hands of later and abler geometers, since it gives a method of approximating, with any desired degree of accuracy, to the area of a circle, and lies at the root of the *method of exhaustion* as established by Eudoxus. As regards Hippocrates's quadrature of lunes, we must, notwithstanding the criticism of Aristotle charging him with a paralogism, decline to believe that he was under any illusion as to the limits of what his method could accomplish, or thought that he had actually squared the circle.

The squaring of the circle.

There is presumably no problem which has exercised such a fascination throughout the ages as that of rectifying or squaring the circle; and it is a curious fact that its attraction has been no less (perhaps even greater) for the non-mathematician than for the mathematician. It was naturally the kind of problem which the Greeks, of all people, would take up with zest the moment that its difficulty was realized. The first name connected with the problem is Anaxagoras, who is said to have occupied himself with it when in prison.[1] The Pythagoreans claimed that it was solved in their school, 'as is clear from the demonstrations of Sextus the Pythagorean, who got his method of demonstration from early tradition'[2]; but Sextus, or rather Sextius, lived in the reign of Augustus or Tiberius, and, for the usual reasons, no value can be attached to the statement.

The first serious attempts to solve the problem belong to the second half of the fifth century B.C. A passage of Aristophanes's *Birds* is quoted as evidence of the popularity of the problem at the time (414 B.C.) of its first representation. Aristophanes introduces Meton, the astronomer and discoverer of the Metonic cycle of 19 years, who brings with him a ruler and compasses, and makes a certain construction 'in order that your circle may become square'.[3] This is a play upon words, because what Meton really does is to divide a circle into four quadrants by two diameters at right angles to one another; the idea is of streets radiating from the agora in the centre

[1] Plutarch, *De exil.* 17, p. 607 F.
[2] Iambl. ap. Simpl. *in Categ.*, p. 192, 16-19 K., 64 b 11 Brandis.
[3] Aristophanes, *Birds* 1005.

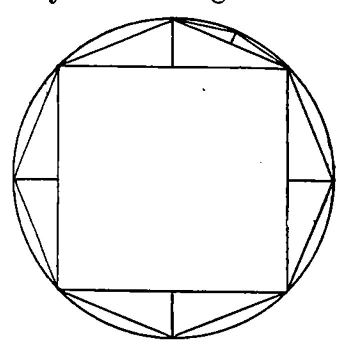
of a town; the word τετράγωνος then really means 'with four (right) angles' (at the centre), and not 'square', but the word conveys a laughing allusion to the problem of squaring all the same.

We have already given an account of Hippocrates's quadratures of lunes. These formed a sort of *prolusio*, and clearly did not purport to be a solution of the problem; Hippocrates was aware that 'plane' methods would not solve it, but, as a matter of interest, he wished to show that, if circles could not be squared by these methods, they could be employed to find the area of *some* figures bounded by arcs of circles, namely certain lunes, and even of the sum of a certain circle and a certain lune.

ANTIPHON of Athens, the Sophist and a contemporary of Socrates, is the next person to claim attention. We owe to Aristotle and his commentators our knowledge of Antiphon's method. Aristotle observes that a geometer is only concerned to refute any fallacious arguments that may be propounded in his subject if they are based upon the admitted principles of geometry; if they are not so based, he is not concerned to refute them:

'thus it is the geometer's business to refute the quadrature by means of segments, but it is not his business to refute that of Antiphon'.[1]

As we have seen, the quadrature 'by means of segments' is probably Hippocrates's quadrature of lunes. Antiphon's method is indicated by Themistius[2] and Simplicius.[3] Suppose there is any regular polygon inscribed in a circle, e.g. a square or an equilateral triangle. (According to Themistius, Antiphon began with an equilateral triangle, and this seems to be the authentic version; Simplicius says he inscribed some one of the regular polygons which can be inscribed



[1] Arist. *Phys.* i. 2, 185 a 14–17.
[2] Them. *in Phys.*, p. 4. 2 sq., Schenkl.
[3] Simpl. *in Phys.*, p. 54. 20–55. 24, Diels.

