM to MA and AM to MH. (5) Therefore ΛM, MA are mean proportionals between ΓM, MH. (6) But as ΓM to MH, ΓΔ to ΔE,[75] (7) while as ΓM to MΛ, AM to MH[76] (8) that is ΓΔ to ΔΘ;[77] therefore ΔΘ is also the second of the means between ΓΔ, ΔE, that which Pappus found as well.



Catalogue: Pappus
Codex D has omitted Z.

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[69] *Elements* VI.2. [70] A close formulaic reference to *Elements* III.3.

[71] *Elements* I.4. [72] *Elements* III.28.

[73] A translation of Diocles' first proof, Step 11, into Pappus' diagram.

[74] *Elements* III.3. [75] Pappus' Step 14.

[76] Two successive applications of *Elements* III.31, VI.8 Cor., parallel in this respect to Steps 7, 8 in Pappus' proof.

[77] A variation on Step 15 of Pappus' proof.

As Sporus[78]

Let the two given unequal lines be AB, BΓ; so it is required to find two mean proportionals in a continuous proportion[79] between AB, BΓ.

(a) Let ΔBE be drawn from B at right angles to AB, (b) and with a center B, and a radius BA, let a semicircle be drawn, <namely> ΔAE, (c) and let a line joined from E to Γ be drawn through to Z, (d) and let a certain line be drawn through from Δ in such a way, that HΘ will be equal to ΘK; (1) for this is possible;[80] (e) and let perpendiculars be drawn from H, K on ΔE, <namely> HΛ, KNM. (2) Now since it is: as KΘ to ΘH, MB to BΛ,[81] (3) and KΘ is equal to ΘH, (4) therefore MB, too, is equal to BΛ; (5) so that the remainder, too, ME,[82] <is equal> to ΛΔ. (6) Therefore the whole ΔM, too, is equal to ΛE, (7) and through this it is: as MΔ to ΔΛ, ΛE to EM.[83] (8) But as MΔ to ΔΛ, KM to HΛ,[84] (9) while as ΛE to EM, HΛ to NM.[85] (10) Again, since it is: as ΔM to MK, KM to ME,[86] (11) therefore as ΔM to ME, so the <square> on ΔM to the <square> on MK,[87] (12) that is the <square> on ΔB to the <square> on BΘ,[88] (13) that is the <square> on AB to the <square> on BΘ; (14) for ΔB is equal to BA.[89] (15) Again, since it is: as MΔ to ΔB, ΛE to EB,[90] (16) but as MΔ to ΔB, KM to ΘB,[91] (17) while as ΛE to EB, HΛ to ΓB,[92] (18) therefore also: as KM to ΘB, HΛ to ΓB; (19) and alternately, as KM to HΛ, ΘB to ΓB.[93] (20)

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78 Apparently Sporus wrote a book called *Honeycombs* (or *Aristotelean honeycombs*?), probably in late antiquity (third century AD?). Our knowledge is a surmise based on indirect evidence from Pappus, in other words our knowledge is minimal. Perhaps he is to be envisaged as a collector of remains from ancient times, mathematical, philosophical and others? In this case, the lack of originality in his solution should not come as a surprise. As usual, consult Knorr (1989) 87–93.

79 "Mean proportionals in a continuous proportion" is an expanded way of saying "mean proportionals."

80 Cf. Eutocius' comment on Philo's solution. Perhaps this sentence, too, is a Eutocian comment, a brief intrusion into the Sporian text.

81 AB is perpendicular on ΔE, just as KM and HΛ are. Hence through *Elements* I.28, VI.2 the set of proportions ΔK:ΔΘ:ΔH::ΔM:ΔB:ΔΛ can be derived, from which, through *Elements* V.17, it is possible to derive, *inter alia*, KΘ:ΘH::MB:BΛ.

82 "Remainder" after MB is taken away from the radius EB.

83 The reasoning is similar to *Elements* V.7.

84 *Elements* VI.2, 4.

85 *Elements* VI.2, 4. Steps 7–9 seem to lead to the conclusion KM:HΛ::HΛ:MN. This conclusion however is not asserted, and is not required in the proof. Steps 7–9 are thus a false start. Is this text an uncorrected draft? Mistaken? Corrupt?

86 *Elements* III.31, VI.8.

87 *Elements* VI. 20 Cor. 2.

88 *Elements* VI.2, 4.

89 Radii in circle.

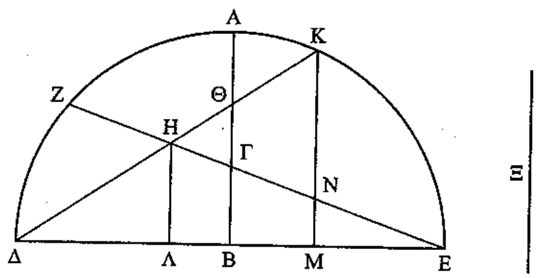
90 The same kind of reasoning as in Step 7.

91 *Elements* VI.2, 4.

92 *Elements* VI.2, 4.

93 *Elements* V.16.

But as KM to HΛ, MΔ to ΔΛ,[94] (21) that is ΔM to ME,[95] (22) that is the <square> on AB to the <square> on ΘB;[96] (23) therefore also: as the <square> on AB to the <square> on ΘB, BΘ to BΓ. (f) Let a mean proportional be taken between ΘB, BΓ, <namely> Ξ. (24) Now since, as the <square> on AB to the <square> on BΘ, ΘB to BΓ, (25) but the <square> on AB has to the <square> on BΘ a ratio duplicate of AB to BΘ, (26) while ΘB has to BΓ a ratio duplicate of ΘB to Ξ,[97] (27) therefore also: as AB to BΘ, BΘ to Ξ. (28) But as ΘB to Ξ, Ξ to BΓ; (29) therefore also: as AB to BΘ, ΘB to Ξ and Ξ to BΓ.



Catalogue: Sporus
Codex D has omitted Θ. Codex E has H instead of N.

And it is obvious that this, too, is the same as that proved by both Pappus and Diocles.

As Menaechmus[98]

Let the two given lines be A, E; so it is required to find two mean proportionals between A, E.

(a) Let it come to be,[99] and let <the mean proportionals> be B, Γ, (b) and let a line be set out, <given> in position, <namely> ΔH

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[94] *Elements* VI.2, 4. [95] Step 5.

[96] Step 13. [97] *Elements* V. Def. 10.

[98] A fourth-century BC mathematician, apparently involved, *inter alia*, with the origins of Conics. Perhaps a student of Eudoxus and an acquaintance of Plato, he seems to have had wide, philosophical interests. Whether he has produced both this solution and its alternative is, however, another question: Toomer has argued in (1976) 169–70 that the alternative solution was, in fact, by Diocles, as a closely related proof is preserved in the Arabic translation of Diocles' *On Burning Mirrors*.

[99] That is, assume the problem solved. The following passage is in an analysis/synthesis structure.

(limited at Δ),[100] (c) and, at Δ, let ΔZ be set equal to Γ, (d) and let ZΘ be drawn at right <angles =to ΔH>, (e) and let ZΘ be set equal to B.[101] (1) Now since three lines <are> proportional, A, B, Γ, (2) the <rectangle contained> by A, Γ is equal to the <square> on B;[102] (3) therefore the <rectangle contained> by a given <line> A, and by Γ, that is ΔZ, (4) is equal to the <square> on B, (5) that is to the <square> on ZΘ. (6) Therefore Θ is on a parabola drawn through Δ.[103] (f) Let parallels ΘK, ΔK be drawn as parallels.[104] (7) And since the <rectangle contained> by B, Γ is given; (8) for it is equal to the <rectangle contained> by A, E;[105] (9) therefore the <rectangle contained> by KΘZ is given as well. (10) Therefore Θ is on a hyperbola in KΔ, ΔZ as asymptotes.[106] (11) Therefore Θ is given;[107] (12) so that Z <is given>, too.