in a circle, 'suppose, if it so happen, that the inscribed polygon is a square'.) On each side of the inscribed triangle or square as base describe an isosceles triangle with its vertex on the arc of the smaller segment of the circle subtended by the side. This gives a regular inscribed polygon with double the number of sides. Repeat the construction with the new polygon, and we have an inscribed polygon with four times as many sides as the original polygon had. Continuing the process,

'Antiphon thought that in this way the area (of the circle) would be used up, and we should some time have a polygon inscribed in the circle the sides of which would, owing to their smallness, coincide with the circumference of the circle. And, as we can make a square equal to any polygon . . . we shall be in a position to make a square equal to a circle.'

Simplicius tells us that, while according to Alexander the geometrical principle hereby infringed is the truth that a circle touches a straight line in one point (only), Eudemus more correctly said it was the principle that magnitudes are divisible without limit; for, if the area of the circle is divisible without limit, the process described by Antiphon will never result in using up the whole area, or in making the sides of the polygon take the position of the actual circumference of the circle. But the objection to Antiphon's statement is really no more than verbal; Euclid uses exactly the same construction in XII. 2, only he expresses the conclusion in a different way, saying that, if the process be continued far enough, the small segments left over will be together less than any assigned area. Antiphon in effect said the same thing, which again we express by saying that the circle is the *limit* of such an inscribed polygon when the number of its sides is indefinitely increased. Antiphon therefore deserves an honourable place in the history of geometry as having originated the idea of *exhausting* an area by means of inscribed regular polygons with an ever increasing number of sides, an idea upon which, as we said, Eudoxus founded his epoch-making *method of exhaustion*. The practical value of Antiphon's construction is illustrated by Archimedes's treatise on the *Measurement of a Circle*, where, by constructing inscribed and circumscribed regular polygons with 96 sides, Archimedes proves that $3\frac{1}{7} > \pi > 3\frac{10}{71}$, the lower limit, $\pi > 3\frac{10}{71}$, being obtained by calculating the

perimeter of the *inscribed* polygon of 96 sides, which is constructed in Antiphon's manner from an inscribed equilateral triangle. The same construction starting from a square was likewise the basis of Vieta's expression for $2/\pi$, namely

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\,.\,\cos\frac{\pi}{8}\,.\,\cos\frac{\pi}{16}\cdots$$
$$= \sqrt{\tfrac{1}{2}}\,.\,\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)}\,.\,\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)}\right)}\cdots \textit{(ad inf.)}$$

BRYSON, who came a generation later than Antiphon, being a pupil of Socrates or of Euclid of Megara, was the author of another attempted quadrature which is criticized by Aristotle as 'sophistic' and 'eristic' on the ground that it was based on principles not special to geometry but applicable equally to other subjects.[1] The commentators give accounts of Bryson's argument which are substantially the same, except that Alexander speaks of *squares* inscribed and circumscribed to a circle[2], while Themistius and Philoponus speak of any polygons.[3] According to Alexander, Bryson inscribed a square in a circle and circumscribed another about it, while he also took a square intermediate between them (Alexander does not say how constructed); then he argued that, as the intermediate square is less than the outer and greater than the inner, while the circle is also less than the outer square and greater than the inner, and as *things which are greater and less than the same things respectively are equal*, it follows that the circle is equal to the intermediate square: upon which Alexander remarks that not only is the thing assumed applicable to other things besides geometrical magnitudes, e.g. to numbers, times, depths of colour, degrees of heat or cold, &c., but it is also false because (for instance) 8 and 9 are both less than 10 and greater than 7 and yet they are not equal. As regards the intermediate square (or polygon), some have assumed that it was the arithmetic mean between the inscribed and circumscribed figures, and others that it was the geometric mean. Both assumptions seem to be due to misunderstanding[4]; for

[1] Arist. *An. Post.* i. 9, 75 b 40.
[2] Alexander on *Soph. El.*, p. 90. 10–21, Wallies, 306 b 24 sq., Brandis.
[3] Them. on *An. Post.*, p. 19. 11–20, Wallies, 211 b 19, Brandis; Philop. on *An. Post.*, p. 111. 20–114. 17 W., 211 b 30, Brandis.
[4] Psellus (11th cent. A.D.) says, 'there are different opinions as to the

the ancient commentators do not attribute to Bryson any such statement, and indeed, to judge by their discussions of different interpretations, it would seem that tradition was by no means clear as to what Bryson actually did say. But it seems important to note that Themistius states (1) that Bryson declared the circle to be greater than *all* inscribed, and less than *all* circumscribed, polygons, while he also says (2) that the assumed axiom is *true*, though not peculiar to geometry. This suggests a possible explanation of what otherwise seems to be an absurd argument. Bryson may have multiplied the number of the sides of both the inscribed and circumscribed regular polygons as Antiphon did with inscribed polygons; he may then have argued that, if we continue this process long enough, we shall have an inscribed and a circumscribed polygon differing so little in area that, if we can describe a polygon intermediate between them in area, the circle, which is also intermediate in area between the inscribed and circumscribed polygons, must be equal to the intermediate polygon.[1] If this is the right explanation, Bryson's name by no means deserves to be banished from histories of Greek mathematics; on the contrary, in so far as he suggested the necessity of considering circumscribed as well as inscribed polygons, he went a step further than Antiphon; and the importance of the idea is attested by the fact that, in the regular method of exhaustion as practised by Archimedes, use is made of both inscribed and circumscribed figures, and this *compression*, as it were, of a circumscribed and an inscribed figure into one so that they ultimately coincide with one another, and with the

proper method of finding the area of a circle, but that which has found the most favour is to take the geometric mean between the inscribed and circumscribed squares'. I am not aware that he quotes Bryson as the authority for this method, and it gives the inaccurate value $\pi = \sqrt{8}$ or 2·8284272.... Isaac Argyrus (14th cent.) adds to his account of Bryson the following sentence: 'For the circumscribed square *seems* to exceed the circle by the same amount as the inscribed square is exceeded by the circle.'

[1] It is true that, according to Philoponus, Proclus had before him an explanation of this kind, but rejected it on the ground that it would mean that the circle must actually *be* the intermediate polygon and not only be equal to it, in which case Bryson's contention would be tantamount to Antiphon's, whereas according to Aristotle it was based on a quite different principle. But it is sufficient that the circle should be taken to be *equal* to any polygon that can be drawn intermediate between the two ultimate polygons, and this gets over Proclus's difficulty.

curvilinear figure to be measured, is particularly characteristic of Archimedes.

We come now to the real rectifications or quadratures of circles effected by means of higher curves, the construction of which is more 'mechanical' than that of the circle. Some of these curves were applied to solve more than one of the three classical problems, and it is not always easy to determine for which purpose they were originally destined by their inventors, because the accounts of the different authorities do not quite agree. Iamblichus, speaking of the quadrature of the circle, said that

'Archimedes effected it by means of the spiral-shaped curve, Nicomedes by means of the curve known by the special name *quadratrix* (τετραγωνίζουσα), Apollonius by means of a certain curve which he himself calls "sister of the cochloid" but which is the same as Nicomedes's curve, and finally Carpus by means of a certain curve which he simply calls (the curve arising) "from a double motion".'[1]

Pappus says that

'for the squaring of the circle Dinostratus, Nicomedes and certain other and later geometers used a certain curve which took its name from its property; for those geometers called it *quadratrix*.'[2]

Lastly, Proclus, speaking of the trisection of any angle, says that

'Nicomedes trisected any rectilineal angle by means of the conchoidal curves, the construction, order and properties of which he handed down, being himself the discoverer of their peculiar character. Others have done the same thing by means of the *quadratrices* of Hippias and Nicomedes.... Others again, starting from the spirals of Archimedes, divided any given rectilineal angle in any given ratio.'[3]

All these passages refer to the *quadratrix* invented by Hippias of Elis. The first two seem to imply that it was not used by Hippias himself for squaring the circle, but that it was Dinostratus (a brother of Menaechmus) and other later geometers who first applied it to that purpose; Iamblichus and Pappus do not even mention the name of Hippias. We might conclude that Hippias originally intended his curve to

[1] Iambl. ap. Simpl. *in Categ.*, p. 192. 19–24 K., 64 b 13–18 Br.
[2] Pappus, iv, pp. 250. 33–252. 3.
[3] Proclus on Eucl. I, p. 272. 1–12.