So it will be constructed like this. Let the two given lines be A, E, and <let> ΔH <be given> in position, limited at Δ, (a) and, through Δ, let a parabola be drawn, whose axis is ΔH, while the *latus rectum*[108] of the figure is A, and let the lines drawn down <from the parabola> in a right angle on ΔH, be in square the rectangular areas applied along A,[109] having as breadths the <lines> taken by them <from the line ΔH>

---

[100] The line ΔH is not so much a magnitude, as a position: it is the line on which Z is situated, ΔK is erected, etc. Hence the strange description, "Given in position, limited at Δ."

[101] ZΘ begins as a position at (d) and becomes a magnitude at (e).

[102] *Elements* VI.17. From this point onwards, A and Δ are consistently inverted in the manuscript's text. It would seem that in Eutocius' original the two given lines were Δ, E and the vertex of the parabola was A. Eutocius inverted A and Δ, in his diagram and at the beginning of his text, but here he forgot about this and just went on copying from his original: let him who has never switched labels in his diagrams cast the first stone. I follow Heiberg's homogenization, keeping Eutocius' inversion (Torelli, following Moerbeke, chose the other way around).

[103] *Conics* I.11. To paraphrase algebraically, Menaechmus notes that there is a constant A satisfying $A*\Delta Z=Z\Theta^2$, so that Θ is on a parabola whose vertex is Δ, its *latus rectum* (see below) A.

[104] ΘK, ΔK are parallel not to each other, but to the already drawn lines, ZΔ, ZΘ.

[105] A, E are the original, *given* lines, so the rectangle they contain is given as well.

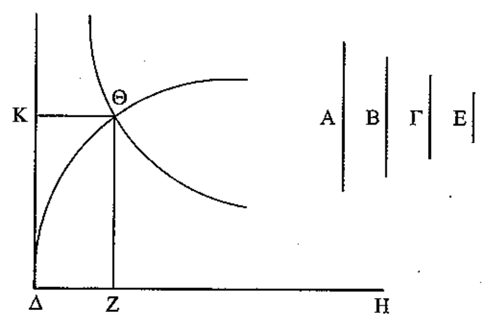
[106] *Conics* II.12. Menaechmus notes that the point Θ determines a constant rectangle intercepted by the two lines KΔ, ΔH, which is a property of the hyperbola.

[107] Θ is now given as the intersection of a given parabola and a given hyperbola.

[108] *Latus rectum*: a technical term. In every parabola, there is a line X such that, for every point on the axis of the parabola (e.g. Z in our case) the rectangle contained by the line from the point to the vertex (e.g. ΔZ in our case) and by the line X, is always equal to the square on what is known as the "ordinate" on that point (e.g. ZΘ in our case). This line X is known as the *Latus rectum*.

[109] That is, rectangles whose one side is A . . .

towards the point Δ.[110] (b) Let it be drawn and let it be the <parabola> ΔΘ, (c) and <let> ΔK <be> right,[111] (d) and let a hyperbola be drawn in KΔ, ΔZ <as> asymptotes, (1) on which <hyperbola>, the <lines> drawn parallel to KΔ, ΔZ make the rectangular area <contained by them> equal to the <rectangle contained> by A, E;[112] (2) so it <=the hyperbola> will cut the parabola. (e) Let it cut <it> at Θ, (f) and let KΘ, ΘZ be drawn as perpendiculars. (3) Now since the <square> on ZΘ is equal to the <rectangle contained> by A, ΔZ,[113] (4) it is: as A to ZΘ, ΘZ to ZΔ.[114] (5) Again, since the <rectangle contained> by A, E is equal to the <rectangle contained> by ΘZΔ, (6) it is: as A to ZΘ, ZΔ to E.[115] (7) But as A to ZΘ, ZΘ to ZΔ; (8) and therefore as A to ZΘ, ZΘ to ZΔ and ZΔ to E. (g) Let B be set equal to ΘZ, (h) while Γ <be> set equal to ΔZ; (9) therefore it is: as A to B, B to Γ and Γ to E. (10) Therefore A, B, Γ, E are continuously proportional; which it was required to find.



Catalogue:
Menaechmus
Codex H has omitted
H, has Λ for A.

In another way

Let the two given lines be (at right <angles> to each other) AB, BΓ, (a) and let their means come to be <as> ΔB, BE, so that it is: as ΓB to BΔ, so BΔ to BE and BE to BA, (b) and let ΔZ, EZ be drawn at right <angles>.[116] (1) Now since it is: as ΓB to BΔ, ΔB to BE, (2) therefore the <rectangle contained> by ΓBE, that is the <rectangle contained>

[110] . . . and whose other side are lines such as ΔZ. This lengthy description unpacks the property of the *latus rectum* – the property of the parabola. The construction of a parabola, given its *latus rectum*, is provided at *Conics* I.52.

[111] That is, ΔK is at right angles to the axis of the parabola.

[112] *Conics* II.12. [113] *Conics* I.11.

[114] *Elements* VI.17. [115] *Elements* VI.16.

[116] "At right angles:" Both to each other and to the original lines EB, BΔ.

by a given <line> and by BE (3) is equal to the <square> on BΔ,[117] (4) that is <to the square on> EZ.[118] (5) Now since the <rectangle contained> by a given <line> and by BE is equal to the <square> on EZ, (6) therefore Z touches[119] a parabola, <namely that> around the axis BE.[120] (7) Again, since it is: as AB to BE, BE to BΔ, (8) therefore the <rectangle contained> by ABΔ, that is the <rectangle contained> by a given <line> and by BΔ, (9) is equal to the <square> on EB,[121] (10) that is <to the square on> ΔZ;[122] (11) therefore Z touches a parabola, <namely that> around the axis BΔ; (12) but it has touched another given <parabola, namely that> around BE; (13) therefore Z is given. (14) And ZΔ, ZE are perpendiculars; (15) therefore Δ, E are given.

And it will be constructed like this. Let the two given lines be (at right <angles> to each other) AB, BΓ, (a) and let them be produced, from B, without limit, (b) and let a parabola be drawn around the axis BE, so that the lines drawn down <from the parabola> on BE are in square the <rectangles applied> along BΓ.[123] (c) Again let a parabola be drawn around the axis ΔB, so that the lines drawn down <from the parabola on the axis> are in square the <rectangles applied> along AB; (1) so the parabolas will cut each other. (d) Let them cut <each other> at Z, (e) and let ZΔ, ZE be drawn from Z as perpendiculars. (2) Now since ZE, that is ΔB[124] (3) has been drawn down in a parabola, (4) therefore the <rectangle contained> by ΓBE is equal to the <square> on BΔ;[125] (5) therefore it is: as ΓB to BΔ, ΔB to BE.[126] (6) Again, since ZΔ, that is EB[127] (7) has been drawn in a parabola, (8) therefore the <rectangle contained> by ΔBA is equal to the <square> on EB;[128] (9) therefore it is: as ΔB to BE, BE to BA.[129] (10) But as ΔB to BE, so ΓB to BΔ; (11) and therefore as ΓB to BΔ, BΔ to BE and EB to BA; which it was required to find.

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117 *Elements* VI.17. 118 *Elements* I.28, 33.

119 "Touches:" a somewhat strange verb to use for a *point*. The claim is that Z is on the parabola.

120 *Conics* I.11. Around a given axis there can be an infinite number of parabolas. Our parabola, however, is uniquely given, since Step 5 effectively defines its *latus rectum*. The same situation is found in Step 11 below.