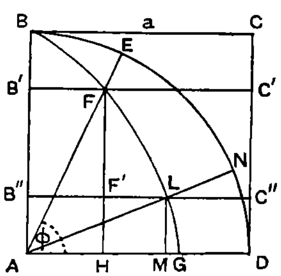
be used for trisecting an angle. But this becomes more doubtful when the passages of Proclus are considered. Pappus's authority seems to be Sporus, who was only slightly older than Pappus himself (towards the end of the third century A.D.), and who was the author of a compilation called Κηρία containing, among other things, mathematical extracts on the quadrature of the circle and the duplication of the cube. Proclus's authority, on the other hand, is doubtless Geminus, who was much earlier (first century B.C.) Now not only does the above passage of Proclus make it possible that the name *quadratrix* may have been used by Hippias himself, but in another place Proclus (i.e. Geminus) says that different mathematicians have explained the properties of particular kinds of curves:

'thus Apollonius shows in the case of each of the conic curves what is its property, and similarly Nicomedes with the conchoids, *Hippias with the quadratrices*, and Perseus with the spiric curves.'[1]

This suggests that Geminus had before him a regular treatise by Hippias on the properties of the *quadratrix* (which may have disappeared by the time of Sporus), and that Nicomedes did not write any such general work on that curve; and, if this is so, it seems not impossible that Hippias himself discovered that it would serve to rectify, and therefore to square, the circle.

(α) *The Quadratrix of Hippias.*

The method of constructing the curve is described by Pappus.[2] Suppose that $ABCD$ is a square, and BED a quadrant of a circle with centre A.



Suppose (1) that a radius of the circle moves uniformly about A from the position AB to the position AD, and (2) that *in the same time* the line BC moves uniformly, always parallel to itself and with its extremity B moving along BA, from the position BC to the position AD.

[1] Proclus on Eucl. I, p. 356. 6–12.

[2] Pappus, iv, pp. 252 sq.

Then, in their ultimate positions, the moving straight line and the moving radius will both coincide with AD; and at any previous instant during the motion the moving line and the moving radius will by their intersection determine a point, as F or L.

The locus of these points is the *quadratrix.*

The property of the curve is that

$$\angle BAD : \angle EAD = (\text{arc } BED) : (\text{arc } ED) = AB : FH.$$

In other words, if ϕ is the angle FAD made by any radius vector AF with AD, ρ the length of AF, and a the length of the side of the square,

$$\frac{\rho \sin \phi}{a} = \frac{\phi}{\frac{1}{2}\pi}.$$

Now clearly, when the curve is once constructed, it enables us not only to *trisect* the angle EAD but also to *divide it in any given ratio.*

For let FH be divided at F' in the given ratio. Draw $F'L$ parallel to AD to meet the curve in L: join AL, and produce it to meet the circle in N.

Then the angles EAN, NAD are in the ratio of FF' to $F'H$, as is easily proved.

Thus the quadratrix lends itself quite readily to the division of any angle in a given ratio.

The application of the *quadratrix* to the rectification of the circle is a more difficult matter, because it requires us to know the position of G, the point where the quadratrix intersects AD. This difficulty was fully appreciated in ancient times, as we shall see.

Meantime, assuming that the quadratrix intersects AD in G, we have to prove the proposition which gives the length of the arc of the quadrant BED and therefore of the circumference of the circle. This proposition is to the effect that

$$(\text{arc of quadrant } BED) : AB = AB : AG.$$

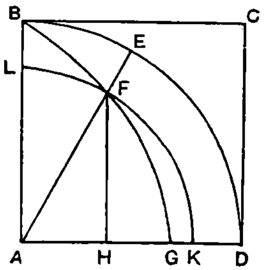
This is proved by *reductio ad absurdum.*

If the former ratio is not equal to $AB : AG$, it must be equal to $AB : AK$, where AK is either (1) greater or (2) less than AG.