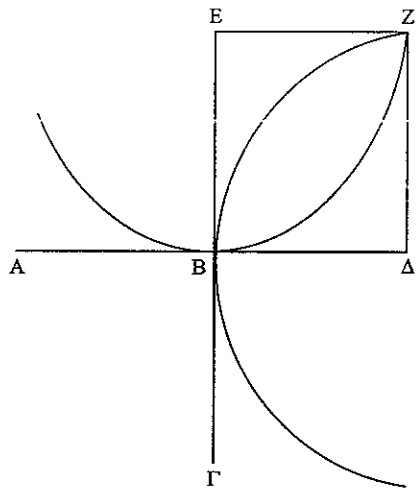
121 *Elements* VI.17. 122 *Elements* I.28, 33.

123 To spell this out: for whatever point Z, EZ^2 shall be equal to EB*BΓ. The text lapses here into the peculiar dense formulae of the theory of conic sections. The construction itself is provided at *Conics* I.52.

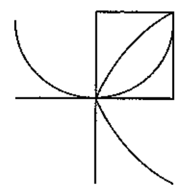
124 *Elements* I.28, 33. 125 *Conics* I.11.

126 *Elements* VI.17. 127 *Elements* I.28, 33.

128 *Conics* I.11. 129 *Elements* VI.17.



Catalogue:
Menaechmus, second diagram
Codex G has BΔ equal to BE, while D has BΔ greater than BE. Related to this, codex D has the right-hand curved line composed of two arcs, not one, as in the thumbnail.



[And the parabola is drawn by the compass invented by our teacher, Isidore the Milesian mechanician, this being proved by him in the commentary which he produced to Hero's *On Vaulting*.][130]

Archytas' solution, according to Eudemus' *History*[131]

Let the two given lines be AΔ, Γ; so it is required to find two mean proportionals between AΔ, Γ.

(a) Let a circle, <namely> ABΔZ, be drawn around the greater <line>, AΔ, (b) and let AB, equal to Γ, be fitted inside, (c) and, produced, let it meet, at Π, the tangent to the circle <drawn> from Δ, (d) and let BEZ be drawn parallel to ΠΔO, (e) and let a right semicylinder be imagined on the semicircle ABΔ, (f) and <let> a right semicircle <be imagined> on the <line> AΔ, positioned in the parallelogram of the semicylinder;[132] (1) so, when this semicircle is rotated, as from Δ to

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[130] Probably an intrusion by the same person as the scholiast writing at the end of the commentaries to Book I and II (see notes there) about whose identity little is known; certainly he belongs to the same general period as that of Eutocius himself. Neither *On Vaulting* nor its commentary are extant.

[131] Archytas was a central figure in early fourth-century BC intellectual life, clearly, among other things, a mathematician. Eudemus, Aristotle's pupil, wrote, near the end of the same century, a history of mathematics. Eutocius writes about a thousand years later, and it is an open question: what did he have as direct evidence for the works of Archytas, or of Eudemus?

[132] "Right" semicircle – i.e., in right angles to the plane of the original circle (the plane of the page, as it were). "In the parallelogram of the semicylinder" – a semicylinder consists of a half of a cylinder, together with a parallelogram where the original cylinder

B (the limit A of the diameter remaining fixed), it will cut the cylindrical surface in its rotation, and will draw in it a certain line.[133] (2) And again, if the triangle AΠΔ is moved in a circular motion (AΔ remaining fixed), opposite <in direction> to that of the semicircle, it will produce, by the line AΠ, a conical surface; which <line>, rotated, will meet the cylindrical line at a certain point;[134] (3) and at the same time B, too,

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was cut. If a cylinder is a circle, extended into space, then a semicylinder is a semicircle – bound by a semi-circumference and a diameter – extended into space, and "the parallelogram of the semicylinder" is the diameter, extended into space. Effectively, what we are asked to do is to take the semicircle ABΔ, and lift it up in space (keeping the line AΔ in place), until it has rotated 90 degrees. More of such spatial thinking is to come.

133 Here verbal and two-dimensional representations almost break down. I will make an effort: keep in mind the semicircle we have just lifted at right angles into space – the semicircle on top of the diameter AΔ. We now detach it from the diameter, keeping however the point A fixed. We rotate it, gliding along the surface of the diagram, keeping its upright position. Learn to do this; glide it in your imagination; imagine it skating along the ice-rink of the original circle (the ice-rink of the diagram, of the page), always keeping one foot firmly on the point A. I shall soon return to this choreography. Now, when your mind is used to this operation, evoke another imaginary object, the semicylinder on top of the original semicircle ABΔ. So we have two objects: the rotating semicircle, and the semicylinder. As the semicircle glides along, it is possible to identify the point where it cuts the semicylinder. For instance, at its Start position, it cuts the semicylinder at Δ. And at the other position depicted in our diagram, it cuts the semicylinder at K. Notice that K is higher than Δ. At its Start position, the semicircle fits snugly inside the semicylinder. As it glides further, parts of it begin to project out of the semicylinder – the second foot is no longer inside the semicylinder – indeed the semicircle will completely emerge out of the semicylinder after a quarter rotation. So look at it, at that other position, AKΔ (another Δ now; this point is allowed – almost uniquely in Greek mathematics – to keep its name while in movement): now the semicircle projects out of the semicylinder, and the point where it cuts the semicylinder is not right at the bottom of the semicylinder (as with the original position of Δ), but a bit higher, K. Move the semicircle further, and the intersection is again a bit higher. So we can imagine the line composed of such points of intersection – and this is finally the line which this Step 1 calls into existence.

134 Here the three-dimensional construction is slightly redundant. We do not require the *cone* as such, but we merely use it as scaffolding for the line AΠ. This line is to be rotated around the diameter AΔ, keeping its head at A and keeping its distance from the diameter AΔ constant. Think of it now as three-dimensional, gravity-free ballet. We have two dancers, a ballerina and a male dancer. The ballerina is the curved line, the arc of the moving semicircle ("Tatiana"). I have discussed Tatiana in the preceding note. We now also have a male dancer, the straight line AΠ ("Eugene"). Both glide effortlessly in space: Tatiana with her two feet on the ground, one foot firmly kept at A, the other rotating; Eugene, even more acrobatically, holds Tatiana at her firm foot A and rotates in space, going round and round, always keeping the same distance from the line AΔ (which happens to be the base position of Tatiana's movement). Even more fantastically, our dancers can intersect with each other and with the stage-props, and go on dancing. Now Eugene, in his movement, keeps intersecting with the (static) semicylinder. At first, he intersects with the semicylinder at the point B. As he moves higher into space,

will, rotating, draw a semicircle in the surface of the cone.[135] (g) So let the moved semicircle[136] have its position at the place of the meeting of the lines,[137] as the <position> of ΔKA, (h) and <let> the contrariwise rotated triangle[138] <have> the <position> of ΔΛA, (i) and let the said point of intersection be K, (j) and, also, let the semicircle drawn by B be BMZ, (k) and let the common section of it <=semicircle BMZ> and of the circle BΔZA be the line BZ, (l) and let a perpendicular be drawn from K on the plane of the semicircle BΔA;[139] (4) so it will fall on the circumference of the circle,[140] (5) through the cylinder's being set up right.[141] (m) Let it <=the perpendicular> fall and let it be KI, (n) and let the <line> joined from I to A meet BZ at Θ, (o) and let AΛ meet the semicircle BMZ at M,[142] (p) and let KΔ, MI, MΘ be joined. (6) Now since each of the semicircles ΔKA, BMZ is right to the

his intersections with the semicylinder move higher, too, as they also move away from the point A. Thus Eugene, too, draws a line of intersections – "Eugene's line," the line drawn by the intersection of the rotating line and the original semicylinder (just as Tatiana had produced her own, "Tatiana's Line," made of her intersections with the cylindrical surface, in the preceding step). We shall soon look, finally, at intersection of those lines of intersections – a second-order intersection – between Tatiana's and Eugene's lines.

135 If we look at a point along the line AΠ, and plot its circular movement as the line AΠ keeps rotating, we will see a circle (or a semicircle if we concentrate, as Archytas does, on the part "above" the original circle, above the plane of the page). Archytas concentrated on the point B, and on the circle BMZ it traces in its movement. This will become important later on in the proof.

Objects, moved, leave a trace, a virtual object. This is the heart of this solution.

136 That is "Tatiana."

137 That is the meeting-point of "Tatiana's and Eugene's lines" – the lines drawn on the semicylinder. This is the second-order intersection between lines of intersections, mentioned in n. 134. That the point of intersection exists, and that it is unique, can be shown by the following topological intuitive argument: Eugene's line starts at B and moves continuously upwards as it moves towards Δ, reaching finally a point above Δ. Tatiana's line starts at Δ and moves continuously upwards as it moves towards B, reaching finally a point above B. This chiastic movement must have a point of intersection.

138 Instead of the line AΠ rotated, we now imagine the entire triangle AΠΔ rotated, its side AΔ remaining fixed as the cone is drawn. Its motion is "contrariwise" to Tatiana's.

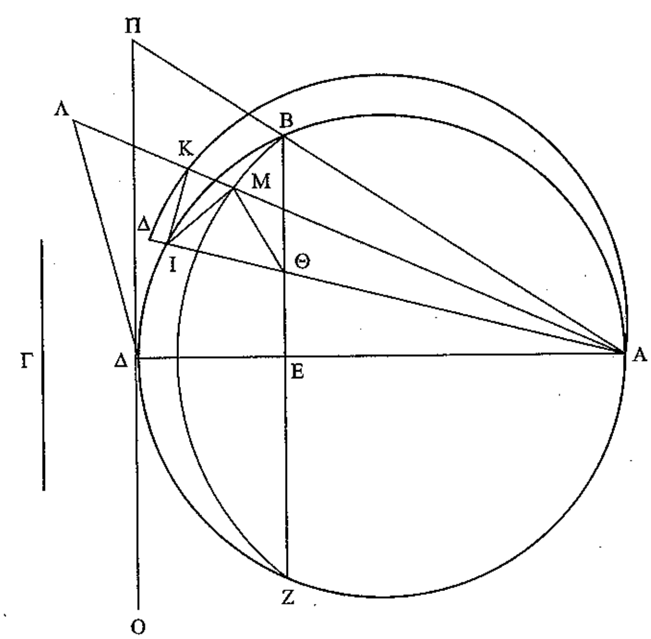
139 That is, on the original plane of the page.

140 That is the original circle ABΔZ.

141 The point K is on the surface of the (right) semicylinder, projecting upwards from the original semicircle ABΔ, and therefore the perpendicular drawn directly downwards from K is simply a line on the semicylinder, and must fall on the circumference of ABΔ.

142 ΔΛA is the position of the rotating triangle ΔΠA when it reaches the point of intersection K. We take two snapshots of the line AΠ: once, resting on the plane of the page, when it is ABΠ; again, when it is stretched in mid-air, passing through the point K. Now Π has become Λ, B has become M, while A remained fixed. The import of Step o is the identification of M as the mapping of B into the line AΛ.

underlying plane,[143] (7) therefore the common section <=of the two semicircles>, too, MΘ, is at right <angles> to the plane of the circle <=ABΔZ>;[144] (8) so that MΘ is right to BZ, as well.[145] (9) Therefore the <rectangle contained> by BΘZ, that is the <rectangle contained> by AΘI,[146] (10) is equal to the <square> on MΘ;[147] (11) therefore the triangle AMI is similar to each of the <triangles> MIΘ, MAΘ, and the <angle contained> by IMA is right;[148] (12) and the <angle contained> by ΔKA is right, too;[149] (13) therefore KΔ, MI are parallel,[150] (14) and it will be proportional: as ΔA to AK, that is KA to AI,[151] (15) so IA to AM, (16) through the similarity of the triangles.[152] (17) Therefore four <lines>, ΔA, AK, AI, AM are continuously proportional, (18) and AM is equal to Γ, (19) since <it is> also <equal> to AB; (20) therefore two mean proportionals have been found, between the given <lines> AΔ, Γ <namely> AK, AI.



Catalogue: Archytas Codex E extends the line ΠO in both directions, beyond the points Π, O. Something went wrong in codex A with the letter Γ. Codices DE have Ϛ (!) instead of Γ, while Codex B may perhaps have omitted it altogether to begin with (mystified by a very unfamiliar Greek character?). Most likely, this was a very badly executed Γ. Codex H positions K on the intersection of ΠO/ΛA.

143 "Underlying plane:" the plane of the original circle; the plane of the diagram/page.

144 *Elements* XI.19. 145 *Elements* XI. Def. 3. 146 *Elements* III.35.

147 *Elements* III.31, VI.8. It should be seen that BZ is the diameter of the circle traced by the point B in its rotating movement.

148 *Elements* VI.8.

149 *Elements* III.31. Remember, once again: ΔKA is a semicircle, and ΔA is a diameter.

150 *Elements* I.28. 151 *Elements* VI.8, 4. 152 *Elements* VI.8, 4.

As Eratosthenes[153]

Eratosthenes to king Ptolemy, greetings.

They say that one of the old tragic authors introduced Minos, building a tomb to Glaucos, and, hearing that it is to be a hundred cubits long in each direction, saying:

> *You have mentioned a small precinct of the tomb royal;*
> *Let it be double, and, not losing its beauty,*
> *Quickly double each side of the tomb.*

He seems, however, to have been mistaken; for, the sides doubled, the plane becomes four times, while the solid becomes eight times. And this was investigated by the geometers, too: in which way one could double the given solid, the solid keeping the same shape; and they called this problem "duplication of a cube:" for, assuming a cube, they investigated how to double it. And, after they were all puzzled by this for a long time, Hippocrates of Chios was the first to realize that, if it is found how to take two mean proportionals, in continuous proportion, between two straight lines (of whom the greater is double the smaller), then the cube shall be doubled, so that he converted the puzzle into another, no smaller puzzle.[154] After a while, they say, some Delians, undertaking to fulfil an oracle demanding that they double one of their altars, encountered the same difficulty, and they sent messengers to the geometers who were with Plato in the Academy, asking of them to find that which was asked. Of those who dedicated themselves to this diligently, and investigated how to take two mean proportionals between two given lines, it is said that Archytas of Tarentum solved this with the aid of semicylinders, while Eudoxus did so with the so-called curved lines;[155] as it happens, all of them wrote demonstratively, and it was

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[153] A third-century BC polymath, the librarian in the Alexandria library; we shall get to see him again in Volume 3 of this translation, as the addressee to one of Archimedes' works, the *Method*. The genuineness of the following letter has been doubted by Wilamowitz (1894). I myself follow Knorr (1989) in thinking this is by Eratosthenes. The treatise is dedicated to Ptolemy III Euergetes (reign 246–221 BC).

[154] Notice that converting X into something, not smaller than X, is the theme of the problem of duplication itself. Eratosthenes' text is shot through with this kind of intelligent play.