(1) Let AK be greater than AG; and with A as centre

and AK as radius, draw the quadrant KFL cutting the quadratrix in F and AB in L.

Join AF, and produce it to meet the circumference BED in E; draw FH perpendicular to AD.



Now, by hypothesis,

$$(\text{arc } BED) : AB = AB : AK$$
$$= (\text{arc } BED) : (\text{arc } LFK);$$

therefore $AB = (\text{arc } LFK)$.

But, by the property of the *quadratrix*,

$$AB : FH = (\text{arc } BED) : (\text{arc } ED)$$
$$= (\text{arc } LFK) : (\text{arc } FK);$$

and it was proved that $AB = (\text{arc } LFK)$;

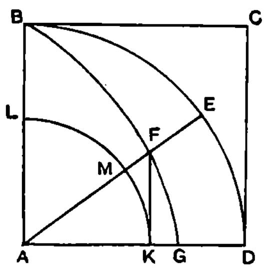
therefore $FH = (\text{arc } FK)$:

which is absurd. Therefore AK is not greater than AG.

(2) Let AK be less than AG.

With centre A and radius AK draw the quadrant KML.

Draw KF at right angles to AD meeting the quadratrix in F; join AF, and let it meet the quadrants in M, E respectively.



Then, as before, we prove that

$$AB = (\text{arc } LMK).$$

And, by the property of the *quadratrix*,

$$AB : FK = (\text{arc } BED) : (\text{arc } DE)$$
$$= (\text{arc } LMK) : (\text{arc } MK).$$

Therefore, since $AB = (\text{arc } LMK)$,

$FK = (\text{arc } KM)$:

which is absurd. Therefore AK is not less than AG.

Since then AK is neither less nor greater than AG, it is equal to it, and

$$(\text{arc } BED) : AB = AB : AG.$$

[The above proof is presumably due to Dinostratus (if not to Hippias himself), and, as Dinostratus was a brother of Menaechmus, a pupil of Eudoxus, and therefore probably

flourished about 350 B.C., that is to say, some time before Euclid, it is worth while to note certain propositions which are assumed as known. These are, in addition to the theorem of Eucl. VI. 33, the following: (1) the circumferences of circles are as their respective radii; (2) any arc of a circle is greater than the chord subtending it; (3) any arc of a circle less than a quadrant is less than the portion of the tangent at one extremity of the arc cut off by the radius passing through the other extremity. (2) and (3) are of course equivalent to the facts that, if α be the circular measure of an angle less than a right angle, $\sin\alpha < \alpha < \tan\alpha$.]

Even now we have only rectified the circle. To square it we have to use the proposition (1) in Archimedes's *Measurement of a Circle*, to the effect that the area of a circle is equal to that of a right-angled triangle in which the perpendicular is equal to the radius, and the base to the circumference, of the circle. This proposition is proved by the method of exhaustion and may have been known to Dinostratus, who was later than Eudoxus, if not to Hippias.

The criticisms of Sporus,[1] in which Pappus concurs, are worth quoting:

(1) 'The very thing for which the construction is thought to serve is actually assumed in the hypothesis. For how is it possible, with two points starting from B, to make one of them move along a straight line to A and the other along a circumference to D in an equal time, unless you first know the ratio of the straight line AB to the circumference BED? In fact this ratio must also be that of the speeds of motion. For, if you employ speeds not definitely adjusted (to this ratio), how can you make the motions end at the same moment, unless this should sometime happen by pure chance? Is not the thing thus shown to be absurd?

(2) 'Again, the extremity of the curve which they employ for squaring the circle, I mean the point in which the curve cuts the straight line AD, is not found at all. For if, in the figure, the straight lines CB, BA are made to end their motion together, they will then coincide with AD itself and will not cut one another any more. In fact they cease to intersect before they coincide with AD, and yet it was the intersection of these lines which was supposed to give the extremity of the

[1] Pappus, iv, pp. 252. 26–254. 22.

curve, where it met the straight line AD. Unless indeed any one should assert that the curve is conceived to be produced further, in the same way as we suppose straight lines to be produced, as far as AD. But this does not follow from the assumptions made; the point G can only be found by first assuming (as known) the ratio of the circumference to the straight line.'