[155] This "it is said" is lovely. The line of myth starts with Minos and tragedy, is stressed by the repeated vague allusions ("one of the tragic authors . . ."), then is reinforced through the Delian oracle; so that now even the fully historical, relatively recent Archytas and Eudoxus may acquire the same literary–mythical aura ("it is said;" and the vague, deliberately tantalizing descriptions: "semicylinders . . . so-called [!] curved lines." Clearly, as the rest of the letter shows, Eratosthenes knew the constructions in full mathematical detail). Eratosthenes writes of mathematics, within *literary* Greek culture.

impossible practically to do this[156] by hand (except Menaechmus, by the *shortness*[157] – and this with difficulty). But we have conceived of a certain easy mechanical way of taking proportionals through which, given two lines, means – not only two, but as many as one may set forth – shall be found. This thing found, we may, generally: reduce a given solid (contained by parallelograms) into a cube, or transform one solid into another, both making it[158] similar[159] and, while enlarging it, maintaining the similitude, and this with both altars and temples;[160] and we can also reduce into a cube, both liquid and dry measures (I mean, e.g., a *metertes*[161] or a *medimnos*[162]), and we can then measure how much the vessels of these liquid or dry materials hold, using the side of the cube.[163] And the conception will be useful also for those who wish to enlarge catapults and stone-throwing machines; for it is required to augment all – the thicknesses and the magnitudes and the apertures and the *choinikids*[164] and the inserted strings – if the throwing-power is to be proportionally augmented, and this can not be done without finding the means. I have written to you the proof and the construction of the said machine.[165]

For let there be given two unequal lines, <namely> AE, ΔΘ, between which it is required to find two mean proportionals in continuous proportion, (a) and, on a certain line, <namely> EΘ, let AE be set at right <angles>, (b) and let three parallelograms, <namely> AZ, ZI, IΘ, be constructed on EΘ, (c) and, in them, let diagonals be drawn: AZ, ΛH, IΘ; (1) so they themselves will be parallel.[166] (d) So, the middle parallelogram (ZI) remaining in place, let AZ be pushed above the middle <parallelogram>, <and let> IΘ <be pushed> beneath it, as in the second figure, until A, B, Γ, Δ come to be on a <single> line,[167] (e) and let a line be drawn through the points A, B, Γ, Δ,

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[156] I.e. duplicating the cube.

[157] Another tantalizing, vague description. I therefore keep the manuscripts' reading against Heiberg's (possible) emendation, "except, to some small extent, Menaechmus."

[158] I.e. the created solid. [159] I.e. to the original solid.

[160] So we round the theme of the Minos/Delos myths.

[161] A liquid measure. [162] A dry measure.

[163] In other words, the two mean proportions will allow us, given a vessel containing X measures, to construct a vessel containing Y measures.

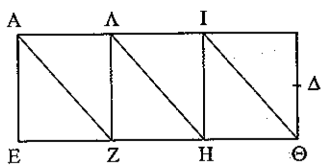
[164] Boxes containing the elastic strings of the throwing machines.

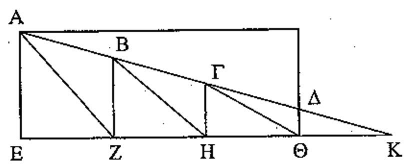
[165] Note the abruptness of the down-to-business move, here between the rhetorical introduction and the mathematical proof. We suddenly see the style of Greek mathematics vividly set against another Greek discourse.

[166] This is true only if EZ=ZH=HΘ (and then *Elements* I.28, I.4), an assumption which is nowhere stated. An oversight by Eratosthenes? A textual corruption?

[167] This is extremely confusing, especially since Eratosthenes (who assumes the reader is acquainted with a model of the machine) did not bother to explain to us that the configuration is, in a way, three dimensional. We must imagine the three parallelograms

(f) and let it meet the <line> EΘ, produced, at K; (2) so it will be: as AK to KB, EK to KZ (in the parallels AE, ZB),[168] (3) and ZK to KH (in the parallels AZ, BH).[169] (4) Therefore as AK to KB, EK to KZ and KZ to KH. (5) Again, since it is: as BK to KΓ, ZK to KH (in the parallels BZ, ΓH),[170] (6) and HK to KΘ (in the parallels BH, ΓΘ), (7) therefore as BK to KΓ, ZK to KH and HK to KΘ. (8) But as ZK to KH, EK to KZ; (9) therefore also: as EK to KZ, ZK to KH and HK to KΘ. (10) But as EK to KZ, AE to BZ, (11) and as ZK to KH, BZ to ΓH, (12) and as HK to KΘ, ΓH to ΔΘ;[171] (13) therefore also: as AE to BZ, BZ to ΓH and ΓH to ΔΘ. (14) Therefore two means have been found between AE, ΔΘ, <namely> both BZ and ΓH.[172]





Catalogue: Eratosthenes Codex D has EZ greater than ZH, both smaller than HΘ, in the left-hand rectangle; a neat trisection in the right-hand rectangle. Codex E has EZ smaller than ZH, both smaller than HΘ, in the left-hand rectangle, a neat trisection in the right-hand triangle.

So these are proved for geometrical surfaces. But so as we may also take the two means by a machine, a box is fixed (made of wood, or ivory, or bronze), holding three equal tablets, as thin as possible. Of these, the middle is fitted in its place, while the other two are moveable along grooves (the sizes and the proportions may be as anyone wishes them; for the arguments of the proof will yield the conclusion in the same way). And, for taking the lines in the most precise way, it must be done with great art, so that when the tablets are simultaneously

AZ, ΛH, IΘ as three sliding doors, each on a different groove (they are on the wall of the tatami room), AZ nearest to us, IΘ farthest from us, ΛH midway between the two. We now slide these sliding doors: AZ to the right ("covering" part of the door ΛH), IΘ to the left (partly covered, then, by the door ΛH). We look throughout at the following two points: the point where the diagonal IΘ (painted on the door IΘ) meets IH (the edge of the door ΛH); and the point where the diagonal ΛH meets ΛZ, (the edge of the door AZ). At first these are the points I, Λ, respectively. As we slide the doors, slowly, through some trial and error, we shall reach a point where the two points (now christened Γ, B) both lie on the line AΔ (itself constantly changing as we slide the doors!). Here we stop. Notice this one crucial point: by sliding the doors to the left or to the right, the painted diagonals remain parallel to each other, as do the edges of the doors. Essentially, before us is a parallelism-preserving machine.

[168] *Elements* VI.2, 4. [169] *Elements* VI. 2, 4.

[170] *Elements* VI.2, 4. [171] Steps 10–12: *Elements* VI.2, 4.

[172] To signal the end of the strictly mathematical discourse, Eratosthenes now redundantly speaks of "both" BZ and ΓH – a redundancy that throws us back into the rhetorical world of the introduction of the letter.

moved they all remain parallel[173] and firm[174] and touching each other throughout.[175]

In the dedication, the machine is made of bronze, and is fitted with lead below the crown of that pillar, and the proof below it (phrased more succinctly), and the figure, and with it the epigram.[176] So let these be written below as well, for you, so that you have, also, just as in the dedication. (Of the two figures, the second is inscribed in the pillar.)

Given two lines, to find two mean proportionals in continuous proportion. Let AE, ΔΘ be given.[177] (a) So I move the tables in the machine together, until the points A, B, Γ, Δ come to be on a <single> line. ((b) So let it be imagined, as in the second figure.) (1) Therefore it is: as AK to KB, EK to KZ (in the parallels AE, BZ),[178] (2) and ZK to KH (in the <parallels> AZ, BH);[179] (3) therefore as EK to KZ, KZ to KH. (4) But as they themselves are to each other, so are both: AE to BZ and BZ to ΓH.[180] (5) And we shall prove in the same way that, also, as ZB to ΓH, ΓH to ΔΘ; (6) therefore AE, BZ, ΓH, ΔΘ are proportional. Therefore two means have been found between the two given <lines>.