The second of these objections is undoubtedly sound. The point G can in fact only be found by applying the method of exhaustion in the orthodox Greek manner; e.g. we may first bisect the angle of the quadrant, then the half towards AD, then the half of that and so on, drawing each time from the points F in which the bisectors cut the quadratrix perpendiculars FH on AD and describing circles with AF as radius cutting AD in K. Then, if we continue this process long enough, HK will get smaller and smaller and, as G lies between H and K, we can approximate to the position of G as nearly as we please. But this process is the equivalent of approximating to π, which is the very object of the whole construction.

As regards objection (1) Hultsch has argued that it is not valid because, with our modern facilities for making instruments of precision, there is no difficulty in making the two uniform motions take the same time. Thus an accurate clock will show the minute hand describing an exact quadrant in a definite time, and it is quite practicable now to contrive a uniform rectilinear motion taking exactly the same time. I suspect, however, that the rectilinear motion would be the result of converting some one or more circular motions into rectilinear motions; if so, they would involve the use of an approximate value of π, in which case the solution would depend on the assumption of the very thing to be found. I am inclined, therefore, to think that both Sporus's objections are valid.

(β) *The Spiral of Archimedes.*

We are assured that Archimedes actually used the spiral for squaring the circle. He does in fact show how to rectify a circle by means of a polar subtangent to the spiral. The spiral is thus generated: suppose that a straight line with one extremity fixed starts from a fixed position (the initial line) and revolves uniformly about the fixed extremity, while a point also moves uniformly along the moving straight line starting from the fixed extremity (the origin) at the commencement of the straight line's motion; the curve described is a spiral.

The polar equation of the curve is obviously $\rho = a\theta$.

Suppose that the tangent at any point P of the spiral is met at T by a straight line drawn from O, the origin or pole, perpendicular to the radius vector OP; then OT is the polar subtangent.

Now in the book *On Spirals* Archimedes proves generally the equivalent of the fact that, if ρ be the radius vector to the point P,

$$OT = \rho^2/a.$$

If P is on the nth turn of the spiral, the moving straight line will have moved through an angle $2(n-1)\pi+\theta$, say.

Hence $$\rho = a\{2(n-1)\pi+\theta\},$$

and $$OT = \rho^2/a = \rho\{2(n-1)\pi+\theta\}.$$

Archimedes's way of expressing this is to say (Prop. 20) that, if p be the circumference of the circle with radius $OP\,(=\rho)$, and if this circle cut the initial line in the point K,

$$OT = (n-1)\,p + \text{arc}\,KP \text{ measured 'forward' from } K \text{ to } P.$$

If P is the end of the nth turn, this reduces to

$$OT = n \text{ (circumf. of circle with radius } OP\text{)},$$

and, if P is the end of the first turn in particular,

$$OT = \text{(circumf. of circle with radius } OP\text{)}. \quad \text{(Prop. 19.)}$$

The spiral can thus be used for the rectification of any circle. And the quadrature follows directly from *Measurement of a Circle*, Prop. 1.

(γ) *Solutions by Apollonius and Carpus.*

Iamblichus says that Apollonius himself called the curve by means of which he squared the circle 'sister of the cochloid'. What this curve was is uncertain. As the passage goes on to say that it was really 'the same as the (curve) of Nicomedes', and the quadratrix has just been mentioned as the curve used by Nicomedes, some have supposed the 'sister of the cochloid' (or conchoid) to be the *quadratrix*, but this seems highly improbable. There is, however, another possibility. Apollonius is known to have written a regular treatise on the *Cochlias*, which was the cylindrical helix.[1] It is conceivable that he might call the *cochlias* the 'sister of the *cochloid*' on the ground of the similarity of the names, if not of the curves. And, as a matter of fact, the drawing of a tangent to the helix enables the circular section of the cylinder to be squared. For, if a plane be drawn at right angles to the axis of the cylinder through the initial position of the moving radius which describes the helix, and if we project on this plane the portion of the tangent at any point of the helix intercepted between the point and the plane, the projection is equal to an arc of the circular section of the cylinder subtended by an angle at the centre equal to the angle through which the plane through the axis and the moving radius has turned from its original position. And this squaring by means of what we may call the 'subtangent' is sufficiently parallel to the use by Archimedes of the polar subtangent to the spiral for the same purpose to make the hypothesis attractive.