And if the given <lines> will not be equal to AE, ΔΘ, then, after we make AE, ΔΘ proportional to them, we shall take the means between them <=AE, ΔΘ>, and return to those <given lines>,[181] and we shall have the task done. And if it is demanded to find several means: we shall insert tablets in the machine, <so that their total is> always more by one than <the number of> the means to be taken; and the proof is the same.

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173 That is, no tilting to the left or to the right.

174 That is, no tilting backwards or forwards.

175 That is, as one tablet slides along another, the two remain constantly in close touch, the three separate planes simulating a single plane to the greatest possible degree.

176 The sense is clear enough: the machine (almost two dimensional) is fixed, very much like a plaque, right below the crown of a certain pillar, and then, going downwards, are, inscribed: a brief proof, a diagram and an epigram. The sentence is difficult, because of the way in which it assumes three hitherto unmentioned objects as part of the universe of discourse: the dedication (which dedication?), the pillar (which pillar?), and the epigram (which epigram?). In short, the author assumes we have seen the pillar.

177 This is supposed to be a report of an inscription, though obviously not an exact one (consider e.g. Step b, referring to the right-hand, preceding diagram, clearly meaningless in the dedication). Of course my lettering and numbering ought to be ignored when the original inscription is envisaged. Anyway, one wonders how tall the pillar was, and how large the letters were (could this be why Ptolemy asked for an explanatory letter?).

178 *Elements* VI.2, 4. 179 *Elements* VI.2, 4. 180 *Elements* VI.2, 4.

181 Suppose the greater given line is twice AE; we take half the smaller given line as our ΔΘ, and then we double the obtained means.

If you plan, of a small cube, its double to fashion,
Or – dear friend – any solid to change to another
In nature: it's yours. You can measure, as well:
Be it byre, or corn-pit, or the space of a deep,
Hollow well.[182] As they run to converge, in between
The two rulers – seize the means by their boundary-ends.[183]
Do not seek the impractical works of Archytas'
Cylinders; nor the three conic-cutting Menaechmics;
And not even that shape which is curved in the lines
That Divine Eudoxus[184] constructed.
By these tablets, indeed, you may easily fashion –
With a small base to start with – even thousands of means.
O Ptolemy, happy! Father, as youthful as son:
You have given him all that is dear to the muses
And to kings.[185] In the future – O Zeus! – may *you* give him,
From your hand, this, as well: a sceptre.[186]
May it all come to pass. And may him, who looks, say:
"Eratosthenes, of Cyrene, set up this dedication."

As Nicomedes[187] in *On Conchoid Lines*

And Nicomedes, too, writes (in the book on conchoids which is written by him[188]) of the construction of a machine accomplishing the same service. From the book the man seems to have prided himself immensely, while making great fun of the solutions of Eratosthenes, as

[182] A georgic touch, this mention of rural measures. Eratosthenes does much more than put geometry in metre; he brings it inside poetic genres.

[183] The rural imagery – now transformed into the shadow of a hunting scene.

[184] Can I have a quadrisyllabic here, please (E-ou-dok-sus, stress on the 'dok' sound)? More important: the poem modulates into the invocation-of-myth theme, and so past mathematicians shade (as they did in the prose letter) into mythical heroes.

[185] Eratosthenes is possibly writing now in the capacity of a tutor to the prince.

[186] Ptolemy gave his son a good education; Zeus would give him the rule over Egypt (note, incidentally, that Eratosthenes is quite proper. There is nothing regicidal in wishing a king to be outlived by his son. But the ground is a bit shaky, hence the "as youthful as son" above).

[187] Otherwise virtually unknown: he may have lived, during the third/second century BC (as the mathematical interests and polemics suggest), in Asia Minor (as the name suggests). Eutocius' text is closely related (especially towards its end) to Pappus, Book IV 26–8 (pp. 242–50).

[188] The "writes"/"written by him" dissonance is in the original.

impractical[189] and at the same time devoid of geometrical skill. So, for the sake of not missing any of those who troubled over this problem, and for comparison with Eratosthenes, we add him too to what we have written so far.[190] He writes, in essence, the following:

(a) One must imagine: two rulers conjoined at right <angles> to each other, in such a way that a single surface keeps hold of them, as are AB, ΓΔ, (b) and, in the <ruler> AB, an axe-shaped groove, inside which a *chelonion*[191] can run freely, (c) and, in the <ruler> ΓΔ, a small cylinder – at the part next to Δ and the middle line dividing the width of the ruler – which is fitted to the ruler and protrudes slightly from the higher surface of the same ruler,[192] (d) and another ruler, as EZ, (e) which has <the following> (<beginning> at some small distance from <its> limit at Z): a cut, as HΘ, which may be mounted on the small cylinder at Δ,[193] (f) and a rounded hole next to E,[194] which will be inserted in a certain axle attached to the freely running *chelonarion*[195] in the axe-shaped groove which is in the ruler AB.[196] (1) So, the ruler EZ fitted (first in the cut HΘ, over the small cylinder next to Δ, and second in the hole E, over the axle attached to the *chelonarion*), if one, taking hold of the K end of the ruler, moves it in the direction of A, then

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189 "Impractical:" in the original, "amechanical," with a nice pun (the solution is bad as a piece of theoretical mechanics, while being impractical).

190 The author of this passage (probably Eutocius, though possibly an earlier compiler) does not approve of polemics in mathematics, in an interesting example of the change of intellectual mores from ancient to late ancient times. As Eutocius implies, this is the concluding solution in this catalogue. The overall structure is clear: Eratosthenes referred to many of the preceding solutions, and so he had to be penultimate; Nicomedes, who referred to Eratosthenes, had to be last. The unintended result is that polemics mark the end of this catalogue: the actors leave the stage bathetically, in a loud, vulgar quarrel.

191 "Chelonion:" a very polysemic noun. The basic meaning is "tortoise-shell" but the Greeks, apparently, saw tortoise-shells everywhere, in parts of the body and in various artificial objects. "Knob" is probably the best stab at what is meant here.

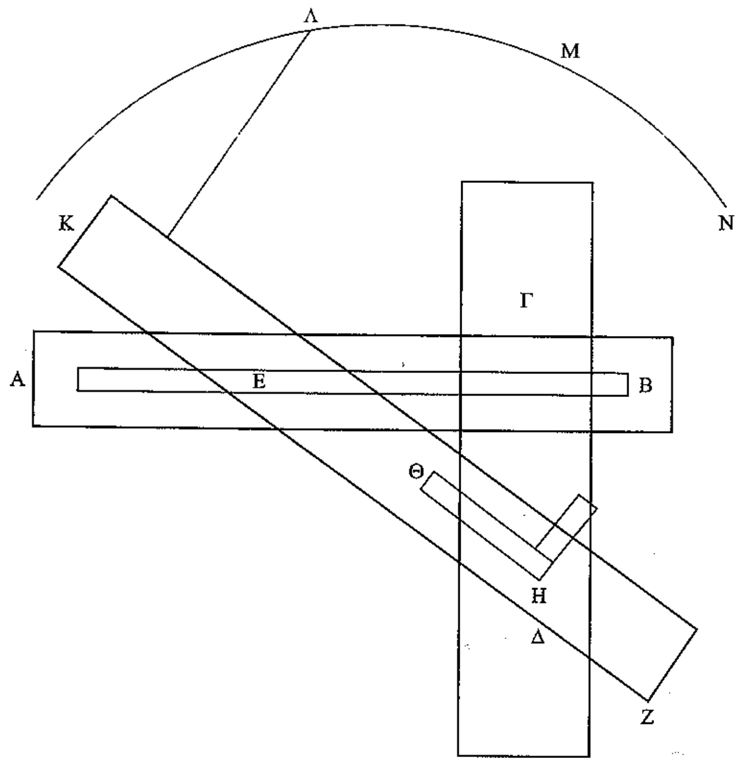
192 In other words: imagine a small cylinder – another knob – this time not freely running (as the knob in the ruler AB), but fixed at Δ.

193 We take the ruler, and cut an internal rectangle away from it, producing a rectangular hole; so that we may now fix this cut ruler, loosely, with its hole upon the cylinder at Δ.

194 Now remove the ruler, and cut it again, this time just with a rounded hole, a hole to be fixed firmly on the knob at E.

195 A variation on the word *chelonion*, clearly meaning the very same object as in Step b.

196 So now all the items of the construction come together. The characters are: three rulers AB, ΓΔ, EZ; an axe-shaped groove in the ruler AB (upon which, a *chelonion*; upon which, again, an axle); a cylinder on the ruler ΓΔ; and the third ruler, EZ, with two holes, one mounted on the axle on the ruler AB, the other mounted on the cylinder on the ruler ΓΔ. None of this can be grasped without the diagram.



Catalogue: Nicomedes Codices GH4 have the upper angle between the rectangle KZ and the line to Λ acute. Codex 4 does not close the small line near Z of the rectangle KZ. Codex A had M instead of H. Codex D has omitted N, and has Λ instead of A, Θ instead of B, and A instead of Δ. Codices EH have omitted Λ.

in that of B, the point E[197] will always be carried on the ruler AB, (2) while the cut HΘ will always be moved on the small cylinder next to Δ (the middle line of the ruler EZ imagined to pass, in its movement, through the axis of the cylinder at Δ[198]), (3) the projection of the ruler, EK,[199] remaining the same.[200] (4) So if we conceive of some writing-tool at K, fixed on the base,[201] a certain line will be drawn, such as ΛMN, which Nicomedes calls "First Conchoid Line," and <he calls>

197 We have moved from imperative to indicative, from construction to argument; to an extent, we have moved from mechanics to geometry, hence the sudden "point."

198 EZ is fitted in such a way, that its middle is exactly on the center of the cylinder at Δ. Also, it is so firmly attached so as not to sway as it moves, always keeping its middle exactly on the center of Δ. That geometrical precision is never perfectly instantiated is acknowledged by the verb "imagined."

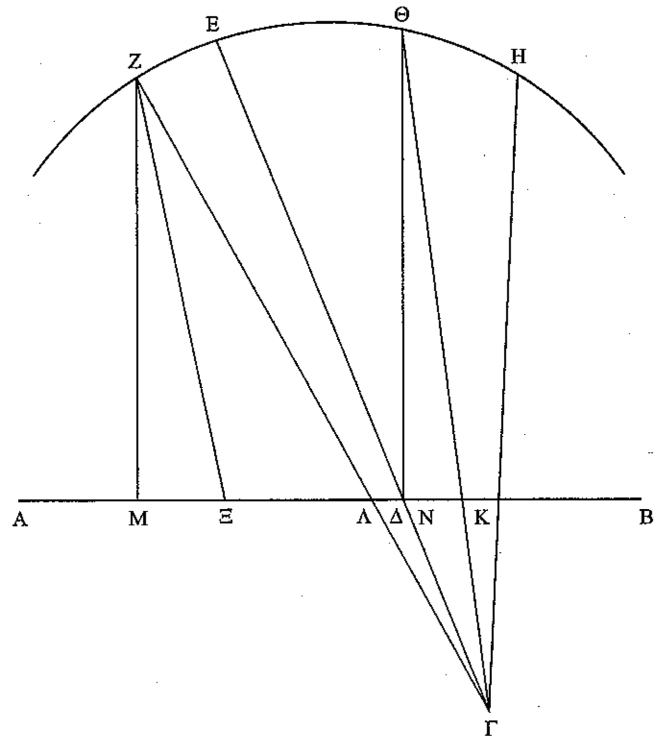
199 That is, a projection beyond the ruler AB.

200 As the ruler runs along the groove, its rigidity is unaffected and its internal distances are kept, including that between the points E and K.

201 "The base of the ruler:" The writing-tool is imagined to move on, say, a piece of papyrus, which is *beneath* the machine.

the magnitude of the ruler EK"Radius[202] of the Line," and <he calls> Δ "Pole."[203]

So he proves for this line: that it has the property that it draws nearer and nearer to the ruler AB; and that if any straight line is drawn between the line and the ruler AB, it will always cut the line. And the first of the properties is best seen on another diagram. Imagining a ruler, AB, and a pole Γ, and a radius ΔE, and a conchoid line ZEH, let the two <lines> ΓΘ, ΓZ be drawn forward from Γ, (1) the resulting <lines> KΘ, ΛZ being, obviously, equal. I say that the perpendicular ZM is smaller than the perpendicular ΘN.[204]



Catalogue: Nicomedes, second diagram
Codex B draws ΘN so that N does not coincide with Δ, but falls on the line AB between the points K, Δ. Codex G has the line ΓH not perpendicular to the line AB, but tilted to the left, the rest following with an even stronger leftward tilt. Codex E had an obvious mistake, corrected perhaps immediately by the same scribe, with the distribution of the letters: at first, it had A instead of M, M instead of Ξ, Ξ instead of Λ. Curiously, codex 4 has a somewhat similar mistake, corrected apparently by the same hand: A instead of M, B instead of Ξ. Codex H has Z instead of Ξ, A instead of Δ. Heiberg has strangely omitted B.

[202] *Diastema*, literally, "interval:" one of the expressions used by the Greeks for our "radius."

[203] Abstracting away the machinery, the conchoid is the locus of points K where a given length EK is on a line passing through a given point Δ, and E is on a given line AB. (There is a further condition, that K is on the other side of AB than Δ.) With the limiting case of Δ being on the line AB itself, the conchoid becomes a circle, accounting for Nicomedes' metaphor of EK as "radius."

[204] It appears that, in the figure, the points Δ, N are taken to coincide (so as to save space and avoid clutter), though – so as to keep the case general – they preserve their separate names and identities.

Again, given an angle, A,[217] and an external point, Γ,[218] to draw ΓH and to make KH[219] equal to a given <line>.[220]

(a) Let a perpendicular, <namely> ΓΘ, be drawn from the point Γ on AB, (b) and let it be produced, and let ΔΘ be equal to the given <line>, (c) and with the pole Γ, and the given radius ΔΘ, and the ruler AB, let a first conchoid line be drawn, EΔZ. (1) Therefore AH cuts it ((2) through what was proved). (d) Let it cut it at H, (e) and let ΓH be joined; (3) therefore KH is equal to the given <line>.

These things proved, let two lines be given at right <angles> to each other, ΓΛ, ΛA, between whom it is required to find continuous two mean proportionals (a) and let the parallelogram ABΓΛ be completed, (b) and let each of AB, BΓ be bisected by the points Δ, E, (c) and, having joined ΔΛ, let it be produced and let it meet ΓB, produced <as well>, at H, (d) and <let> EZ <be drawn> at right <angles> to BΓ, (e) and let ΓZ be produced, being equal to AΔ,[221] (f) and let ZH be joined, (g) and <let> ΓΘ <be drawn> parallel to it <ZH>, (1) and, there being an angle, (the <one contained> by KΓΘ), (h) let ZΘK be drawn through, from Z, (a given <point>), making ΘK equal to AΔ or to ΓZ; (2) for it was proved through the conchoid that this is possible;[222] (i) and, having joined KΛ, let it be produced and let it meet AB, produced <as well>, at M. I say, that it is: as ΓΛ to KΓ, KΓ to MA and MA to AΛ.