Nothing whatever is known of Carpus's curve 'of double motion'. Tannery thought it was the cycloid; but there is no evidence for this.

(δ) *Approximations to the value of π.*

As we have seen, Archimedes, by inscribing and circumscribing regular polygons of 96 sides, and calculating their perimeters respectively, obtained the approximation $3\frac{1}{7} > \pi > 3\frac{10}{71}$ (*Measurement of a Circle*, Prop. 3). But we now learn[2] that, in a work on *Plinthides and Cylinders*, he made a nearer approximation still. Unfortunately the figures as they stand in the Greek text are incorrect, the lower limit being given as the ratio of $\overset{\kappa\alpha}{\mu},\alpha\omega o\epsilon$ to $\overset{ϛ}{\mu},\zeta\upsilon\mu\alpha$, or 211875 : 67441 (= 3·141635), and the higher limit as the ratio of $\overset{\iota\theta}{\mu},\zeta\omega\pi\eta$ to $\overset{ϛ}{\mu},\beta\tau\nu\alpha$ or 197888 : 62351 (= 3·17377), so that the lower limit

[1] Pappus, viii, p. 1110. 20; Proclus on Eucl. I, p. 105. 5.
[2] Heron, *Metrica*, i. 26, p. 66. 13–17.

as given is greater than the true value, and the higher limit is greater than the earlier upper limit $3\frac{1}{7}$. Slight corrections by Tannery ($\overset{\kappa\alpha}{\mu},\alpha\omega o\beta$ for $\overset{\kappa\alpha}{\mu},\alpha\omega o\epsilon$ and $\overset{\iota\theta}{\mu},\epsilon\omega\pi\beta$ for $\overset{\iota\theta}{\mu},\zeta\omega\pi\eta$) give better figures, namely

$$\frac{195882}{62351} > \pi > \frac{211872}{67441}$$

or $$3\cdot1416016 > \pi > 3\cdot1415904\ldots.$$

Another suggestion[1] is to correct $\overset{ϛ}{\mu},\zeta\upsilon\mu\alpha$ into $\overset{ϛ}{\mu},\zeta\upsilon\mu\delta$ and $\overset{\iota\theta}{\mu},\zeta\omega\pi\eta$ into $\overset{\iota\theta}{\mu},\epsilon\omega\pi\eta$, giving

$$\frac{195888}{62351} > \pi > \frac{211875}{67444}$$

or $$3\cdot141697\ldots > \pi > 3\cdot141495\ldots.$$

If either suggestion represents the true reading, the mean between the two limits gives the same remarkably close approximation 3·141596.

Ptolemy[2] gives a value for the ratio of the circumference of a circle to its diameter expressed thus in sexagesimal fractions, γ η λ, i.e. $3 + \frac{8}{60} + \frac{30}{60^2}$ or 3·1416. He observes that this is almost exactly the mean between the Archimedean limits $3\frac{1}{7}$ and $3\frac{10}{71}$. It is, however, more exact than this mean, and Ptolemy no doubt obtained his value independently. He had the basis of the calculation ready to hand in his Table of Chords. This Table gives the lengths of the chords of a circle subtended by arcs of $\frac{1}{2}°$, 1°, $1\frac{1}{2}°$, and so on by half degrees. The chords are expressed in terms of 120th parts of the length of the diameter. If one such part be denoted by 1^p, the chord subtended by an arc of 1° is given by the Table in terms of this unit and sexagesimal fractions of it thus, $1^p\,2'\,50''$. Since an angle of 1° at the centre subtends a side of the regular polygon of 360 sides inscribed in the circle, the perimeter of this polygon is 360 times $1^p\,2'\,50''$ or, since $1^p = 1/120$th of the diameter, the perimeter of the polygon expressed in terms of the diameter is 3 times $1\;2'\,50''$, that is $3\;8'\,30''$, which is Ptolemy's figure for π.

[1] J. L. Heiben in *Nordisk Tidsskrift for Filologi*, 3e Sér. xx. Fasc. 1–2.
[2] Ptolemy, *Syntaxis*, vi. 7, p. 513. 1–5, Heib.

There is evidence of a still closer calculation than Ptolemy's due to some Greek whose name we do not know. The Indian mathematician Aryabhaṭṭa (born A.D. 476) says in his *Lessons in Calculation*:

> 'To 100 add 4; multiply the sum by 8; add 62000 more and thus (we have), for a diameter of 2 myriads, the approximate length of the circumference of the circle';

that is, he gives $\frac{62832}{20000}$ or 3·1416 as the value of π. But the way in which he expresses it points indubitably to a Greek source, 'for the Greeks alone of all peoples made the myriad the unit of the second order' (Rodet).

This brings us to the notice at the end of Eutocius's commentary on the *Measurement of a Circle* of Archimedes, which records[1] that other mathematicians made similar approximations, though it does not give their results.

> 'It is to be observed that Apollonius of Perga solved the same problem in his Ὠκυτόκιον ("means of quick delivery"), using other numbers and making the approximation closer [than that of Archimedes]. While Apollonius's figures seem to be more accurate, they do not serve the purpose which Archimedes had in view; for, as we said, his object in this book was to find an approximate figure suitable for use in daily life. Hence we cannot regard as appropriate the censure of Sporus of Nicaea, who seems to charge Archimedes with having failed to determine with accuracy (the length of) the straight line which is equal to the circumference of the circle, to judge by the passage in his *Keria* where Sporus observes that his own teacher, meaning Philon of Gadara, reduced (the matter) to more exact numerical expression than Archimedes did, I mean in his $\frac{1}{7}$ and $\frac{10}{71}$; in fact people seem, one after the other, to have failed to appreciate Archimedes's object. They have also used multiplications and divisions of myriads, a method not easy to follow for any one who has not gone through a course of Magnus's *Logistica*.'

It is possible that, as Apollonius used myriads, 'second myriads', 'third myriads', &c., as orders of integral numbers, he may have worked with the fractions $\frac{1}{10000}$, $\frac{1}{10000^2}$, &c.;

[1] Archimedes, ed. Heib., vol. iii, pp. 258–9.

in any case Magnus (apparently later than Sporus, and therefore perhaps belonging to the fourth or fifth century A.D.) would seem to have written an exposition of such a method, which, as Eutocius indicates, must have been very much more troublesome than the method of sexagesimal fractions used by Ptolemy.

The Trisection of any Angle.

This problem presumably arose from attempts to continue the construction of regular polygons after that of the pentagon had been discovered. The trisection of an angle would be necessary in order to construct a regular polygon the sides of which are nine, or any multiple of nine, in number. A regular polygon of seven sides, on the other hand, would no doubt be constructed with the help of the first discovered method of dividing any angle in a given ratio, i.e. by means of the *quadratrix*. This method covered the case of trisection, but other more practicable ways of effecting this particular construction were in due time evolved.

We are told that the ancients attempted, and failed, to solve the problem by 'plane' methods, i.e. by means of the straight line and circle; they failed because the problem is not 'plane' but 'solid'. Moreover, they were not yet familiar with conic sections, and so were at a loss; afterwards, however, they succeeded in trisecting an angle by means of conic sections, a method to which they were led by the reduction of the problem to another, of the kind known as νεύσεις (*inclinationes*, or *vergings*).[1]

(α) *Reduction to a certain νεῦσις, solved by conics.*

The reduction is arrived at by the following analysis. It is only necessary to deal with the case where the given angle to be trisected is acute, since a right angle can be trisected by drawing an equilateral triangle.

Let ABC be the given angle, and let AC be drawn perpendicular to BC. Complete the parallelogram $ACBF$, and produce the side FA to E.

[1] Pappus, iv, p. 272. 7–14.