(1) Since BΓ has been bisected by E, (2) and KΓ is added to it, (3) therefore the <rectangle contained> by BKΓ with the <square> on ΓE is equal to the <square> on EK.[223] (4) Let the <square> on EZ be added <as> common; (5) therefore the <rectangle contained> by BKΓ with the <squares> on ΓEZ,[224] that is <with> the <square> on ΓZ[225] (6) is equal to the <squares> on KEZ, (7) that is the <square> on KZ.[226] (8) And since, as MA to AB, MΛ to ΛK,[227] (9) but as MΛ to ΛK, so BΓ to ΓK,[228] (10) therefore also: as MA to AB, so BΓ to ΓK.

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[217] "An angle, A:" by this Nicomedes refers to the angle BAH.

[218] Γ is *external* in the sense that it does not fall in the section of the plane intercepted by the angle BAH. It is external to this angle.

[219] KH is implicitly defined roughly as follows: "the section of the line drawn from Γ, which is intercepted by the angle BAH."

[220] Confusingly, the text does not move on directly to the problem of finding two mean proportionals, but adds a final lemma, this time a problem solved with the conchoid.

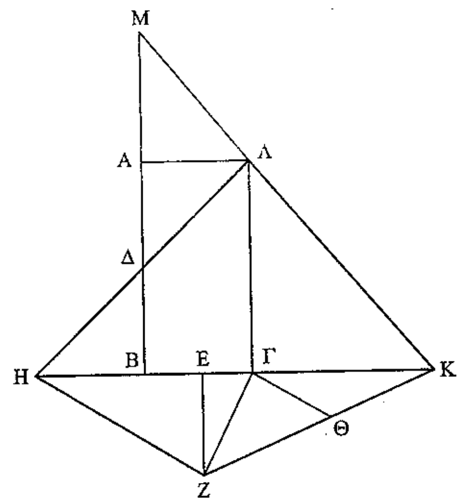
[221] Step e is where the point Z is completely determined (Step d merely set the line on which it is located).

[222] In the preceding lemma. [223] *Elements* II.6.

[224] "The <squares> on ΓEZ:" A way of saying "the squares on ΓE, EZ."

[225] *Elements* I.47. [226] *Elements* I.47.

[227] *Elements* VI.2. [228] *Elements* VI.2.



Catalogue: Nicomedes, fifth diagram
Codex D has ΔB greater than ΔA; codex G has ΔA greater than ΔB.

(11) And ΑΔ is half AB, (12) while ΓH is twice BΓ ((13) since ΓΛ, too, <is twice> ΔB[229]); (14) therefore it shall also be: as MA to ΑΔ, so HΓ to KΓ.[230] (15) But as HΓ to ΓK, so ZΘ to ΘK ((16) through the parallels HZ, ΓΘ[231]); (17) therefore compoundly, too: as MΔ to ΔA, ZK to KΘ.[232] (18) But ΑΔ, in turn, is assumed equal to ΘK, (19) since ΑΔ is equal to ΓZ, as well;[233] (20) therefore MΔ, as well, is equal to ZK; (21) therefore the <square> on MΔ, too, is equal to the <square> on ZK. (22) And the <rectangle contained> by BMA with the <square> on ΔA is equal to the <square> on MΔ,[234] (23) while the <rectangle contained> by BKΓ, with the <square> on ΓZ, was proved equal to the <square> on ZK, (24) of which, the <square>

229 Step 12 derives from Step 13 through *Elements* VI.2.

230 The move from Steps 10–12 to Step 14 is interesting. The intuition is that, halving the consequent of a ratio, and doubling the antecedent of a ratio, are equivalent operations. If you take 17:12, you may "double" it in two different ways: either by doubling the antecedent (so you get 34:12), or by halving the consequent (so you get 17:6). Ratios yield a binary structure, a Noah's ark of paired actions and counter-actions.

231 And then apply *Elements* VI.2. From Step 14 (MA:ΑΔ::HΓ:KΓ) and 15 (HΓ:ΓK::ZΘ:ΘK) one expects: Step 16* (MA:ΑΔ::ZΘ:ΘK). This is not asserted, but is understood as the starting-point for Step 17.

232 *Elements* V.18.

233 That the text derives Step 18 – whose claim was explicitly stated in Step h – is very jarring. (Most probably, the final clause in Step h, "or to ΓZ," is a later gloss added on the basis of Step e, anticipating Step 18.)

234 *Elements* II.6. The original reads "to the square . . . is equal the rectangle with the square," so the topic of the Greek sentence is the square on MΔ, connecting all Steps 20–2.

ΑΔ[235] is equal to the <square> on ΓΖ;[236] (25) for ΑΔ was assumed equal to ΓΖ; (26) therefore the <rectangle contained> by ΒΜΑ, too, is equal to the <rectangle contained> by ΒΚΓ. (27) Therefore as ΜΒ to ΒΚ, ΚΓ to ΑΜ.[237] (28) But as ΒΜ to ΒΚ, ΓΛ to ΓΚ;[238] (29) therefore also: as ΛΓ to ΓΚ, ΓΚ to ΑΜ. (30) And it is also: as ΛΓ to ΓΚ, ΜΑ to ΑΛ;[239] (31) therefore also: as ΛΓ to ΓΚ, ΓΚ to ΑΜ and ΑΜ to ΑΛ.

To the second theorem

Arch. 192 "And compoundly, as ΔΘ to ΘΓ, ΓΑ to ΑΕ, that is the <square> on ΓΒ to the <square> on ΒΕ." For, as on the same diagram in the said:[240] (1) since, in a right angled triangle – ΓΒΑ – a perpendicular has been drawn from the right <angle> on the base, (2) the triangles next to the perpendicular are similar both to the whole and to each other,[241] (3) and through this it is: as ΓΑ to ΑΒ, ΒΑ to ΑΕ (4) and ΓΒ to ΒΕ; (5) so that also: as the <square> on ΓΑ to the <square> on ΑΒ, so the <square> on ΓΒ to the <square> on ΒΕ. (6) But as the <square> on ΓΑ to the <square> on ΑΒ, so ΓΑ to ΑΕ;[242] (7) for as the first to the third, so the <square> on the first to the <square> on the second.[243] (8) Therefore as ΓΑ to ΑΕ, so the <square> on ΓΒ to the <square> on ΒΕ.

Arch. 193 Through the same it is proved, that it is: as ΓΑ to ΓΕ, so the <square> on ΑΒ to the <square> on ΒΕ.[244] (1) For through the similarity of the triangles, (2) it is, again: as ΑΓ to ΓΒ, so ΒΓ to ΓΕ,[245] (3) that is, as the <square> on ΑΓ to the <square> on ΓΒ, so ΑΓ to ΓΕ;[246] (4) while as the <square> on ΑΓ to the <square> on ΓΒ, so

235 The preposition "on" is omitted in the manuscripts. Probably it should be reinstated (as in Heiberg, following Moerbeke), but I keep the manuscripts' reading, to point at the possibility that someone along the chain of transmission thought of this square in an abstract, very modern way.

236 Starting from the implicit result of Steps 21–3: ((rect. ΒΜΑ)+(sq. ΔΑ))=((rect. ΒΚΓ)+(sq. ΓΖ)) we further notice in Step 24 that (sq. ΔΑ)=(sq. ΓΖ), whence Step 26 is obvious.

237 *Elements* VI.16. 238 *Elements* VI.2. 239 *Elements* VI.2, applied twice.

240 Proposition 2 has two diagrams so one has to distinguish between the two. Eutocius explains he refers to the diagram in the text from which the preceding quotation is taken.

241 *Elements* VI.8.

242 Can be deduced from *Elements* VI.8 Cor., 20 Cor. 2.

243 The ordinals are terms in a continuous proportion first:second::second:third. The reference is to *Elements* VI.20 Cor. 2.

244 *SC* II.2, Step 31. 245 *Elements* VI.8 Cor. 246 *Elements* VI. 20 Cor. 2.